Jesús Garcia-Falset, Khalid Latrach

Nonlinear Functional Analysis and Applications

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To our beloved families with affection and gratitude

Preface

Nonlinear functional analysis is a new area that was born and has matured from abundant research developed in studying nonlinear problems. In the past 30 years, nonlinear analysis has undergone rapid growth; it has become part of the mainstream research fields in contemporary mathematical analysis. It is now an important field of study in both pure and applied mathematics. Many nonlinear analysis problems have their roots in geometry, astronomy, fluid and elastic mechanics, physics, chemistry, biology, control theory, image processing, optimization, game theory, and economics. Their development provided nonlinear functional analysis with new concepts, tools, and theories that enriched the subject considerably. Nowadays nonlinear functional analysis is a well-established mathematical discipline, which is characterized by a remarkable mixture of analysis, topology, and applications. It is exactly the fact that the subject combines in a beautiful way these items that makes it attractive to mathematicians. The notions and techniques of nonlinear functional analysis provide appropriate tools to develop more realistic and accurate models describing various phenomena. Today the more theoretically inclined nonmathematicians (engineers, economists, biologists, or chemists) need a working knowledge of at least a part of nonlinear analysis in order to be able to conduct a complete qualitative analysis of their models. Of course, the subject is actually vast and it is not possible to include in a book all its theoretical and applied parts. Thus, in this manuscript, we have selected some techniques and applications where we have been working recently. We have focused essentially on two topics which are pertinent to evolution equations and boundary value problems: accretive operators and fixed point theory.

The theory of generation of semigroups of linear contractions, which is the basis of the theory of evolution equations governed by linear operators, was developed by E. Hille and K. Yosida in 1948, and by W. Feller, I. Miyadera, and R. S. Phillips for the general case. Theorems of generation of semigroups of Hille–Yosida and Lumer–Phillips and the Hille–Yosida exponential formula are presented. The theory of evolution equations governed by accretive operators started in the late 1960s by the works of Y. Komura who established a generation theorem of nonlinear semigroups in a Hilbert space in 1967. Afterwards, in 1971, the theory was extended to general Banach spaces with the seminal work by M. G. Crandall and T. Liggett. We present an account of various aspects of the (m-)accretive operator theory (old and recent results). The exponential formula of M. G. Crandall and T. Liggett and the generation theorem for accretive operators in Banach spaces are established. The case of quasiaccretive operators is also considered. Crandall–Liggett's generation result is applied to discuss the well-posedness of Cauchy problems (existence and uniqueness of solutions, and their continuous dependence of the initial data) governed by accretive operators. The concepts of mild, weak, and strong solutions are discussed. These results are applied to concrete evolution problems arising in population dynamics and neutron transport theory.

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For stationary problems, the fixed point theory plays a crucial role because, in general, the solutions of such problems can be expressed as fixed points of operators derived from these problems. We present the remarkable Banach's fixed point theorem known as the Banach contraction principle. It is fundamental in fixed point theory and it is a simple tool in establishing existence and uniqueness results for functional and operator equations. In the literature there are numerous generalizations of Banach's fixed point theorem for various kind of contractive mappings (we refer, for example, to the works by E. Rakotch, V. M. Sehgal, D. W. Boyd, and J. S. W. Wong, M. Edelstein, L. Janos, A. Meir, and E. Keeler, R. Kannan, S. Reich, D. Wardowski, S. P. Singh, L. B. Ciric, Y. Liu, and Z. Li, T. Suziki, etc.). We discuss the fixed point property for nonexpansive mappings in some classes of Banach spaces such as Hilbert spaces, reflexive Banach spaces with the Opial property, uniformly convex Banach spaces, reflexive Banach spaces with normal structure. A detailed discussion of fixed points for ϕ-expansive mappings and the zeros of m-accretive mappings is presented.

A detailed exposition of topological fixed point theory for the strong, as well as the weak, topology is given. For the strong topology, the classical results such as Brouwer's fixed point theorem, Schauder's fixed point theorem, Darbo's fixed point theorem, and Sadovskii's fixed point theorem for set contractive mappings (with respect to a measure of noncompactness) are presented. Afterwards, some recents results of Dardo–Sadovskii's type are discussed. For the weak topology, we start with the Tychonoff fixed point theorem in a Hausdorff locally convex topological vector space and its formulation in Banach spaces equipped with the weak topology, together with its extension for weakly sequentially mappings in metrizable locally convex topological vector spaces. We introduce the classes of ws-compact and ww-compact operators and derive some results related to them. Various fixed point theorems of Dardo–Sadovskii's type for set contractive mappings (with respect to a measure of weak noncompactness) are shown. Afterwards, we present fixed point theorems of Schaefer's and Leray–Schauder's type. Next, fixed points results for sums of mappings in bounded sets, unbounded sets, and in Banach algebras are discussed. These results are applied to concrete boundary value problems arising in neutron transport theory.

The book consists of eight chapters. Chapter 1 collects preliminary ideas and gathers most of definitions and concepts which will be needed throughout the book, i. e., elements of topological linear vector spaces, linear operators theory, weak topology, geometry of Banach spaces, Bochner integral, duality mapping, etc. In order to avoid countless referrals to other works, we give definitions and precise statements of many results of functional analysis.

In Chapter 2 we expose an introduction to the theory of C_0-semigroups for linear operators. We introduce the concepts, such as the infinitesimal generator, semigroup of contractions, Yosida approximation, exponential formula, required in Chapters 3 and 4. Theorems of generation (Hille–Yosida's theorem and Lumer–Phillips' theorem) are established.

Chapter 3 deals with the concept of accretive operators in Banach spaces. We give the main properties of accretive operators and m-accretive operators. We present a proof of Crandall–Liggett's exponential formula. We close this chapter by introducing the class of quasiaccretive operators.

In Chapter 4, we deal with abstract Cauchy problems governed by accretive operators. We discuss existence and uniqueness of (strong, integral, weak, mild) solutions to both homogeneous and inhomogeneous Cauchy problems, as well as the relationship between them. We end this chapter by discussing abstract Cauchy problems governed by quasiaccretive and m-quasiaccretive operators and derive some of their elementary properties.

In Chapter 5, we present several applications of general theory to nonlinear Cauchy problems governed by m-accretive or m-quasiaccretive operators, illustrating the ideas and general existence theory developed in the previous chapter.

Chapter 6 is dedicated to the metric fixed point theory. The first part is concerned with the Banach contraction principle and some of its generalizations. In Section 6.5 we have gathered some results about the fixed point property for nonexpansive mappings in some classes of Banach spaces. Section 6.6 is devoted to fixed point properties of ϕ-expansive mappings while Sections 6.7 and 6.8 treat zeros of m-accretive and ϕ-expansive operators. The results of the last three sections of this chapter are recent.

In Chapter 7, we consider the topological fixed point theory in normed spaces. Section 7.2 is concerned with the Brouwer and Schauder fixed point theorems while Section 7.3 is dedicated to the study of set contractive mappings with respect to a measure of noncompactness. In addition to the classical Darbo's and Sadovskii's theorems, we present some recent results of Darbo's and Sadovskii's type. In Section 7.4 we give two recent results concerning the existence of an approximate fixed point sequence for a mapping f such that $I - f$ is ϕ-expansive. In Section 7.5, we present Thychonoff's fixed point theorem and its formulation for weakly sequential maps. We introduce in Section 7.6 the classes of ws-compact and ww-compact mappings. In Sections 7.7, 7.8, and 7.10 the concept of a measure of weak noncompactness is considered, and we give some fixed point results for set contractive mappings with respect to a measure of weak noncompactness. In Section 7.11, our goal is to present some fixed point theorems of Leray–Schauder's type for the weak topology while Section 7.12 deals with fixed point theorems for sums of mappings involving the weak topology in bounded, as well as unbounded, sets. The results presented in Sections 7.11 and 7.12 are recent.

In Chapter 8, we present some examples of nonlinear boundary value problems to illustrate the field of applications of the results presented in Chapters 6 and 7.

Valencia, January 2023 — Jesus Garcia-Falset
Clermont-Ferrand, January 2023 — Khalid Latrach

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1 Fundamentals of functional analysis

1.1 Topological spaces

1.1.1 Topology and base for a topology

Definition 1.1.1. We call a couple (X, Θ), where X is a set and Θ is a collection of subsets of X, a topological space if:

(O1) For any subcollection $(O_\alpha)_{\alpha\in J}$ of Θ, the union $\bigcup_{\alpha\in J} O_\alpha \in \Theta$.

(O2) For any finite subcollection $\{O_1, O_2, \ldots, O_n\}$ of Θ, the intersection $O_1 \cap O_2 \cap \cdots \cap O_n \in \Theta$.

(O3) $\emptyset$ and X are in Θ.

We call a member of Θ an open set in X. We say that F is *closed set in* X if $F^c = X\backslash F$ is open, that is, $X\backslash F \in \Theta$.

We often write (X, Θ) for a topological space X with topology Θ; sometimes, if the topology is obvious or irrelevant, we also may simply write X. A set may be both open and closed, or it may be neither. In particular, $\emptyset$ and X are both open and closed.

Note that a nontrivial set X can have many different topologies. The family of all topologies on X is partially ordered by the set inclusion. If $\Theta' \subseteq \Theta$, that is, if every Θ'-open set is also Θ-open, then we say that Θ' is weaker or coarser than Θ, and that Θ is stronger or finer than Θ'.

It is easy to see that the intersection of a family of topologies on a set is again a topology. If $\mathcal{U}$ is an arbitrary nonempty family of subsets of a set X, then there exists the smallest (with respect to set inclusion) topology that includes $\mathcal{U}$. It is the intersection of all topologies that include $\mathcal{U}$. (Note that the discrete topology always includes $\mathcal{U}$.) This topology is called the topology generated by $\mathcal{U}$ and consists precisely of $\emptyset$ and X, and all sets of the form $\bigcup_\alpha U_\alpha$ where each U_α is a finite intersection of sets from $\mathcal{U}$.

If Y is a subset of a topological space (X, Θ), then the collection Θ_Y of subsets of Y, defined by

$$\Theta_Y = \{V \cap Y : V \in \Theta\},$$

is a topology on Y. This topology is called the *relative topology or the topology induced* by Θ on Y. When $Y \subset X$ is equipped with its relative topology, we call Y a *(topological) subspace* of X. A set in Θ_Y is called *(relatively) open* in Y. For example, since $X \in \Theta$ and $Y \cap X = Y$, the set Y is relatively open in itself. Note that the relatively closed subsets of Y are of the form

$$Y\backslash(Y \cap V) = Y\backslash V = Y \cap (X\backslash V),$$

where $V \in \Theta$.

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Definition 1.1.2. Let (X, Θ) be a topological space. A *base* for the topology Θ is a subfamily $\mathcal{B}$ of Θ such that each $U \in \Theta$ is a union of members of $\mathcal{B}$. Equivalently, $\mathcal{B}$ is a base for Θ if for every $x \in X$ and every open set U containing x, there is an open set $V \in \mathcal{B}$ satisfying $x \in V \subseteq U$.

Let (X, Θ) be a topological space and $\mathcal{B} \subset \Theta$. The following two assertions are equivalent:

(a) Family $\mathcal{B}$ is a base for the topology Θ of X.
(b) For all $U \in \Theta$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Obviously, a topological space can have many bases. Any base $\mathcal{B}$ has the following properties:

(a) For every $U_1, U_2 \in \mathcal{B}$ and every point $x \in U_1 \cap U_2$, there exists a $U \in \mathcal{B}$ such that $x \in U \subset U_1 \cap U_2$.
(b) For every $x \in X$, there exists a $U \in \mathcal{B}$ such that $x \in U$.

Indeed, the first property follows from the fact that $U_1 \cap U_2$ is an open set, and the second follows from the fact that X is open.

With the notion of openness, we can define neighborhoods in topological spaces.

Definition 1.1.3. Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$.

(a) A subset N of X is called a neighborhood of x if there is an open subset O of X with $x \in O \subset N$.
(b) The collection of all neighborhoods of x, denoted by $\mathcal{N}_x$, is called the neighborhood system of x.

In the following proposition, we recall the properties of neighborhoods.

Proposition 1.1.1. *Let (X, Θ) be a topological space and $x \in X$.*

(a) *A subset $O \subset X$ is a neighborhood of all of its points if and only if O is open.*
(b) *For $U \in \mathcal{N}_x$ and $U \subset U'$, we have $U' \in \mathcal{N}_x$.*
(c) *For $U_1, \dots, U_n \in \mathcal{N}_x$, we have $U_1 \cap U_2 \cap \dots \cap U_n \in \mathcal{N}_x$.*
(d) *For $U \in \mathcal{N}_x$, there exists $V \in \mathcal{N}_x$ such that $V \subset U$.*

Let (X, Θ) be a topological space and $A \subset X$. A point x is an interior point of A (or interior to A), if A is a neighborhood of x. The set of all interior points of A is called the interior of A and is denoted by $\operatorname{int}(A)$. The interior of A is the largest open set contained inside A. Every point in the interior of A has a neighborhood contained inside A. An exterior point of A is an interior point to $A^c = X \backslash A$. The set of all points exterior to A is called the exterior of A and is denoted $\operatorname{Ext}(A)$. It is the largest open set inside $X \setminus A$. A point x is called a point of closure of (or adherent to) A if every neighborhood of x contains at least one element of A. The set of all adherent points to A is called the closure (or adherence) of A and is denoted by $\overline{A}$. A point x is called a limit point (or accumulation point, or cluster point) of A if every neighborhood of x intersects A in at least one point other than x. Define the frontier (or boundary) of A to be the set $\overline{A} \setminus \mathrm{int}(A)$. It is denoted by ∂A. A subset A of X is said to be *dense* (in X) if the intersection of every nonempty open set with A is nonempty. Equivalently, A is dense in X if and only if the only closed subset of X containing A is X itself.

This can also be expressed by saying that the closure of A is X, or the interior of the complement of A is empty. It is clear that if (X, Θ) is a topological space, then X is dense in X. The space X is called *separable* if it contains a countable, dense subset; that is, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence; X is *connected* if X is not the union of two disjoint nonempty open subsets (otherwise, X is disconnected).

1.1.2 Continuity of mappings

One of the most important duties of topologies is to define the class of continuous mappings.

Let (X, Θ) and (Y, Θ') be topological spaces and let $f : X \to Y$ be a mapping. Then f is continuous at $x \in X$ if for each neighborhood V of $y = f(x)$, $f^{-1}(V)$ is a neighborhood of x; f is continuous on X into Y (briefly, continuous) if f is continuous at each $x \in X$ (equivalently, if $f^{-1}(G)$ is open in X for each open $G \subset Y$). If Z is also a topological space and $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is a continuous function. If $f : X \to Y$ is a continuous map and A a subset of X, then the restriction of f to A, $f_{|A}$, is also a continuous map with respect to the relative topology on A. Let $A \subset X$ be closed, and $f : A \to Y$ a continuous function. A continuous function $F : X \to Y$ such that $F_{|A} = f$ is called an extension of f.

Theorem 1.1.1. *Let X and Y be topological spaces and $f : X \to Y$ a function. Then the following statements are equivalent:*
(a) *f is continuous.*
(b) *The inverse image of each closed set is closed.*
(c) *For each $x \in X$, the inverse image of every neighborhood of $f(x)$ is a neighborhood of x.*
(d) *For each $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.*
(e) *For each $B \subset Y$, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.*

A homeomorphism is a continuous one-to-one map from a topological space X onto a topological space Y such that f^{-1} is continuous; X and Y are homeomorphic if there exists a homeomorphism of X onto Y. A homeomorphism defines a one-to-one correspondence between the points of the spaces and the open sets of the two spaces. From the topological point of view, two homeomorphic spaces are identical, only the names of the points have been changed. Any topological property possessed by one space is also possessed by the other.

A topological space is said to be a *Hausdorff* space if given any points $x, y \in X$ with $x \neq y$, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$. We note that if X is Hausdorff, then each point, considered as a set, is closed.

Definition 1.1.4. A topological space X is said to be *normal* if it is a Hausdorff space and if given any two closed subsets $A, B \subset X$ such that $A \cap B = \emptyset$, there exist open subsets U, V of X such that $A \subset U$ and $B \subset V$, and $U \cap V = \emptyset$. ☐

Theorem 1.1.2 (Urysohn's lemma). *Let X be a normal topological space, and let A and B be closed disjoint subsets of X. Then there exists a continuous function $f : X \to [0,1]$ such that*

$$A \subset f^{-1}(\{0\}), \quad B \subset f^{-1}(\{1\}).$$

1.1.3 Compact spaces

The concept of compactness was introduced in topology with the intention of generalizing the properties of closed and bounded subsets of $\mathbb{R}^n$.

Let X be a Hausdorff topological space. A subset K of X is called compact if every open cover of K has a finite subcover in X, that is, K has the finite open cover property:

(FOC) Whenever $\{D_i\}_{i \in I}$ is a collection of open sets such that $K \subset \bigcup_{i \in I} D_i$, there exists a finite subcollection $D_{i_1}, \dots, D_{i_n}$ such that $K \subset D_{i_1} \cup \dots \cup D_{i_n}$.

We say that a topological space (X, Θ) is compact if X itself is a compact set. Let K be a subset of X equipped with the induced topology $\Theta_{|K}$. Then it is straightforward from the previous definition that K is compact as a subset in (X, Θ) if and only if $(K, \Theta_{|K})$ is a compact topological space. It is not difficult check that any finite union of compact sets is compact. A Hausdorff topological space is called locally compact if each of its points possesses a compact neighborhood.

Proposition 1.1.2. *Let (X, Θ) and $(Y, \mathcal{S})$ be topological spaces and $f : X \to Y$ a continuous map. If K is a compact subset of X, then $f(K)$ is a compact subset of Y.*

The following result is a consequence of Proposition 1.1.2.

Corollary 1.1.1. *Let (X, Θ) and $(Y, \mathcal{S})$ be topological spaces such that X is compact and Y is Hausdorff, and let $f : X \to Y$ be a bijective and continuous map. Then f is a homeomorphism, that is, $f^{-1} : Y \to X$ is continuous.*

1.2 Metric spaces

1.2.1 Generalities

A metric space is a nonempty set X together with a map $d : X \times X \to \mathbb{R}$ that has the following properties:
(a) $d(x,y) \geq 0$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$,
(b) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(c) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

The function d is said to be a metric on X. The property (a) is known as nonnegativity and (b) is called symmetry. The final property (c) is called the triangle inequality. A metric is a measure of distance on the set X. For $x,y \in X$, the number $d(x,y)$ is called the distance between x and y. We often denote a metric space X with a metric d by (X,d); sometimes, if the metric is obvious or irrelevant, we may also simply write X.

Given a metric space (X,d) and an arbitrary real number $r > 0$, the open ball of radius r and center x is the set defined by

$$B(x,r) = \{y \in X : d(x,y) < r\}.$$

We say that a subset U of X is open if, given any point $x \in U$, there exist $r > 0$ and an open ball $B(x,r)$ such that $B(x,r) \subset U$. It is clear from the definition of open sets that any open ball is an open set. The collection $\mathcal{O}$ of all open sets of X satisfies the properties **(O1)**, **(O2)**, and **(O3)** of Definition 1.1.1, that is, it is stable with respect to taking arbitrary unions, finite intersections, and contains X and $\emptyset$. This particular topology defined from open balls is called the topology generated by the metric d. A metric space endowed with this topology is a topological space. The topology generated by a metric is a Hausdorff topology. If U is an open subset of a metric space X, then $U^c = X \setminus U$ is closed. In other words, a set is closed if and only if its complement is open. The closed ball of radius r centered at x is the set

$$B_f(x,r) = \{y \in X : d(x,y) \leq r\}.$$

In a metric space, closed balls are closed sets. Let $x \in X$, balls centered at x form a base of neighborhoods of x. We can also take balls with center x and radius the inverse of an integer, which shows that in a metric space each point has a countable neighborhood base. By definition, $\overline{B(x,r)}$ is the smallest closed set (in the sense of inclusion) containing $B(x,r)$. Since the closed ball $B_f(x,r)$ contains $B(x,r)$, we have $\overline{B(x,r)} \subseteq B_f(x,r)$. This inclusion may be strict.

Notations

We shall adopt the following notations:

$$\mathbb{B}_r := B_f(0,r) \quad \text{and} \quad \mathbb{B}_r(x) := B_f(x,r).$$

For a nonempty subset A of a metric space X, the distance map $d(\cdot, A)$ on X is defined by

$$X \ni x \mapsto d(x,A) := \inf\{d(x,y) : y \in A\}.$$

The diameter of A is defined by

$$\operatorname{diam}(A) := \sup\{d(x,y) : x,y \in A\}.$$

It is clear that $\operatorname{diam}(A) \in \mathbb{R}^+ \cup \{+\infty\}$. We say that A is bounded if $\operatorname{diam}(A) < +\infty$.

1.2.2 Completeness

We recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space $(X, \mathcal{T})$ converges to a point $l \in X$ if, for each neighborhood V of l, there exists an integer $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $x_n \in V$. If X is a metric space, then a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a limit $l \in X$ if and only if the sequence of real numbers $(d(x_n, l))_{n\in\mathbb{N}}$ converges to 0, or equivalently,

$$(\forall \varepsilon > 0) \quad (\exists N(\varepsilon) \in \mathbb{N}) \quad (\forall n \in \mathbb{N}) \quad (n \geq N(\varepsilon) \implies d(x_n, l) < \varepsilon).$$

Since metric spaces are Hausdorff spaces, if a sequence has a limit, it is unique. A subsequence (or extracted sequence) of a sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space X is any sequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ where φ is a strictly increasing mapping from $\mathbb{N}$ into $\mathbb{N}$. We say that l is a limit point of $(x_n)_{n\in\mathbb{N}}$ if there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} x_{n_k} = l$.

A *Cauchy sequence* in a metric space (X, d) is a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$(\forall \varepsilon > 0) \quad (\exists N(\varepsilon) \in \mathbb{N}) \quad (\forall m, n \in \mathbb{N}) \quad (m, n > N(\varepsilon) \implies d(x_m, x_n) < \varepsilon).$$

A metric space X is complete if every Cauchy sequence of points of X converges in X. In a complete metric space, each closed subset is complete, any intersection of complete subspaces is complete, and all finite unions of complete metric subspaces are complete.

Theorem 1.2.1 (Cantor). *Let (X, d) be a complete metric space, and let $(F_n)_{n\in\mathbb{N}}$ be a sequence of nonempty closed subsets of X such that $F_1 \supset F_2 \supset F_3 \supset \cdots$ and $\lim_{n\to+\infty} \operatorname{diam}(F_n) = 0$. Then the intersection $\bigcap_{n=1}^{+\infty} F_n$ contains exactly one point.*

1.2.3 Compactness and completeness

Let (X, d) be a metric space and A a subset of X. Then A is said to be *totally bounded* if, for each $\varepsilon > 0$, there exists a finite set of points $\{x_1, \ldots, x_n\} \subset A$ such that $A \subset \bigcup_{i=1}^{n} \mathbb{B}_\varepsilon(x_i)$; A is called *sequentially compact* if every sequence of points of A has a convergent subsequence.

If a set is totally bounded, then so are its closure and all of its subsets. Every compact metric space is obviously totally bounded and sequentially compact. It is easy to see that totally bounded metric spaces are separable which implies that compact metric spaces are separable. The following result relates compactness, total boundedness, and sequential compactness.

Theorem 1.2.2. *Let (X, d) be a metric space. The following assertions are equivalent:*
(a) *X is compact.*
(b) *X is complete and totally bounded.*
(c) *X is sequentially compact.*

By Theorem 1.2.2, it is obvious that a metric space is totally bounded if and only if its completion is compact.

Corollary 1.2.1 (Heine). *Every continuous function from a compact metric space into a metric space is uniformly continuous.*

Let (X, d) and (Y, δ) be two metric spaces and denote by $C(X, Y)$ the space of all continuous functions from X into Y endowed with the uniform convergence topology. A subset $\mathcal{F}$ of $C(X, Y)$ is said to be equicontinuous if

$$\forall x \in X, \forall \varepsilon > 0, \quad \exists \eta > 0, \forall f \in \mathcal{F}, \forall y \in X \quad \big(d(x,y) < \eta \Longrightarrow \delta(f(x), f(y)) < \varepsilon\big).$$

We have the following properties:

- Every subset of an equicontinuous set is equicontinuous.
- A finite union of equicontinuous subsets is equicontinuous.
- Every sequence of functions of $C(X, Y)$ which is uniformly convergent is equicontinuous.

Theorem 1.2.3 (Arzelà–Ascoli). *Let (X, d) be a compact metric space, (Y, δ) a complete metric space, and $C(X, Y)$ the space of continuous functions from X into Y endowed with the uniform topology. A subset $\mathcal{F}$ of $C(X, Y)$ is relatively compact if and only if the following conditions are satisfied:*
(a) *$\mathcal{F}$ is equicontinuous,*
(b) *For all $x \in X$, $\mathcal{F}(x) := \{f(x) : f \in \mathcal{F}\}$ is a relatively compact subset of Y.*

In a Hausdorff topological vector space, relatively compact subsets are bounded, but bounded subsets are not necessarily relatively compact. However, we know from Riesz's theorem (cf. Theorem 1.4.2) that, in a finite-dimensional vector space, compact subsets are exactly those which are closed and bounded, hence, if Y is a finite-dimensional topological vector space, then we can replace condition (b) by the following: for all $x \in X$, $\mathcal{F}(x) := \{f(x) : f \in \mathcal{F}\}$ is bounded in Y.

1.2.4 Baire theorem

This famous theorem has important applications in analysis and elsewhere. We remind the reader that a G_δ-set is a countable intersection of open sets. It is obvious that all open sets are G_δ-sets, but not all G_δ-sets are open. Correspondingly, an F_σ-set is a countable union of closed sets. Naturally, all closed sets are F_σ-sets, but not all F_σ-sets are closed.

Theorem 1.2.4 (Baire). *Suppose (X, d) is a complete metric space. If $(O_n)_{n\geq 1}$ is a sequence of dense open subsets of X, then $\bigcap_{n\geq 1} O_n$ is dense in X.*

Proof. Let (X, d) be a complete metric space and suppose $(O_n)_{n\geq 1}$ is a sequence of dense open subsets of X. Our goal is to show that $\bigcap_{n\geq 1} O_n$ is also dense in X.

For any $z \in X$, and $\varepsilon > 0$, let $\mathbb{B}_\varepsilon(z) = \{x \in X : d(x, z) < \varepsilon\}$ be the open ball about z of radius ε. Let U be a nonempty open set in X. We will show that $\bigcap_{n\geq 1} O_n$ and U have nonempty intersection. Pick any $x_0 \in U$ and $\varepsilon_0 \in (0, 1)$ such that $\mathbb{B}_{\varepsilon_0}(x_0)$ subset U. By assumption, the set O_1 is dense in X, and so $\mathbb{B}_{\frac{\varepsilon_0}{2}}(x_0) \cap O_1 \neq \emptyset$. Thus, there exist a point $x_1 \in O_1$ and an $\varepsilon_2 \leq \frac{\varepsilon 0}{2}$ such that $\mathbb{B}_{\varepsilon_1}(x_1) \subset \mathbb{B}_{\frac{\varepsilon_0}{2}}(x_0) \cap O_1 \subset U \cap O_1$. By assumption, as before, the set O_2 is dense in X, and so $\mathbb{B}_{\frac{\varepsilon_1}{2}}(x_1) \cap O_2 \neq \emptyset$. Arguing as before, there exist a point $x_2 \in O_2$ and an $\varepsilon_2 \leq \frac{\varepsilon_1}{2}$ such that $\mathbb{B}_{\varepsilon_2}(x_2) \subset \mathbb{B}_{\frac{\varepsilon_1}{2}}(x_1) \cap O_2 \subset U \cap O_1 \cap O_2$. Continuing inductively, we construct sequences $(x_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$ such that $\varepsilon_n \leq \frac{\varepsilon_{n-1}}{2}$ and $x_n \in U \cap (O_1 \cap O_2 \cap \cdots \cap O_n)$, with the further property that

$$\mathbb{B}_{\varepsilon_n}(x_n) \subset \mathbb{B}_{\frac{\varepsilon_{n-1}}{2}}(x_{n-1}) \cap O_n \subset U \cap (O_1 \cap O_2 \cap \cdots \cap O_n),$$

for all $n \in \mathbb{N}$. Observe that $\varepsilon_n < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Suppose $m > n$. By the triangle inequality, $d(x_m, x_n) \leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n)$. The sequence $(x_n)_{n\geq 1}$ was chosen such that $d(x_{n+1}, x_n) \leq \frac{\varepsilon_n}{2}$. Consequently,

$$d(x_m, x_n) \leq \frac{\varepsilon_{m-1}}{2} + \cdots + \frac{\varepsilon_n}{2} < \frac{1}{2^m} + \cdots + \frac{1}{2^{n+1}} = \frac{1}{2^n} - \frac{1}{2^m} < \frac{1}{2^n}.$$

It follows that $(x_n)_{n\geq 1}$ is a Cauchy sequence. Hence, by completeness, there exists a point $x \in X$ such that $x = \lim_{n\to+\infty} x_n$. If $m > n$, then $x_m \in \mathbb{B}_{\varepsilon_n}(x_n)$, by construction. We conclude that $d(x_m, x_n) < \varepsilon_n$ for all $m > n$, and hence $d(x, x_n) \leq \varepsilon_n$ for all $n \in \mathbb{N}$. Thus

$$x \in \overline{\mathbb{B}_{\varepsilon_n}(x_n)} \subset \overline{\mathbb{B}_{\varepsilon_{n-1}}(x_n)} \subset \overline{\mathbb{B}_{\varepsilon_{n-1}}(x_{n-1})}$$

and so $x \in U \cap (O_1 \cap O_2 \cap \cdots \cap O_{n-1})$ for all $n \in \mathbb{N}$. It follows that $x \in U \cap (\bigcap_{n\geq 1} O_n)$.

We have shown that the intersection of $(\bigcap_{n\geq 1} O_n)$ with any open set U in X is nonempty. Therefore the set $(\bigcap_{n\geq 1} O_n)$ is dense in X. ☐

Remark 1.2.1. The Baire theorem is often used in the following form. Let X be a nonempty complete metric space. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets such that

$$\bigcup_{n\geq 1} F_n = X.$$

Then there exists some n_0 such that $\operatorname{int}(F_{n_0}) \neq \emptyset$.

As a consequence of Theorem 1.2.4, we have

Corollary 1.2.2. *If $(G_n)_{n\geq 1}$ is a sequence of dense G_δ-sets, then $\bigcap_{n\geq 1} G_n$ is also a G_δ-set.*

Proof. By assumption, for each $n \in \mathbb{N}$, the set G_n can be written as an intersection of countably many open sets, say $G_n = \bigcap_{m\geq 1} O_{mn}$. Since G_n is assumed to be dense, it must be the case that O_{mn} is dense for each m and n in $\mathbb{N}$. By Theorem 1.2.4, the set $\bigcap_{n\geq 1} G_n = \bigcap_{n\geq 1}\bigcap_{m\geq 1} O_{mn}$ is a dense set. Since the sequence of open sets $(O_{mn})_{m,n\geq 1}$ is countable, the set $\bigcap_{n\geq 1} G_n$ is a G_δ-set. ☐

Let X be a topological space. A set $E \subset X$ is said to be *nowhere dense* if the closure of E in X has empty interior; that is to say, if $\overset{\circ}{\overline{E}} = \emptyset$. A set $G \subset X$ is called a *first category* set (also known as *meager*) if there exists a sequence $(E_n)_{n\geq 1}$ of nowhere dense sets such that $G \subset \bigcup_{n\geq 1} E_n$. A set is called a *second category* set if it is not a first category set. A second category set is also known as *nonmeager*. The complement of a meager set is called a *residual* set.

Proposition 1.2.1. *A countable union of first category sets is a first category set.*

Proof. A countable union of countably many sets is a countable union of sets. ☐

1.3 Topological vector spaces

1.3.1 Vector spaces

Let $\mathbb{K}$ denote the scalar field (in practice $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

Definition 1.3.1. Let X be a vector space over $\mathbb{K}$. A subset V of X is called a vector subspace of X if V is also a vector space with the same operations as those of X. If $V \neq X$, we say that V is a proper subspace of X.

A useful characterization of vector subspaces of a vector space X is as follows: Let V be a subset of a vector space X, V is a vector subspace of X if

$$\forall x, y \in V, \quad \forall \lambda, \mu \in \mathbb{K}, \quad \lambda x + \mu y \in V.$$

It is easy to check that the intersection of any family of vector subspaces of a vector space X is a vector subspace of X.

Definition 1.3.2. Let X be a vector space and let V be a subset of X. The linear span of V, denoted by $\operatorname{span}(V)$, is the set of all finite linear combinations of vectors in V.

We note that $\operatorname{span}(V)$ is a vector subspace of X and $V \subseteq \operatorname{span}(V)$. It is obvious that any subspace of X that contains V must contain $\operatorname{span}(V)$.

1.3.2 Topology on a vector space

Let X be a vector space endowed with a topology Θ. The pair (X, Θ) is a *topological vector space* if the topology Θ in X is compatible with the linear structure of X, that is, the operations

$$X \times X \to X \quad (x, y) \to x + y \quad \text{and} \quad \mathbb{K} \times X \to X \quad (\alpha, y) \to \alpha \times x$$

are continuous.

To each $a \in X$, we associate the *translation operator*, $T_a : X \to X$, defined by

$$T_a(x) = x + a,$$

and, to each $0 \neq \alpha \in \mathbb{K}$, we associate the *multiplication operator*, $M_\alpha : X \to X$, defined by

$$M_\alpha(x) = \alpha x.$$

Proposition 1.3.1. *Both T_a and M_α are homeomorphisms of X onto X.*

Thus the whole topological structure of X is determined by a base of neighborhoods of the origin. We therefore work mainly with neighborhoods of the origin, and we call them simply neighborhoods. If U is a neighborhood (of the origin), $U + a$ is the corresponding neighborhood of a, and $x \in U + a$ if and only if $x - a \in U$.

In topological vector spaces, the term *local base* will always mean a local base at 0. A local base of a topological vector space X is a collection $\mathcal{B}$ of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathcal{B}$.

Definition 1.3.3. Let X be a topological vector space and let A be a subset of X.

- A is said to be *bounded* if $(\forall V \in \mathcal{N}_0)\ (\exists s \in \mathbb{R})\ (\forall t \in \mathbb{R})\ (t > s \implies A \subset tV)$.
- A is said to be *balanced* if, for all $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, we have $\alpha A \subset A$.
- A is said to be *symmetric* if $x \in A$ implies $(-x) \in A$, or equivalently, $(-A) = A$.
- A is said to be *convex* if $tA + (1 - t)A \subset A$ $(0 \leq t \leq 1)$. In other words, it is required that A contains $x + (1 - t)y$ if $x \in A, y \in A$, and $t \in [0, 1]$.

It is obvious that the topological notion of boundedness may not coincide with the metric notion of boundedness. The notion of being balanced is purely algebraic. When we talk about balanced neighborhoods, we connect the algebraic concept to the topological concept. Note that in $\mathbb{C}$ the only balanced sets are discs and the whole $\mathbb{C}$. Symmetry is also an algebraic concept. Every balanced set is symmetric and the opposite is not true.

Definition 1.3.4. Let (X, Θ) be a topological vector space.
(a) X is *locally convex* if there exists a local base at 0 whose members are convex.
(b) X is *locally bounded* if 0 has a bounded neighborhood.
(c) X is *locally compact* if 0 has a neighborhood whose closure is compact.

Local convexity is important because it amounts to the topology being generated by a family of seminorms. Local convexity is also the minimum requirement for the validity of geometric Hahn–Banach properties. Weak topologies, which we will see later, are always locally convex.

Definition 1.3.5. Let C be any subset of a vector space X. The convex hull of C, denoted by $\operatorname{co}(C)$, is the intersection of all convex subsets of X containing C, i. e., the smallest convex subset of X containing C. The closed convex hull of C is the smallest closed convex subset of X containing C. It is denoted by $\overline{\operatorname{co}}(C)$.

Proposition 1.3.2. *Let A, B be arbitrary subsets of a topological vector space X. The following hold:*
(a) $\operatorname{co}(A)$ *is convex and* $A \subset \operatorname{co}(A) \subset \overline{\operatorname{co}}(A)$.
(b) *A is convex if and only if* $A = \operatorname{co}(A)$.
(c) *If $A \subset B$, then* $\operatorname{co}(A) \subset \operatorname{co}(B)$.

1.3.3 Separation properties

Proposition 1.3.3. *Suppose K and C are subsets of a topological vector space X, K is compact, C is closed, and $K \cap C = \emptyset$. Then* 0 *has a neighborhood V such that*

$$(K + V) \cap (C + V) = \emptyset.$$

Note that $K + V$ is a union of translates $x + V$ of V $(x \in K)$. Hence $K + V$ is an open set that contains K. The proposition thus implies the existence of disjoint open sets that contain K and C, respectively. Since $K + V$ is open, it is even true that the closure of $K + V$ does not intersect $C + V$ and, in particular, the closure of $K + V$ does not intersect C.

In the following proposition, we gather relations between sets, their closures, and their interiors.

Proposition 1.3.4. *Let X be a topological vector space.*
(a) *If $A \subset X$, then $\overline{A} = \bigcap_{V \in \mathcal{N}_0}(A + V)$ where V runs through all neighborhoods of* 0.
(b) *If C is a convex subset of X, then so are $\overline{C}$ and* $\operatorname{int}(C)$.
(c) *If B is balanced subset of X, then so is $\overline{B}$.*
(d) *If B is balanced and* $0 \in \operatorname{int}(B)$, *then* $\operatorname{int}(B)$ *is balanced.*

Proposition 1.3.5. *In a topological vector space X, every neighborhood of* 0 *contains a balanced neighborhood of* 0*, and every convex neighborhood of* 0 *contains a balanced convex neighborhood of* 0.

Proposition 1.3.5 can be restated in terms of local bases. Let us say that a local base $\mathcal{B}$ is balanced if its members are balanced sets, and let us call $\mathcal{B}$ convex if all its members are convex sets.

Corollary 1.3.1. *Every topological vector space has a balanced local base and every locally convex topological vector space has a balanced convex local base.*

Lemma 1.3.1. *Let $\mathcal{B}$ be a base of neighborhoods (of the origin) of a topological vector space. Then, for each $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V + V \subset U$.*

Proof. If $f(x,y) = x+y$, then f is continuous at $x = 0, y = 0$ and so there are neighborhoods V_1 and V_2 with $x+y \in U$ for $x \in V_1$ and $y \in V_2$. There exists $V \in \mathcal{B}$ with $V \subset V_1 \cap V_2$, and then $V + V \subset U$. ☐

Proposition 1.3.6. *If $\mathcal{B}$ is a base of neighborhoods in a topological vector space X, then X is separated if and only if*

$$\bigcap_{U\in\mathcal{B}} U = \{0\}.$$

Proof. If X is separated and $x \neq 0$, then there is some $U \in \mathcal{B}$ with $x \notin U$, and so

$$\bigcap_{U\in\mathcal{B}} U = \{0\}.$$

Conversely, if the latter condition holds and $x \neq y$, then there is some U with $x - y \notin U$. By Lemma 1.3.1, there is a balanced neighborhood V with $V + V \subset U$. Then $x + V$ and $y + V$ are disjoint neighborhoods of x and y, for if $z \in (x + V) \cap (y + V)$ then

$$x - y = (z - y) - (z - x) \in V - V = V + V \subseteq U.$$

Therefore X is separated. ☐

1.3.4 Metrizable topological vector spaces

A topological vector space (X, Θ) is said to be *metrizable* if there is a metric d on X which is compatible with Θ, i. e., the topology defined by d coincides with Θ. In that case, the balls with radius $1/n$ centered at x form a local base at x. This gives a necessary condition for metrizability which, for topological vector spaces, turns out to be also sufficient.

Theorem 1.3.1. *If X is a topological vector space with a countable local base, then there is a metric d on X such that*

(a) *d is compatible with the topology of X.*
(b) *The open balls centered at* 0 *are balanced.*
(c) *d is invariant, that is, $d(x+z, y+z) = d(x,y)$ for $x, y, z \in X$.*
(d) *If further X is locally convex, then d can be chosen so as to satisfy (a), (b), (c); and also all open balls are convex.*

Obviously, a sequence $(x_n)_{n\in\mathbb{N}}$ in a metrizable topological vector space X is a d-Cauchy sequence if and only if it is a Θ-Cauchy sequence.

1.3.5 Seminorms and local convexity

Definition 1.3.6. A *seminorm* on a vector space X is a function $p : X \to \mathbb{R}$ such that for all x and y in X and all scalars α, we have
(a) $p(x+y) \le p(x) + p(y)$,
(b) $p(\alpha x) = |\alpha| p(x)$. ☐

It follows from (a) and (b) that $p(x) \ge 0$. Moreover, if $p(x) > 0$ for all $x \in X \setminus \{0\}$, then p is a norm.

Let $\mathcal{P} = \{p_i; i \in I\}$ be a family of seminorms on a vector space X. Consider, for every $x \in X$, the family of subsets of X defined by

$$\mathcal{N}(x) = \{V_{i_1,i_2,\dots,i_k,\varepsilon}(x) : k \in \mathbb{N}^*, i_1, i_2, \dots, i_k \in I, \varepsilon > 0\}, \quad x \in X,$$

where

$$V_{i_1,i_2,\dots,i_k,\varepsilon}(x) = \{u \in X : p_{i_j}(u-x) < \varepsilon, \forall j = 1, 2, \dots, k\}.$$

We can easily see that $\{V(x) : x \in X\}$ is a base of neighborhoods for a locally convex topology $T_{\mathcal{P}}$ on X. The topological properties for $T_{\mathcal{P}}$ can be characterized analytically by means of seminorms of $\mathcal{P}$.

Theorem 1.3.2. *The locally convex topology $T_{\mathcal{P}}$ is the coarsest linear topology on X for which all seminorms of the family $\mathcal{P}$ are continuous.*

A particular class of topological vector spaces with richer properties is the class of locally convex spaces; these are topological vector spaces with the property that, for every element, there exists a base of neighborhoods consisting of convex sets. It is well known that any locally convex topology on a vector space may be generated by a family of seminorms.

The locally convex topology $T_{\mathcal{P}}$ is separated if and only if the family of seminorms $\mathcal{P}$ possesses the following property:

$$\text{for each } x \in X \setminus \{0\} \text{ there is } p \in \mathcal{P} \text{ such that } p(x) \ne 0.$$

A family $\mathcal{P}$ of seminorms on X satisfying the previous property is said to be *separating.*

1.4 Normed vector spaces

1.4.1 Generalities

Let X be a vector space over the scalar field $\mathbb{K}$. A norm on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all vectors x, y and $\alpha \in \mathbb{K}$:

(a) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(b) $\|\alpha x\| = |\alpha|\|x\|$,

(c) $\|x + y\| \leq \|x\| + \|y\|$.

The ordered pair $(X, \|\cdot\|)$ is called a normed vector space, or simply a normed space. In particular, we can obtain the topology of a linear normed space if we take $\mathcal{P} = \{\|\cdot\|\}$. A norm on X defines a metric d on X given by

$$d(x,y) = \|x - y\| \quad \forall x, y \in X. \tag{1.1}$$

It is called the metric induced by the norm. The norm topology of X is the topology obtained from this metric, it is compatible with the vector space structure. In this way, for linear normed spaces, the metric properties interweave with the topological properties of a locally convex space. In fact, the topology generated by the seminorm $\mathcal{P}$ is locally convex and metrizable. The interest in the metrizability of this topology lies in the fact that all topological properties can be characterized by sequences.

Let X be a normed space. The set $\mathbb{B}_1 = \{x : x \in X, \|x\| \leq 1\}$ denotes the closed unit ball of X, and $\mathbb{S}_1 = \partial\mathbb{B}_1 = \{x : x \in X, \|x\| = 1\}$ is the unit sphere of X. The open unit ball of X is the set $\mathring{\mathbb{B}}_1 = \{x : x \in X, \|x\| < 1\}$. In general, we denote by $\mathbb{B}_r(x) = \{y \in X : \|y - x\| \leq r\}$ the closed ball with center x and radius r and by $\mathbb{S}_r(x) = \partial\mathbb{B}_r(x) = \{x : x \in X, \|x\| = r\}$ the sphere with center x and radius r. The open ball with center x and radius r, $\{y \in X : \|y - x\| < r\}$, is $\mathbb{B}_1(r) \setminus \partial\mathbb{B}_1(r)$ and can denoted by $\operatorname{int}(\mathbb{B}_r(x))$. In a normed space, the closure of the open ball centered at x with radius r is precisely the closed ball of center x and radius r, that is, for all $x \in X$ and $r > 0$, we have $\overline{\mathring{\mathbb{B}}_r(x)} = \mathbb{B}_r(x)$. It follows from the definition of the metric of X that

$$\mathbb{B}_r(x) = x + \mathbb{B}_r(0) = x + r\mathbb{B}_1.$$

Definition 1.4.1. A subset A of a normed space $(X, \|\cdot\|)$ is called bounded if there exists $r > 0$ such that $A \subset r\mathbb{B}_1$. Equivalently, A is bounded if there is some real M such that $\|x\| \leq M$ for all $x \in A$.

We note that all topological notions in a normed vector space are related to the canonical metric (1.1).

Definition 1.4.2. A complete normed space with respect to the canonical metric defined by (1.1) is called a *Banach space*.

Let Y be a subspace of a normed space $(X, \|\cdot\|)$. We denote by $(Y, \|\cdot\|)$ the subspace Y endowed with the norm $\|\cdot\|$ (the restriction of the norm to Y).

Proposition 1.4.1. *Let Y be a subspace of a Banach space X. Then Y is a Banach space if and only if Y is closed in X.*

Let x_0 be an element of a normed space X and let α be a nonzero scalar. Then the maps $x \mapsto x + x_0$ and $x \mapsto \alpha x$ are homeomorphisms from X onto itself. Consequently, if A is a subset of X which is open, closed, or compact, then $x_0 + A$ and αA also have that property. If A and U are subsets of X and U is open, then $A + U$ is open.

If X is a normed space, the following facts hold true:

- If C is a convex subset of X, then both $\overline{C}$ and $\operatorname{int}(C)$ are convex.
- If A is a subset of X, then $\overline{\operatorname{co}}(A) = \overline{\operatorname{co}(A)}$.
- Every ball centered at the origin, whether open or closed, is convex, balanced, and absorbing.
- Every closed, convex, absorbing subset of a Banach space includes a neighborhood of the origin.

If X is a vector space over $\mathbb{K}$ and $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X, we say they are equivalent norms if they define the same convergent sequences in X, that is, $\|x_n - x\|_1 \to 0$ if and only if $\|x_n - x\|_2 \to 0$. The fact that convergent sequences are the same for the two norms means that the two metrics induced on X by these norms define the same collection of open sets. Hence, a set is open relative to the first norm if and only if it is open relative to the second norm. The equivalence of two norms is characterized by the following: Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space are equivalent if and only if there are constants C_1 and C_2 such that

$$C_1\|x\|_1 \le \|x\|_2 \le C_2\|x\|_1 \quad \text{for all } x \in X.$$

We conclude this subsection by the following result due to S. Mazur.

Theorem 1.4.1 (Mazur). *Let X be a Banach space X and let $A \subset X$. If A is compact, then $\overline{\operatorname{co}}(A)$ is compact.*

1.4.2 Finite-dimensional normed spaces

A vector space X has dimension n ($\dim X = n$) if X has a basis $\{x_1, x_2, \ldots, x_n\}$, this means that every $x \in X$ has a unique representation of the form

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, \quad \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}.$$

A vector space X is said to have finite dimension if $\dim X = n$ for some $n \in \mathbb{N}$.

If X is a topological vector space over $\mathbb{C}$, and $\dim X = n$, then every basis of X induces an isomorphism of X onto $\mathbb{C}^n$.

Proposition 1.4.2. *On a finite-dimensional space, any two norms are equivalent.*

Corollary 1.4.1. *A finite-dimensional subspace of any normed space is closed.*

We conclude this subsection by recalling the Riesz theorem.

Theorem 1.4.2 (Riesz). *A normed space X is finite dimensional if and only if its closed unit ball $\mathbb{B}_1$ is compact.*

In fact, Theorem 1.4.2 says that a Banach space is finite dimensional if and only if it is locally compact.

1.5 Linear operators

1.5.1 Generalities

Let X and Y be vector spaces (both real or complex). Let T be a map with domain $D(T) \subseteq X$ and range $\mathrm{R}(T) = T(D(T)) \subseteq Y$. We say that T is a *linear map*, or a *linear operator* if, whenever $x, y, z \in X$ and $\alpha \in \mathbb{K}$, we have

$$T(x + y) = T(x) + T(y), \quad T(\alpha z) = \alpha T(z).$$

The *kernel* (or *null space*) of a linear operator $T : D(T) \to Y$ is a subset of $D(T)$ defined by

$$\ker(T) := \{x \in D(T) \text{ such that } T(x) = 0\}.$$

It is readily verified that $\ker(T)$ and the domain of T, $D(T)$, are vector subspaces of X. Likewise, range of T,

$$\mathrm{R}(T) := \{y \in Y : y = Tx, x \in D(T)\},$$

is a vector subspace of Y.

We note that a linear operator T is one-to-one if and only if $\ker(T) = \{0\}$. If S is a linear operator from X into Y and T is a linear operator from Y into a vector space Z, then the *product* (or *composite*) TS of S and T is the linear operator from X into Z formed by letting $TS(x) = T(S(x))$ for each x in X.

If $D(T)$ is dense in X, we say that T is densely defined. If $D(T) = X$, T is said to be defined on X. If $X = Y$, we say that T is an operator on X.

If a linear operator is a one-to-one map on $D(T)$ onto $\mathrm{R}(T)$, then the inverse map T^{-1} is a linear operator from $\mathrm{R}(T)$ onto $D(T)$ and

$$T^{-1}Tx = x \quad \text{for } x \in D(T) \quad \text{and} \quad TT^{-1}y = y \quad \text{for } y \in \mathrm{R}(T).$$

The operator T^{-1} is the *inverse* of T.

Let X and Y be linear spaces over the same field $\mathbb{K}$, the product $X \times Y$ is a linear space with respect to the operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad \lambda(x, y) = (\lambda x, \lambda y).$$

Further, if X and Y are normed spaces, then $X \times Y$ is also a normed space by the norm $\|(x, y)\| = \|x\| + \|y\|$ or any equivalent norm.

Let T be a linear operator with domain $D(T) \subseteq X$ and range $\mathrm{R}(T) \subseteq Y$. The subset $G(T) = \{(x, T(x)) : x \in D(T)\}$ of $X \times Y$ is called the *graph* of T.

Definition 1.5.1. Let X and Y be normed spaces and let T be a linear operator with domain $D(T) \subseteq X$ and range $\mathrm{R}(T) \subseteq Y$. Then T is said to be closed whenever $G(T)$ is a closed subspace of $X \times Y$.

Equivalently, T is closed if and only if the following condition is satisfied: if $(x_n)_{n \in \mathbb{N}}$ is a sequence in $D(T)$ such that $\lim_{n \to +\infty} x_n = x$ and $\lim_{n \to +\infty} Tx_n = y$ then $x \in D(T)$ and $Tx = y$.

We denote by $\mathcal{C}(X, Y)$, the class of all closed, densely defined linear operators from X into Y. Let $T \in \mathcal{C}(X, Y)$; for $x \in D(T)$, we define the graph norm of x by

$$\|x\|_T = \|x\| + \|Tx\|.$$

It follows from the closedness of T that $D(T)$ endowed with the norm $\|\cdot\|_T$ is a Banach space. In this new space, denoted by X_T, the operator T satisfies $\|Tx\| \leq \|x\|_T$ and, consequently, is a bounded operator (acting from X_T into Y).

1.5.2 Continuity of linear operators

Let X and Y be normed spaces. A linear operator $T : D(T) \subset X \to Y$ is continuous at $x_0 \in D(T)$ if $\|x_n - x_0\| \to 0, x_n \in D(T)$ implies $\|Tx_n - Tx_0\| \to 0$. Since $T(x_n) - T(x_0) = T(x_n - x_0)$, it follows that T is continuous everywhere on $D(T)$ if and only if it is continuous at $x_0 = 0$.

A nonzero linear operator $T : D(T) \subset X \to Y$ cannot map $D(T)$ into a bounded subset of Y simply because $\|T(\alpha x)\|_Y$ and $|\alpha|$ can be arbitrarily large. However, we can consider *boundedness*, in another sense, by examining the relationship between the size of $\|x\|_X$ and $\|T(x)\|_Y$. Hence, if T is continuous (that is, continuous everywhere on $D(T)$), then there exists $\eta > 0$ such that $\|x\| < \eta$ implies $\|Tx\| \leq 1$. By homogeneity, we have

$$\|Tx\| \leq M\|x\| \quad \text{for every } x \in D(T), \tag{1.2}$$

where $M = 1/\eta$.

A linear operator with property (1.2) is said to be bounded. The smallest number M with this property is called the bound of T and is denoted by $\|T\|$. However, the terminology *bounded linear operator* will be reserved to operators where their domain is the whole space, that is, $D(T) = X$, and they satisfy (1.2).

The following obvious result makes clear the reason for introducing boundedness into this discussion.

Proposition 1.5.1. *Let X and Y be normed spaces and let $T : X \to Y$ be a linear operator. The following assertions are equivalent:*

(a) *T is continuous at the origin.*

(b) *T is continuous.*

(c) *There exists $M \geq 0$ such that, for each $x \in X$, $\|Tx\| \leq M\|x\|$.*

A linear mapping $T : X \to Y$ between two normed spaces X and Y is called bounded if $T(\mathbb{B}_1)$ is bounded in Y. The collection of all bounded linear operators between X and Y is denoted by $\mathcal{L}(X, Y)$. If $X = Y$, we shortly denote $\mathcal{L}(X) = \mathcal{L}(X, X)$. It is clear that $\mathcal{L}(X, Y)$ is a vector space and the map

$$\|\cdot\|_{\mathcal{L}(X,Y)} : \mathcal{L}(X, Y) \to \mathbb{R}^+, \quad T \mapsto \|T\|_{\mathcal{L}(X,Y)} := \sup\{\|T(x)\| : x \in \mathbb{B}_1\} \tag{1.3}$$

is a norm on $\mathcal{L}(X, Y)$. We recall the following classical result.

Proposition 1.5.2. *Let X and Y be normed spaces. If Y is a Banach space, then $\mathcal{L}(X, Y)$ is also a Banach space.*

1.5.3 Uniform boundedness principle

In this subsection, we will investigate some implications of category in Banach spaces. In particular, we will introduce and prove the Uniform Boundedness Principle which is one of the pillars of functional analysis.

Definition 1.5.2. Let X, Y be normed spaces and $\mathcal{A} \subset \mathcal{L}(X, Y)$. We say that $\mathcal{A}$ is pointwise bounded if $\sup\{\|T(x)\|_Y : T \in \mathcal{A}\} < +\infty$ for all $x \in X$.

If $\mathcal{A}$ is bounded in $\mathcal{L}(X, Y)$ equipped with the operator norm, that is, there is $C > 0$ such that $\|T\| \leq C$, then $\mathcal{A}$ is pointwise bounded. Indeed, for $x \in X$, we have $\|T(x)\|_Y \leq \|T\|\|x\|_X \leq C\|x\|_X$, so $\sup\{\|T(x)\|_Y : T \in \mathcal{A}\} \leq C\|x\|_X$. The opposite direction is often called the Banach–Steinhaus Uniform Boundedness Principle.

Theorem 1.5.1 (Uniform Boundedness Principle). *Let X, Y be Banach spaces and $\mathcal{A} \subset \mathcal{L}(X, Y)$. If $\mathcal{A} \subset \mathcal{L}(X, Y)$ is pointwise bounded, then $\mathcal{A}$ is bounded in $\mathcal{L}(X, Y)$.*

Proof. For $n \in \mathbb{N}$, set $V_n = \{x \in X : \sup_{T\in\mathcal{A}} \|T(x)\|_Y \le n\}$.

We claim that V_n is closed, convex, and balanced in X. The balancedness of V_n is obvious. To check closedness, let $x_k \in V_n$ and $x_k \to x$. Given $T \in \mathcal{A}$, we have $\|T(x_k)\| \le n$, so $\|T(x)\|_Y \le n$ by continuity. To see that V_n is convex, let $x, y \in V_n$ and $t \in [0,1]$. Then for every $T \in \mathcal{A}$, we have

$$\|T(tx + (1-t)y)\|_Y \le t\big\|T(x)\big\|_Y + (1-t)\big\|T(y)\big\|_Y \le tn + (1-t)n = n.$$

Since for every $x \in X$ we have $\sup_{T\in\mathcal{A}} \|T(x)\|_Y < \infty$, there is some $n \in \mathbb{N}$ greater than the supremum, hence $x \in V_n$. So $\bigcup_{n\ge 0} V_n = X$. By the Baire category theorem (Theorem 1.2.4, in fact here we use the formulation introduced in Remark 1.2.1), there is an n_0 such that the set V_{n_0} contains an interior point x_0. Thus there is $\delta > 0$ such that $x_0 + \delta\mathbb{B}_1^X \subset V_{n_0}$. Because of the symmetry of V_{n_0} we have $-x_0 + \delta\mathbb{B}_1^X \subset V_{n_0}$. Hence $\delta\mathbb{B}_1^X \subset V_{n_0}$. Consequently, given $T \in \mathcal{A}$, for every $x \in \mathbb{B}_1^X$ we have $\|T(\delta x)\|_Y \le n_0$, that is, $\|T\| \le \frac{n_0}{\delta}$. This means that $\sup_{T\in\mathcal{A}} \|T\| \le \frac{n_0}{\delta}$. This ends the proof. ☐

The following theorem is a consequence of the Uniform Boundedness Principle.

Theorem 1.5.2 (Banach–Steinhaus theorem). *Let X and Y be Banach spaces. Suppose $(T_n)_{n\in\mathbb{R}}$ is a sequence of bounded linear operators from X to Y. If $\lim_{n\to+\infty} T_n(x)$ exists for each $x \in X$, then*

(a) $\sup_{n\in\mathbb{N}} \|T_n\| < +\infty$, *and*

(b) *if $Tx = \lim_{n\to+\infty} T_n(x)$ for all $x \in X$, then T is a bounded linear operator, and*

$$\|T\| \le \liminf_{n\to+\infty} \|T_n\| < +\infty.$$

Proof. (a) If $(T_n)_{n\in\mathbb{R}}$ converges, then $\sup_{n\in\mathbb{N}} \|T_n(x)\| < +\infty$ and therefore, the Uniform Boundedness Principle implies that $\sup_{n\in\mathbb{N}} \|T_n\| < +\infty$.

(b) The linearity of T is easy:

$$\begin{aligned} T(x+y) &= \lim_{n\to+\infty} T_n(x+y) = \lim_{n\to+\infty} \big(T_n(x) + T_n(y)\big) = \lim_{n\to+\infty} T_n(x) + \lim_{n\to+\infty} T_n(y) \\ &= T(x) + T(y). \end{aligned}$$

To show that T is bounded, observe that

$$\big\|T(x)\big\| \le \sup_{n\in\mathbb{N}}\big\|T_n(x)\big\| \le \Big(\sup_{n\in\mathbb{N}} \|T_n\|\Big)\|x\|.$$

Taking the supremum over $x \in \mathbb{B}_1^X$ provides the desired result. Next, if we pose $C := \liminf_{n\to+\infty} \|T_n\|$, then

$$\big\|T(x)\big\| = \lim_{n\to+\infty}\big\|T_n(x)\big\| = \liminf_{n\to+\infty}\big\|T_n(x)\big\| \le \liminf_{n\to+\infty}(\|T_n\|\|x\|).$$

This proves that $\|T(x)\| \le C\|x\|$, that is, T is bounded and $\|T\| \le C$. ☐

1.5.4 The open mapping theorem

Before starting we recall the definition of open maps. Let X and Y be topological spaces. A map $T : X \to Y$ is called *open* (or an open map) if $T(U)$ is open in Y whenever U is open in X.

Proposition 1.5.3. *Let X and Y be Banach spaces and suppose $T : X \to Y$ is a bounded linear operator. The map T is open map if and only if there exists a $\delta > 0$*

$$\delta \mathbb{B}_1^Y \subset T(\mathbb{B}_1^X). \tag{1.4}$$

Proof. Assume T is an open map. The set $\operatorname{int}(\mathbb{B}_1^X)$ is open in X, and so $T(\operatorname{int}(\mathbb{B}_1^X))$ is open in Y. By the linearity of T, we have $0 \in T(\operatorname{int}(\mathbb{B}_1^X))$. It follows that $\operatorname{int}(\mathbb{B}_1^X)$ contains a basic neighborhood of 0. Therefore, there exists a $\delta > 0$ such that $\delta \mathbb{B}_1^Y \subset T(\operatorname{int}(\mathbb{B}_1^X)) \subset T(\mathbb{B}_1^X)$.

Conversely, assume there exists a $\delta > 0$ such that $\delta \mathbb{B}_1^Y \subset T(\mathbb{B}_1^X)$. We wish to show that $T(U)$ is open in Y whenever U is an open set in X. To that end, let U be open in X and let $y \in T(U)$. There exists some $x \in U$ such that $Tx = y$. Since $x \in U$, and U is open in X, there is some $\eta > 0$ such that $x + \eta \mathbb{B}_1^X \subset U$. It follows that $y + \eta T(\mathbb{B}_1^X) \subset T(U)$. By our assumption, this implies that $y + \eta\delta \mathbb{B}_1^Y \subset T(U)$. Consequently, $y \in \operatorname{int}(T(U))$, and so the set $T(U)$ is open in Y. Therefore, T is an open map, as required. □

Corollary 1.5.1. *Let X and Y be Banach spaces and suppose $T : X \to Y$ is a bounded linear operator. If T is open, then T maps X onto Y. Moreover, if T is one-to-one, then T is invertible and T^{-1} is continuous.*

Proof. Suppose T is an open map. By Proposition 1.5.3, there exists a $\delta > 0$ such that $\delta \mathbb{B}_1^Y \subset T(\mathbb{B}_1^X)$. Let $y \in Y$. It obvious that $y \in \|y\| \mathbb{B}_1^Y$, and so $y \in \frac{\|y\|}{\delta} \delta \mathbb{B}_1^Y \subset \frac{\|y\|}{\delta} T(\mathbb{B}_1^X)$. This yields that $y = T(\frac{\|y\|}{\delta} x)$ for some $x \in \mathbb{B}_1^X$, and so T is onto.

Assume further that T is one-to-one. Since it is open, it follows from the first assertion that T in onto, so the inverse operator T^{-1} is well defined. To show that $T^{-1} : Y \to X$ is continuous, let U be an open set in X. Since $(T^{-1})^{-1}(U) = T(U)$ is open in Y (because T is an open map), we see that the preimage of an open set is open which proves that T^{-1} is continuous. □

Let X and Y be Banach spaces and suppose $T : X \to Y$ is a bounded linear operator. The map T is said to be *almost open* if there exists a $\delta > 0$ such that $\delta \mathbb{B}_1^Y \subset \overline{T(\mathbb{B}_1^X)}$.

We give the following result without proof.

Proposition 1.5.4. *Let X and Y be Banach spaces and suppose $T : X \to Y$ is a bounded linear operator. If T is almost open, then T is open.*

Theorem 1.5.3 (Open mapping theorem). *Suppose X and Y are Banach spaces. If $T : X \to Y$ is a bounded linear surjective operator, then T is an open map.*

Proof. Note that $Y = T(X) = \bigcup_{n\geq 1} n\overline{T(\mathbb{B}_1^X)}$. Hence, by Remark 1.2.1, the set $\overline{T(\mathbb{B}_1^X)}$ has nonempty interior. So, there exists an element $y \in Y$ and a number $\delta > 0$ such that

$$y + \delta\mathbb{B}_1^Y \subset \overline{T(\mathbb{B}_1^X)}.$$

A simple calculation reveals that $-y + \delta\mathbb{B}_1^Y \subset \overline{T(\mathbb{B}_1^X)}$ as well, and so it must be the case that $\delta\mathbb{B}_1^Y \subset \overline{T(\mathbb{B}_1^X)}$. Therefore, T is almost open. So, by Proposition 1.5.4, it follows that T is an open map. ☐

Corollary 1.5.2 (Inverse mapping theorem). *If X and Y are Banach spaces and $T : X \to Y$ is a continuous linear bijection, then $T^{-1} : Y \to X$ is continuous.*

Proof. By assumption, the map T is a continuous bijection. Since T is surjective, by Theorem 1.5.3, it is open. Because T is an injective map, it follows from Corollary 1.5.1 that T^{-1} is bounded which is required. ☐

1.5.5 The closed graph theorem

Let X and Y be two vector spaces. The algebraic direct sum $X \oplus Y$ is the vector space of all ordered pairs $(x,y), x \in X, y \in Y$ with the vector operations defined coordinatewise. The spaces X and Y are algebraically isomorphic to the subspaces $\{(x,0) : x \in X\}$ and $\{(0,y) : y \in Y\}$ of $X \oplus Y$, respectively.

Definition 1.5.3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. The algebraic direct sum $X \oplus Y$ of X and Y becomes a normed space, called the topological direct sum of X and Y and still denoted $X \oplus Y$, when it is endowed with the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$.

We note that X and Y are isometric to the subspaces $\{(x,0) : x \in X\}$ and $\{(0,y) : \mathrm{Y} \in Y\}$ of $X \oplus Y$, respectively. If X and Y are Banach spaces, then so is $X \oplus Y$.

Theorem 1.5.4 (Closed graph theorem). *Let X, Y be Banach spaces and let T be an operator from X into Y. Then T is a bounded operator if and only if its graph $G := \{(x, Tx) : x \in X\}$ is closed in $X \oplus Y$.*

Recall that a sequence $((x_n,y_n))_{n\in\mathbb{N}}$ of points of $X \oplus Y$ converges to (x,y) if and only if $x_n \to x$ and $y_n \to y$. We also notice that the graph G of T is a subspace of $X \oplus Y$.

Proof. If T is continuous and $(x_n, Tx_n) \to (x_0,y_0)$, then $y_0 = Tx_0$. Indeed, we have $x_n \to x_0$ and $Tx_n \to y_0$, while the continuity of T implies that $Tx_n \to Tx_0$. This means that (x_0,y_0) is in the graph of T, showing that G is closed.

If G is closed in $X \oplus Y$, then G is a Banach space in the norm induced from $X \oplus Y$. Consider the mapping $p(x, Tx) = x$. By the definition of the norm of the space $X \oplus Y$, we see that p is continuous, maps G onto X, and is one-to-one. By Corollary 1.5.2, $p^{-1} : x \to$

(x, Tx) is a continuous mapping from X onto G. Since the map $q : X \oplus Y \to Y$, $q(x,y) := y$, is continuous and $T = q \circ p^{-1}$, T must be continuous. ☐

1.5.6 Linear functionals

Let X be a vector space over $\mathbb{K}$. A hyperplane in X is a linear subspace M of X such that $\dim(X/M) = 1$. If $f : X \to \mathbb{K}$ is a linear functional and $f \neq 0$, then $\ker f$ is a hyperplane of X. In fact, f induces an isomorphism between $X/\ker f$ and $\mathbb{K}$. Conversely, if M is a hyperplane in X, then there is an isomorphism $\rho : X/M \to \mathbb{K}$ so that $f(x) = \rho(x + M)$ defines a linear functional with $\ker f = M$.

Suppose now that f and g are linear functionals on X such that $\ker f = \ker g$. Let $x_0 \in X$ be such that $f(x_0) = 1$, so $g(x_0) \neq 0$. If $x \in X$ and $\alpha = f(x)$, then $x - \alpha x_0 \in \ker f = \ker g$. So $0 = g(x) - \alpha g(x_0)$, or $g(x) = (g(x_0))\alpha = (g(x_0))f(x)$. Thus $g = \beta f$ for a scalar β. This is summarized as follows.

Proposition 1.5.5. *A linear manifold in a vector space X is a hyperplane if and only if it is the kernel of a nonzero linear functional. Two linear functionals have the same kernel if and only if one is a nonzero multiple of the other.*

Hyperplanes in a normed space fall into one of the two following categories.

Proposition 1.5.6. *Let X be a normed space. A hyperplane in X is either closed or it is dense. If f is a linear functional on X, then f is bounded if and only if $\ker f$ is closed.*

Note that if f is a linear functional, then it is a linear operator and so Proposition 1.5.1 applies. Continuous linear functionals are also called bounded linear functionals and

$$\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}. \tag{1.5}$$

Let X^* be the collection of all bounded linear functionals on X. It is clear that if $f, g \in X^*$ and $\alpha \in \mathbb{K}$, then $(\alpha f + g)(x) = \alpha f(x) + g(x) \in X^*$, hence X^* is a vector space. The space X^* equipped with the norm (1.5) is a normed space, called the *dual space* of X. Note that X^* is nothing else but $\mathcal{L}(X, \mathbb{K})$.

Proposition 1.5.7. *If X is normed space, then X^* is a Banach space.*

Proof. In practice, the scalar field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Since $\mathbb{R}$ and $\mathbb{C}$ are complete, the result follows from Proposition 1.5.2. ☐

Notation

Let X be a normed space. Given $f \in X^*$ and $x \in X$, we shall often write $\langle f, x\rangle$ or $\langle f, x\rangle_{X^* \times X}$ instead of $f(x)$.

1.5.7 The adjoint of a linear operator

In this subsection we are concerned with the adjoint of bounded linear operators. For unbounded linear operators, we refer, for example, to [46] or [138].

Definition 1.5.4. Let X and Y be normed spaces and let $T : X \to Y$ be a bounded linear operator. The map $T^* : Y^* \to X^*$ defined by

$$(T^* y^*)(x) = (y^* \circ T)(x), \quad x \in X, y^* \in Y^*,$$

is called the adjoint of T.

Proposition 1.5.8. *Let X and Y be two normed spaces. If $T : X \to Y$ is a bounded linear operator, then the adjoint T^* is a bounded linear operator and $\|T^*\| = \|T\|$.*

Proof. It is not hard to show T^* is linear. To show T^* is bounded, let $y^* \in Y^*$. Then

$$\|T^* y^*\|_{X^*} = \sup_{\|x\| \le 1} |(T^* y^*)(x)| = \sup_{\|x\| \le 1} |y^*(Tx)|.$$

Therefore,

$$\|T^* y^*\|_{X^*} \le \sup_{\|x\| \le 1} \|y^*\|_{Y^*} \|Tx\|_Y = \|T\| \|y^*\|_{Y^*}.$$

Hence, T is bounded and $\|T^*\| \le \|T\|$.

To prove the reverse inequality, we begin by letting $\varepsilon > 0$. There exists $x \in X$ such that $\|x\|_X \le 1$ and $\|Tx\|_Y > \|T\| - \varepsilon$. By Corollary 1.6.2, there exists $y^* \in Y^*$ such that $\|y^*\|_{Y^*} = 1$ and $y^*(Tx) = \|Tx\|_Y$. Hence

$$\|Tx\|_Y = (T^* y^*)(x) \le \|T^* y^*\|_{X^*} \|x\|_X \le \|T^* y^*\|_{X^*} \le \|T^*\|.$$

Consequently, $\|T\| < \|T^*\| + \varepsilon$. Due to the arbitrary choice of ε, we conclude that $\|T\| \le \|T^*\|$. ☐

Corollary 1.5.3. *Let X and Y be normed spaces. The map taking T to T^* is a linear isometry from $\mathcal{L}(X, Y)$ to $\mathcal{L}(Y^*, X^*)$.*

Proof. This follows from Proposition 1.5.8. ☐

Proposition 1.5.9. *Let X, Y, and Z be normed spaces, and suppose both $S : X \to Y$ and $T : Y \to Z$ are bounded linear operators. If $TS = T \circ S : X \to Z$, then the adjoint map $(TS)^* : Z^* \to X^*$ is given by $(TS)^* = S^* T^*$.*

Proof. If $x \in X$ and $z^* \in Z^*$, then

$$\langle x, (TS)^*(z^*) \rangle = \langle TSx, z^* \rangle = \langle Sx, T^* z^* \rangle = \langle x, S^* T^* z^* \rangle,$$

from which it follows that $(TS)^* = S^* T^*$. ☐

1.6 Hahn–Banach theorem

1.6.1 The analytic form of Hahn–Banach theorem

Definition 1.6.1. Suppose X is a vector space. A map $p : X \to \mathbb{R}$ is called a sublinear functional if

(a) $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$ and $x \in X$,

(b) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$. □

Note that condition (a) implies that $p(0) = 0$. Condition (b) is called the triangle inequality. It is clear that any real linear functional is a sublinear functional. If X is a normed space, then the norm function defined by $p(x) = \|x\|$ for $x \in X$ is a sublinear functional.

Theorem 1.6.1 (Hahn–Banach, analytic form). *Let X be a real vector space and p a sublinear functional on X. Let f be a linear functional which is defined on a subspace Y of X and satisfies*

$$f(x) \leq p(x) \quad \text{for all } x \in Y. \tag{1.6}$$

Then f has a linear extension $\tilde{f}$ from Y to X satisfying

$$\tilde{f}(x) \leq p(x) \quad \text{for all } x \in X, \tag{1.7}$$

that is, $\tilde{f}$ is a linear functional on X satisfying (1.7) *on X and $\tilde{f}(x) = f(x)$ for every $x \in Y$.*

Theorem 1.6.2 (Hahn–Banach theorem for normed spaces). *Let X be a real normed vector space and let Y be a linear subspace of X. If $f \in Y^*$, then there exists a bounded linear extension functional $\tilde{f}$ on X such that*

$$\|\tilde{f}\|_X = \|f\|_Y.$$

Proof. Define $p(x) = \|f\|\|x\|$ for $x \in X$. Then p is a continuous seminorm defined on X such that $|f(x)| \leq p(x)$ for all $x \in Y$. Therefore, by Theorem 1.6.1, there is a linear extension $\tilde{f}$ of f defined on the whole space X such that $|\tilde{f}(x)| \leq p(x)$. This yields $\|\tilde{f}\| \leq \sup_{\|x\|\leq 1} p(x) = \|f\|$. On the other hand, since $\tilde{f}$ is an extension of f, we must have $\|\tilde{f}\| \geq \|f\|$ and so we obtain $\|\tilde{f}\| = \|f\|$. □

Corollary 1.6.1. *Let X be a normed space. For every $x_0 \in X$, there exists $f_0 \in X^*$ such that*

$$\|f_0\| = \|x_0\| \quad \text{and} \quad \langle f_0, x_0 \rangle = \|x_0\|^2.$$

Proof. Use Theorem 1.6.2 with $Y = \mathbb{R}x_0$ and $f(tx_0) = t\|x_0\|^2$ so that $\|f\|_{Y^*} = \|x_0\|^2$. □

Remark 1.6.1. The element f_0 given by Corollary 1.6.1 is in general not unique. However, if X is strictly convex (for example, if X is a Hilbert space or $X = L^p(\Omega)$ with $1 < p < +\infty$), then f_0 is unique.

Corollary 1.6.2. *Let X be a normed space. For every $x \in X$, we have*

$$\|x\| = \sup\{|\langle f,x\rangle| : f \in X^* \textit{ with } \|f\| \le 1\} = \max\{|\langle f,x\rangle| : f \in X^* \textit{ with } \|f\| \le 1\}.$$

Proof. We may always assume that $x \neq 0$. It is clear that $\sup_{f\in X^*,\|f\|\le 1} |\langle f,x\rangle| \le \|x\|$. By Corollary 1.6.1, there exists $f_0 \in X^*$ such that $\|f_0\| = \|x\|$ and $\langle f_0,x\rangle = \|x\|^2$. Set $f_1 = f_0/\|x\|$, so that $\|f_1\| = 1$ and $\langle f_1,x\rangle = \|x\|$. ☐

1.6.2 Geometric forms of Hahn–Banach theorem

Definition 1.6.2. Let A and B be two subsets of a real vector space X and f a real linear functional on X. We say that a hyperplane $H = \{x \in X : f(x) = \alpha\}$ separates A and B if

$$f(x) \le \alpha \quad \forall x \in A \quad \text{and} \quad f(x) \ge \alpha \quad \forall x \in B.$$

We say that H strictly separates A and B if there exists some $\varepsilon > 0$ such that

$$f(x) \le \alpha - \varepsilon \quad \forall x \in A \quad \text{and} \quad f(x) \ge \alpha + \varepsilon \quad \forall x \in B.$$ ☐

Geometrically, separation means that A lies in one of the half-spaces determined by H, and B lies in the other.

To prove separation theorems of convex sets, the following two lemmas are required.

Lemma 1.6.1. *Let X be a normed space and let C be a convex neighborhood of* 0. *For every $x \in X$, define*

$$\mu_c(x) = \inf\{\alpha > 0 : x \in \alpha C\}. \tag{1.8}$$

Then $\mu_C(\cdot)$ is a finite nonnegative positively homogeneous subadditive continuous functional. Moreover, $\{x : \mu_C(x) < 1\} = \operatorname{int}(C) \subset C \subset \overline{C} = \{x : \mu_C(x) \le 1\}$.

The map $\mu_c(\cdot)$ is called the Minkowski functional.

Proof. Let $x \in X$. Note first that, since the map $\lambda \mapsto \lambda x$, $\mathbb{R} \to X$, is continuous at $\lambda = 0$, there is some $\delta > 0$ such that $\lambda x \in C$ whenever $|\lambda| < \delta$. This shows that $x \in \eta C$ whenever $|\eta| < \delta^{-1}$. Thus $\mu_c(x)$ is well defined; $0 \le \mu_c(x) < \infty$. Moreover, since $0 \in C$, the point 0 is in λC for every $\lambda > 0$ and thus $\mu_C(0) = 0$.

Now let $x,y \in X$, and set $r = \mu_c(x)$, as well as $s = \mu_C(y)$. Then for each $\varepsilon > 0$, there exist α and β such that $0 < \alpha < r + \varepsilon/2$, $0 < \beta < s + \varepsilon/2$, $x \in \alpha C$, and $y \in \beta C$. Then, using the fact that C is convex, we get

$$x+y \in \alpha C + \beta C = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta}C + \frac{\beta}{\alpha+\beta}C\right) \subseteq (\alpha+\beta)C.$$

Thus $\mu_C(x+y) \leq \alpha+\beta < r+s+\varepsilon$. Since this holds for all $\varepsilon > 0$, we get $\mu_C(x+y) \leq \mu_C(x)+\mu_C(y)$. For each $\lambda > 0$, we have $\lambda x \in \alpha\lambda C$ if and only if $x \in \alpha C$, and so it follows that $\mu_C(\lambda x) = \lambda\mu_C(x)$ (≥ 0). This shows that μ_C is a sublinear functional on X.

Let $\mathbb{B}_\delta \subset C$ for some $\delta > 0$. Given $x \in X \setminus \{0\}$, we get $\delta\frac{x}{\|x\|} \in \mathbb{B}_\delta \subset C$, so $x \in \frac{\|x\|}{\delta}C$. Thus

$$0 \leq \mu_C(x) \leq \frac{\|x\|}{\delta} < \infty.$$

This estimate shows that μ_C is continuous at 0. Moreover, since μ_C is subadditive, for $x,y \in X$, we have $-\mu_C(y-x) \leq \mu_C(x)+\mu_C(y) \leq \mu_C(x-y)$. This yields that μ_C is continuous if (and only if) it is continuous at 0. Hence μ_C is continuous on X because it is continuous at 0.

By the continuity of μ_C, the set $\{x \in X : \mu_C(x) < 1\}$ is open. Using the convexity of C and the fact that $0 \in C$, it is not difficult to see that if, for some $x \in X$, $\mu_C(x) < \lambda$, then $x \in \lambda C$. So, by this observation we conclude that $\{x \in X : \mu_C(x) < 1\}$ is a subset of C, hence a subset of $\operatorname{int}(C)$. It follows that if $\mu_C(x) = 1$ and $0 < s < 1 < t$ then $sx \in \operatorname{int}(C)$ and $tx \neq C$. Therefore, if $\mu_C(x) = 1$ then $x \in \partial C$, and if $\mu_C(x) > 1$ then, again by the continuity of μ_C, $x \in \operatorname{Ext}(C)$. □

Lemma 1.6.2. *Let X be a real normed space and let C be an open convex subset of X. If $x_0 \notin C$, then there is an $f \in X^*$ such that $f(x) < f(x_0)$ $\forall x \in C$. In particular, the hyperplane $[f = f(x_0)]$ separates $\{x_0\}$ and C.*

Proof. We may assume without loss of generality that $0 \in C$, otherwise consider $(C-x)$ and $x_0 - x$ for some $x \in C$. Hence, C is open, convex, and contains 0. Let μ_C be the Minkowski functional of C. Since C is open and $x_0 \notin C$, we have $\mu_C(x_0) \geq 1$. Define a linear functional on $\operatorname{span}\{x_0\}$ by $f(\lambda x_0) = \lambda\mu_C(x_0)$. Then on $\operatorname{span}\{x_0\}$ we have $f(\lambda x_0) \leq \mu_C(\lambda x_0)$. For $\lambda \geq 0$, it is clear from the definition of f that, for $\lambda < 0$, we have $f(\lambda x_0) = \lambda\mu_C(x_0) < 0$ while $\mu_C(\lambda y_0) \geq 0$. By Lemma 1.6.1, μ_C is continuous, hence, by construction, f is also continuous. Extend f onto X by Theorem 1.6.2 and denote this extension by f again. Then $f(x) \leq \mu_C(x)$ for every $x \in X$. Since $f(x_0) = \mu_C(x_0) > 1$ as $x_0 \notin C$ and $f(x_0) = \mu_C(0)$, we get $f(x_0) > 1$ and so $f(x_0) > \sup\{f(x) : x \in C\}$, and the statement follows. □

Theorem 1.6.3. *Let $A, B \subset X$ be two nonempty convex subsets such that $A \cap B = \emptyset$. Assume that A is open. Then there exists a closed hyperplane that separates A and B.*

Proof. Set $C = A-B$, so that C is convex, open (since $C = \bigcup_{y\in B}(A-y)$), and $0 \notin C$ (because $A \cap B = \emptyset$). Applying Lemma 1.6.2 to the open convex set C and to $x_0 := 0$, we obtain f such that $f(x) < f(0) = 0$ for $x \in A-B$. Thus $f(a) < f(b)$ $\forall a \in A$, $b \in B$. It follows that $f(a) \leq \inf\{f(b) : b \in B\}$ for $a \in A$. If $a \in A$ is such that $f(a) = \inf\{f(b) : b \in B\}$, then from the openness of A we get $f(a+h) > \inf\{f(b) : b \in B\}$ for some $a+h \in A$, a contradiction. Therefore, $f(a) < \inf\{f(b) : b \in B\}$ for all $a \in A$. □

We also have the following separation theorem which was obtained independently by J. W. Tukey (1942) and V. L. Klee (1951).

Theorem 1.6.4. *Let $A, B \subset X$ be two nonempty convex subsets such that $A \cap B = \emptyset$. Assume that A is closed and B is compact. Then there exists a closed hyperplane that strictly separates A and B.*

Proof. Set $C = A - B$. It is clear that C is convex, closed, and $0 \notin C$. So, there exists some $r > 0$ such that $\mathbb{B}_r \cap C = \emptyset$. Applying Theorem 1.6.3, we infer that there exists a closed hyperplane that separates $\mathbb{B}_r$ and C. Hence, there is some $f \in X^*, f \neq 0$ such that

$$f(x-y) \le f(rz) \quad \forall x \in A,\ \forall y \in B,\ \forall z \in \mathbb{B}_1.$$

It follows that $f(x-y) \le -r\|f\|\ \forall x \in A, \forall y \in B$. Letting $\varepsilon = \frac{1}{2}r\|f\| > 0$, we obtain

$$f(x) + \varepsilon \le f(y) - \varepsilon \quad \forall x \in A,\ \forall y \in B.$$

Choosing α such that

$$\sup_{x\in A} f(x) + \varepsilon \le \alpha \le \inf_{y\in B} f(y) - \varepsilon,$$

we see that the hyperplane $[f = \alpha]$ strictly separates A and B. □

1.7 Weak topology

Let $(X, \|\cdot\|)$ be a normed space. We recall that X^* denotes the vector space of all continuous linear functionals on X, endowed with the dual norm (1.5). We know from Proposition 1.5.7 that the dual space X^* is always complete, even if X is not. Since the norm on the scalar field $\mathbb{K}$ is simply the absolute value, the operator norm of a linear functional f can be written in this notation as $\|f\| = \sup_{\|x\|=1} |\langle f, x\rangle|$.

Note that, according to Hahn–Banach theorem (Theorem 1.6.2), the dual space X^* does not reduce to $\{0\}$ if X is different from $\{0\}$. If $\dim(X) = n$ for some $n \in \mathbb{N}$, then $\dim(X^*) = n$, and if X is infinite-dimensional, so is X^*.

Let X be a Banach space and let $f \in X^*$. We denote by $\pi_f : X \to \mathbb{K}$ the linear functional defined by $\pi_f(x) = \langle f, x\rangle$. As f runs through X^*, we obtain a collection $(\pi_f)_{f\in X^*}$ of maps from X into $\mathbb{K}$. We now ignore the usual topology on X (associated to $\|\cdot\|$) and define a new topology on X as follows.

Definition 1.7.1. The weak topology $\sigma(X, X^*)$ on X is the coarsest topology for which all functionals $(\pi_f)_{f\in X^*}$ are continuous.

The weak topology of a normed space is the smallest topology for the space such that every member of the dual space is continuous with respect to that topology. Note that, since every map π_f is continuous for the usual topology, the weak topology is weaker than the usual topology. More precisely, open (resp. closed) sets in the weak topology $\sigma(X, X^*)$ are always open (resp. closed) in the strong topology. In any infinite-dimensional space, the weak topology is strictly coarser than the strong topology, i. e., there exist open (resp. closed) sets in the strong topology that are not open (resp. closed) in the weak topology.

Lemma 1.7.1. *Let Y be a topological space and let φ be a map from Y to X (X is equipped with the weak topology). Then φ is continuous if and only if $\varphi \circ \pi_f$ is continuous from Y into $\mathbb{K}$ for each $f \in X^*$.*

Proof. If ψ is continuous, then $f \circ \psi$ is also continuous for each $f \in X^*$. Conversely, let U be an open subset of X, we have to prove that $\psi^{-1}(U)$ is open in Y. But we know that U has the form

$$U = \bigcup_{\text{abitrary}} \bigcap_{\text{finite}} f^{-1}(\theta_f) \quad \text{with } \theta_f \text{ being an open subset of } \mathbb{K}.$$

Therefore

$$\psi^{-1}(U) = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \psi^{-1}(f^{-1}(\theta_f)) = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} (f \circ \psi)^{-1}(\theta_f),$$

which is open in Y since every map $f \circ \psi$ is continuous. ☐

Proposition 1.7.1. *The weak topology $\sigma(X, X^*)$ is a Hausdorff topology.*

Proof. Let x and y be two elements of X with $x \neq y$. We have to find two open sets U and V for the weak topology $\sigma(X, X^*)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. By Theorem 1.6.4, there exists a closed hyperplane strictly separating $\{x\}$ and $\{y\}$. Hence there exist some linear functional $f \in X^*$ and some $\zeta \in \mathbb{R}$ such that $\langle f, x\rangle < \zeta < \langle f, y\rangle$. Set

$$\begin{aligned} U &= \{z \in X : \langle f, z\rangle < \zeta\} = \pi_f^{-1}((-\infty, \zeta)), \\ V &= \{z \in X : \langle f, z\rangle > \zeta\} = \pi_f^{-1}((\zeta, +\infty)). \end{aligned}$$

It is clear that U and V are open in the topology $\sigma(X, X^*)$ and satisfy the properties listed above. ☐

We recall that, for $x_0 \in X$, $\varepsilon > 0$, and a finite set $\{f_1, f_2, \ldots, f_n\}$ in X^*, we can define

$$V = V(f_1, f_2, \ldots, f_n; \varepsilon) = \{x \in X : |\langle f_i, x - x_0\rangle| < \varepsilon \text{ for all } i = 1, 2, \ldots, n\}.$$

Then V is a neighborhood of x_0 for the topology $\sigma(X, X^*)$. Moreover, we obtain a basis of neighborhoods of x_0 for $\sigma(X, X^*)$ by varying ε, n, and the functionals f_i's in X^*.

Notations

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X, $x \in X$, and A a subset of X.

- With the symbol $x_n \rightharpoonup x$ we denote the convergence of $(x_n)_{n\in\mathbb{N}}$ to x for the topology $\sigma(X, X^*)$.
- We denote by $\overline{A}^w$ the closure of A in the weak topology $\sigma(X, X^*)$.

Evidently in the case where $(x_n)_{n\in\mathbb{N}}$ converges to x in the norm topology, we say that $(x_n)_{n\in\mathbb{N}}$ converges strongly to x and we use the notation $x_n \to x$.

Remark 1.7.1.

(a) Let X be a normed space. We note that the norm topology and the weak topology $\sigma(X, X^*)$ of X are the same if and only if the space is finite-dimensional. Hence the weak topology of an infinite-dimensional normed space is not induced by the norm of the space. Moreover, $\sigma(X, X^*)$ is, in general, not metrizable. It is metrizable only if the space is finite-dimensional.

(b) Completeness is yet another property that the weak topology cannot have unless the normed space is finite-dimensional. Hence, the weak topology of a normed space is complete if and only if the space is finite-dimensional.

Let X be a normed space and let A be a subset of X. According to Definition 1.3.3(a), A is *weakly bounded* if, for each weak neighborhood U of 0 in X, there is $\tau_U \geq 0$ such that $A \subset tU$ whenever $t > \tau_U$.

Proposition 1.7.2. *A subset of a normed space is bounded if and only if it is weakly bounded.*

We have also the following useful corollaries.

Corollary 1.7.1. *A subset A of a normed space X is bounded if and only if, for each f in X^*, $f(A)$ is a bounded set of scalars.*

Corollary 1.7.2. *In a normed space, weakly compact subsets, weakly Cauchy sequences, and weakly convergent sequences are bounded.*

Definition 1.7.2. Let X be normed space. A mapping $f : X \to X$ is said to be weakly continuous if it is continuous for the weak topology $\sigma(X, X^*)$ of X.

It is clear that weakly continuous maps are not necessarily continuous. However, for linear operators, we have the following result.

Theorem 1.7.1. *Let X and Y be two normed spaces and let T be a linear operator from X into Y. Then T is norm-to-norm continuous if and only if it is weak-to-weak continuous.*

Proof. In view of Lemma 1.7.1, it suffices to check that for every $f \in Y^*$ the map $x \mapsto \langle f, Tx\rangle$ is continuous from $(X, \sigma(X, X^*))$ into $\mathbb{K}$. But the map $x \mapsto \langle f, Tx\rangle$ is a continuous linear functional on X. Therefore, it is also continuous in the weak topology $\sigma(X, X^*)$.

Conversely, suppose that T is continuous from X weak into Y weak. Then $G(T)$ is closed in $X \times Y$ with the product topology $\sigma(X,X^*) \times \sigma(Y,Y^*)$, which is the same as $\sigma(X\times Y,(X\times Y)^*)$. It follows that $G(T)$ is strongly closed (any weakly closed set is strongly closed). Next, using the closed graph theorem (Theorem 1.5.4), we conclude that T is continuous from X strong into Y strong. □

Note that, in contrast to linear operators, nonlinear maps that are continuous from $(X,\|\cdot\|)$ into $(Y,\|\cdot\|)$ are not necessarily continuous from $(X,\sigma(X,X^*))$ into $(Y,\sigma(Y,Y^*))$. This produces many difficulties in nonlinear problems.

It is not difficult to see that, when X is an infinite-dimensional normed space, open subsets for the weak topology are unbounded. In fact, all weakly open subsets contain linear vector subspaces of X and so they are not bounded. In particular, bounded balls in X cannot be weakly open since they are bounded. Because of this, one asks whether closed convex subsets of an arbitrary normed space are weakly closed or not. However, a theorem of S. Mazur (1933) ensures that this must be the case.

Theorem 1.7.2 (Mazur). *Let X be a normed space and let C be a convex subset of X. Then C is closed for the strong topology if and only if it is closed for the weakly topology $\sigma(X,X^*)$.*

Proof. Assume that C is closed in the strong topology and let us prove that C is closed in the weak topology. We shall check that the complement $C^c = X \setminus C$ of C is open in the weak topology. To this end, let $x_0 \notin C$. By Theorem 1.6.4, there exists a closed hyperplane strictly separating $\{x_0\}$ and C. So, there exist some linear functional $f \in X^*$ and some $\zeta \in \mathbb{R}$ such that $\langle f,x\rangle < \zeta < \langle f,y\rangle$ for all $y \in C$. Set

$$U = \{z \in X : \langle f,z\rangle < \zeta\}$$

so that $x_0 \in U$, $U \cap C = \emptyset$, that is, $U \subset C^c$ and U is open in the weak topology of X. □

Proposition 1.7.3. *Let X be normed space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then*
(a) $x_n \rightharpoonup x$ *($(x_n)_{n\in\mathbb{N}}$ converges weakly to x) if and only if, $\langle f,x_n\rangle \to \langle f,x\rangle$ for every $f \in X^*$.*
(b) *If $x_n \to x$, then $x_n \rightharpoonup x$.*
(c) *If $x_n \rightharpoonup x$, then $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded and $\|x\| \le \liminf_{n\to+\infty} \|x_n\|$.*
(d) *If $x_n \rightharpoonup x$ and $f_n \to f$ in X^*, then $\langle f_n,x_n\rangle \to \langle f,x\rangle$.*

Definition 1.7.3. Let X be normed space. A mapping $f : X \to X$ is said to be weakly sequentially continuous at a point $x_0 \in X$ if, for each sequence $(x_n)_{n\in\mathbb{N}}$ which converges weakly to x_0, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges weakly to $f(x_0)$, that is, $x_n \rightharpoonup x_0$ implies $f(x_n) \rightharpoonup f(x_0)$. If f is weakly sequentially continuous at each point of X, then we say that f is weakly sequentially continuous on X.

We end this section by recalling some results related to the weak compactness required in the sequel. Their proofs require heavy preparations so they are dropped.

Theorem 1.7.3 (Eberlein–Šmulian). *Let A be a subset of a Banach space X. Then the following statements are equivalent:*

(a) *A is weakly sequentially compact, that is, any sequence in A has a subsequence which converges weakly to an element of X;*

(b) *the closure of A in the topology $\sigma(X, X^*)$, $\overline{A}^w$, is compact.*

For a proof of this result we refer, for example, to [89, 92, 173] or [243].

Let us notice that an immediate consequence of Theorem 1.7.3 is that if K is a weakly compact subset of a Banach space, then any weakly sequentially continuous map $f : K \to X$ is weakly continuous (cf. [21, p. 274]).

As an application of the Eberlein–Šmulian theorem (Theorem 1.7.3), we have

Theorem 1.7.4 (Krein–Šmulian). *Let X be a Banach space and let K be a subset of X. If K is weakly compact, then $\overline{\text{co}}(K)$ is weakly compact.*

For a proof of this result we refer, for example, to [89, p. 434].

Let Ω be a nonempty open subset of $\mathbb{R}^n$, not necessarily bounded, and (Ω, Σ, μ) a measure space where μ is a positive measure on Σ. Let $L^1(\Omega, d\mu)$ be the space of integrable functions from Ω into $\mathbb{R}$.

Theorem 1.7.5. *A subset M of $L^1(\Omega, d\mu)$ is weakly compact if and only if M is bounded and, for any decreasing sequence of measurable subsets $(M_n)_{n\in\mathbb{N}}$ of Σ such that $\bigcap_{n\geq 0} M_n = \emptyset$, we have*

$$\lim_{n\to+\infty} \int_{M_n} f(x)\, d\mu(x) = 0, \text{ uniformly in } f \in M.$$

For a proof see, for example, [89, Theorem 9, p. 292].

The following result is owing to J. Dieudonné [80, Theorem 4, p. 93]. It provides a characterization of weakly compact subsets in L^1-spaces.

Theorem 1.7.6. *Let Ω be a nonempty measurable subset of $\mathbb{R}^n$ (not necessary bounded). A bounded subset M of $L^1(\Omega; d\mu)$ is relatively weakly compact in $L^1(\Omega; d\mu)$ if and only if*

(a) *for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(A) \leq \delta$ implies $\int_A |f|\, d\mu < \varepsilon$, for all $f \in M$,*

(b) *for any $\varepsilon > 0$, there exists a compact subset $C \subset \Omega$ such that $\int_{\Omega\setminus C} |f|\, d\mu < \varepsilon$, for all $f \in M$.*

1.8 Weak* topology

Let X be a Banach space. It is obvious from the preceding section, that we have two topologies on X^*: the usual (strong) topology associated to the norm of X^* and the weak topology $\sigma(X^*, X^{**})$ obtained by using on X^* the construction of Section 1.7.

For every $x \in X$, define the functional $\varphi_x : X^* \to \mathbb{K}$ by $f \mapsto \varphi_x(f) = \langle f, x\rangle$. It is clear that when varying x through X we obtain a collection $(\varphi_x)_{x\in X}$ of maps from X^* to $\mathbb{K}$. Using the family of functions $(\varphi_x)_{x\in X}$, we are going to define a third topology on X^*.

Definition 1.8.1. The weak* topology $\sigma(X^*, X)$ on X^* is the coarsest topology for which all functionals $(\varphi_x)_{x\in X}$ are continuous.

Since X can be embedded in X^{**}, it is obvious that the topology $\sigma(X^*, X)$ is coarser than the topology $\sigma(X^*, X^{**})$.

Proposition 1.8.1. *The* weak* *topology is Hausdorff.*

Proof. Given f_1, f_2 with $f_1 \neq f_2$, there exists some $x \in X$ such that $\langle f_1, x\rangle \neq \langle x, f_2\rangle$. Assume, for example, that $\langle f_1, x\rangle < \langle f_2, x\rangle$ and choose α such that $\langle f_1, x\rangle < \alpha < \langle x, f_2\rangle$. Set

$$U_1 = \{f \in X^* : \langle f, x\rangle < \alpha\} = \varphi_x^{-1}((-\infty, \alpha))$$

and

$$U_2 = \{f \in X^* : \langle f, x\rangle > \alpha\} = \varphi_x^{-1}((\alpha, +\infty)).$$

Then U_1 and U_2 are open sets in $\sigma(X^*, X)$ such that $f_1 \in U_1, f_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. ☐

For $f_0 \in X^*$, $\varepsilon > 0$ and a finite set $\{x_1, \dots, x_n\}$ in X, we can define

$$V = V(x_1, \dots, x_n; \varepsilon) = \{f \in X^* : |\langle f - f_0, x_i\rangle| < \varepsilon \quad \forall i = 1, 2, \dots, n\}.$$

Then V is a neighborhood of f_0 for the topology $\sigma(X^*, X)$. Moreover, we obtain a basis of neighborhoods of f_0 for $\sigma(X^*, X)$ by varying ε, n, and the x_i's in X.

Remark 1.8.1. In the case where X is a finite-dimensional normed space, the three topologies, strong, weak, and weak*, coincide on X^*.

While the unit ball in a Banach space can be compact in the norm topology only if the space is finite-dimensional, the unit ball in the weak* topology will always be compact. We recall this result known as Banach–Alaoglu theorem.

Theorem 1.8.1 (Banach–Alaoglu). *If X is a Banach space, then $\mathbb{B}_1^{X^*}$ is compact in the* weak* *topology on X^*.*

The next result due to Helly is required.

Lemma 1.8.1 (Helly). *Let X be a Banach space. Let $f_1, \dots, f_k$ be given in X^* and let $\zeta_1, \dots, \zeta_k$ be given in $\mathbb{R}$. The following properties are equivalent:*

(a) $\forall \varepsilon > 0\ \exists x_\varepsilon \in X$ *such that* $\|x_\varepsilon\| \leq 1$ *and*

$$|\langle f_i, x_\varepsilon\rangle - \zeta_i| < \varepsilon \quad \forall i = 1, 2, \dots, k,$$

(b) $|\sum_{i=1}^k \beta_i\zeta_i| \le \|\sum_{i=1}^k \beta_i f_i\| \quad \forall \beta_1, \beta_2, \dots, \beta_k \in \mathbb{R}$.

Proof. (a) $\Longrightarrow$ (b) Fix $\beta_1, \beta_2, \dots, \beta_k$ in $\mathbb{R}$ and let $S = \sum_{i=1}^k |\beta_i|$. It follows from (a) that

$$\left|\sum_{i=1}^k \beta_i \langle f_i, \varkappa_\varepsilon\rangle - \sum_{i=1}^k \beta_i \zeta_i\right| \le \varepsilon S$$

and therefore

$$\left|\sum_{i=1}^k \beta_i\zeta_i\right| \le \left\|\sum_{i=1}^k \beta_i f_i\right\| \|x_\varepsilon\| + \varepsilon S \le \left\|\sum_{i=1}^k \beta_i f_i\right\| + \varepsilon S.$$

Since this holds for every $\varepsilon > 0$, we obtain (b).

(b) $\Longrightarrow$ (a) Set $\zeta = (\zeta_1, \dots, \zeta_k) \in \mathbb{R}^k$ and let $\psi : X \to \mathbb{R}^k$ be the map defined by

$$\psi(x) = (\langle f_1, x\rangle, \dots, \langle f_k, x\rangle).$$

Property (a) says precisely that $\zeta \in \overline{\psi(\mathbb{B}_1)}$. Suppose, by contradiction, that (a) fails, so that $\zeta \notin \overline{\psi(\mathbb{B}_1)}$. Hence $\{\zeta\}$ and $\overline{\psi(\mathbb{B}_1)}$ are separated strictly by some hyperplane. This means that there exists some $\beta = (\beta_1, \beta_2, \dots, \beta_k) \in \mathbb{R}^k$ and some $\alpha \in \mathbb{R}$ such that

$$\beta \cdot \psi(x) < \alpha < \beta \cdot \zeta \quad \forall x \in \mathbb{B}_1.$$

It follows that

$$\left\langle \sum_{i=1}^k \beta_i f_i, x \right\rangle < \alpha < \sum_{i=1}^k \beta_i \zeta_i \quad \forall x \in \mathbb{B}_1,$$

and then

$$\left\|\sum_{i=1}^k \beta_i f_i\right\| \le \alpha < \sum_{i=1}^k \beta_i \zeta_i,$$

which contradicts (b). ☐

Given a normed space $(X, \|\cdot\|)$, let X^* be its dual space endowed with dual norm. By $X^{**} = (X^*)^*$ we denote the second dual of X; it is equipped with the norm dual of X^* which is defined, for each $\zeta \in X^{**}$, by

$$\|\zeta\| = \|\zeta\|_{X^{**}} = \sup\{|\langle \zeta, f\rangle| : f \in X^*, \|f\| \le 1\}.$$

Let $i_c : X \to X^{**}$ be the canonical embedding of X into X^{**} defined in the following way. Let x be a fixed point in X, the map $f \mapsto \langle f, x\rangle$ from X^* into $\mathbb{R}$ is a linear continuous functional on X^*, that is, it is an element of X^{**} denoted by $i_c x$. Then we have

$$\langle i_c x, f\rangle_{X^{**},X^*} = \langle f, x\rangle_{X^*,X} \quad \forall x \in X, \quad \forall f \in X^*.$$

It is clear that i_c is linear. Further, it follows from Corollary 1.6.2 that

$$\|i_c(x)\|_{X^{**}} = \sup_{\|f\|\le 1} |\langle f, x\rangle| = \sup_{\|f\|\le 1} |\langle f, x\rangle| = \|x\|,$$

which proves that i_c is an isometry.

Using this natural identification, for $x \in X$ we often write $x \in X^{**}$ instead of $i_c(x) \in X^{**}$, and we identify X with $i_c(X) \subset X^{**}$. Hence X can be thought of as a subspace of its bidual X^{**}.

Remark 1.8.2. The space X^{**} is the dual space for X^*, and as such can be given a weak* topology. The weak* topology on X^{**} is the weakest topology under which elements of X^* define continuous functions on X^{**}. If we restrict to the subspace X, then the weakest topology under which elements of X^* are continuous is the weak topology on X. Therefore

$$(X^{**}, \sigma(X^{**}, X^*))_{|X} = (X, \sigma(X, X^*)).$$

In other words, the restriction of the weak* topology on X^{**} to X is the weak topology on X.

We are now ready to state and prove Goldstine's theorem.

Theorem 1.8.2 (Goldstine). *If X is a Banach space, then $i_c(\mathbb{B}_1)$ is dense in $\mathbb{B}_1^{X^{**}}$ with respect to the topology $\sigma(X^{**}, X^*)$, and consequently $i_c(X)$ is dense in X^{**} in the topology $\sigma(X^{**}, X^*)$.*

Proof. Let $\tau \in \mathbb{B}_1^{X^{**}}$ and let V be a neighborhood of τ for the topology $\sigma(X^{**}, X^*)$. We must prove that $V \cap i_c(\mathbb{B}_1) \neq \emptyset$. We may assume that V is of the form

$$V = \{\eta \in X^{**} : |\langle \eta - \tau, f_i\rangle| < \varepsilon \quad \forall i = 1, 2, \dots, k\}$$

for some given elements $f_1, f_2, \dots, f_k$ in X^* and some $\varepsilon > 0$. We have to find some $x \in \mathbb{B}_1$ such that $i_c(x) \in V$, that is,

$$|\langle f_i, x\rangle - \langle \tau, f_i\rangle| < \varepsilon \quad \forall i = 1, 2, \dots, k.$$

Set $\zeta_i = \langle \tau, f_i\rangle$. According to Lemma 1.8.1, it suffices to check that

$$\left|\sum_{i=1}^k \beta_i \zeta_i\right| \le \left\|\sum_{i=1}^k \beta_i f_i\right\|,$$

which is clear since $\sum_{i=1}^k \beta_i \zeta_i = \langle \tau, \sum_{i=1}^k \beta_i f_i\rangle$ and $\|\tau\| \le 1$. □

1.9 Some classes of Banach spaces

1.9.1 Reflexive spaces

We know from the previous section that, if X is a normed space, we can identify X with $i_c(X)$, which is a subspace of X^{**}.

Definition 1.9.1. We say that a Banach space X is reflexive if the canonical embedding i_c is surjective, that is, $i_c(X) = X^{**}$. □

The following result provides a characterization of reflexive Banach spaces.

Theorem 1.9.1 (Kakutani). *A Banach space X is reflexive if and only if the closed unit ball $\mathbb{B}_1$ of X is weakly compact.*

Proof. Assume first that X is reflexive. Then $\mathbb{B}_1 = \mathbb{B}_1^{X^{**}}$. By Banach–Alaoglu theorem (Theorem 1.8.1), the set $\mathbb{B}_1^{X^{**}}$ is compact in the weak* topology on X^{**}. Since X is reflexive, according to Remark 1.8.2, the weak* topology on X^{**} coincides with the weak topology on X. Therefore, $\mathbb{B}_1$ is compact in the weak topology on X.

Conversely, assume that $\mathbb{B}_1$ is compact in the weak topology on X. By Remark 1.8.2, the weak topology on X is the restriction of the weak* topology on X^{**}, and so $\mathbb{B}_1$ is compact in the (and hence closed) in the weak* topology on X^{**}. By Goldstine's theorem (Theorem 1.8.2), we conclude that $\mathbb{B}_1 = \mathbb{B}_1^{X^{**}}$ since $\mathbb{B}_1$ is closed and dense in $\mathbb{B}_1^{X^{**}}$. Therefore, X is reflexive. □

1.9.2 Uniformly convex Banach spaces

Definition 1.9.2. A normed space X is called uniformly convex if, for all $\varepsilon \in (0,2]$, there exists $\delta(\varepsilon) > 0$ such that

$$(\forall x, y \in X \text{ such that } \|x\| = 1,\ \|y\| = 1, \text{ and } \|x - y\| > \varepsilon) \implies \left(\left\| \frac{x+y}{2} \right\| < 1 - \delta(\varepsilon) \right).$$

Thus, a normed space is uniformly convex if for any two distinct points x and y on the unit sphere centered at the origin the midpoint of the line segment joining x and y is never on the sphere but is close to the sphere only if x and y are sufficiently close to each other.

This allows us to introduce the following definition.

Definition 1.9.3. The modulus of uniform convexity of a Banach space X is the function $\delta_X : (0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

Obviously, a space X is uniformly convex if and only if its modulus of convexity satisfies $\delta_X(\varepsilon) > 0$ for each $\varepsilon \in (0, 2]$. Moreover, if X is a uniformly convex Banach space, then the function $\delta_X(\cdot)$ is continuous on $[0, 2]$.

Theorem 1.9.2. *L^p-spaces, $1 < p < +\infty$, are uniformly convex.*

For the proof of this result we refer, for example, to [46, p. 60].

An interesting result about the uniform convexity is Milman–Pettis theorem. The proof of this result is due to J. Ringrose. Here we follow the exposition of J. Diestel [78].

Theorem 1.9.3 (Milman–Pettis). *If X is a uniformly convex Banach space, then X is reflexive.*

Uniform convexity is a geometric property of the norm, that is, this property is not preserved by equivalent renormings. On the other hand, reflexivity is a topological property: a reflexive space remains reflexive for any equivalent norm. Theorem 1.9.3 is remarkable because it expresses the fact that a geometric property implies a topological one. However, there are some reflexive spaces that admit no uniformly convex equivalent norm.

Proof. Suppose that X is a nonreflexive, uniformly convex Banach space. Then, for some $\varepsilon > 0$, there exists x^{**} in X^{**} with $\|x^{**}\| = 1$ and such that the distance between x^{**} and $\mathbb{B}_1$, the closed unit ball of X, is 2ε. Let δ be chosen such that if x and y are in X with $\|x\| \leq 1$, $\|y\| \leq 1$, and $2 - \delta \leq \|x + y\|$ then $\|x - y\| \leq \varepsilon$. Take x^* in X^*, with $\|x^*\| = 1$ such that $\langle x^{**}, x^* \rangle = 1$.

Let V be the weak* neighborhood of x^{**} given by

$$V = \left\{ u^{**} \in X^{**} : \left| \langle x^*, u^{**} \rangle - 1 \right| < \frac{1}{2} \right\}.$$

If x and y are in the closed unit ball of X belonging to V (under identification by the canonical injection), then $|\langle x^*, x + y \rangle| > 2 - \delta$, so $2 - \delta \leq \|x + y\|$. Hence $\|x - y\| \leq \varepsilon$. Fixing x, we conclude that $V \cap \mathbb{B}_1 \subset x + \varepsilon \mathbb{B}_1^{**}$ ($\mathbb{B}_1^{**}$ denotes the closed unit ball of X^{**}). By Goldstine's theorem (Theorem 1.8.2), we know that $V \cap \mathbb{B}_1$ is weak* dense in $V \cap \mathbb{B}_1^{**}$ which, since $x + \varepsilon \mathbb{B}_1^{**}$ is weak* closed, yields that x^{**} belongs to $x + \varepsilon \mathbb{B}_1^{**}$. But this means that the distance between x^{**} and $\mathbb{B}_1$ is less than or equal to ε, contradicting our choice of x^{**}. ☐

We now give the following useful property of uniformly convex spaces.

Proposition 1.9.1. *Let X be a uniformly convex Banach space. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X such that $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $x_n \to x$ strongly.*

Definition 1.9.4. A Banach space X is said to have Kadec–Klee property if, for every sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$.

Remark 1.9.1. By Proposition 1.9.1, every uniformly convex Banach space has Kadec–Klee property, but there are Banach spaces, for instance, $(\mathbb{R}^n, \|\cdot\|_\infty)$, which possess Kadec–Klee property without being uniformly convex. This property is studied, for example, in [78].

We recall that the dual space of a uniformly convex Banach space is called a *uniformly smooth Banach space* and it can be characterized as follows: a Banach space X is uniformly smooth if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$, with $\|x\| = 1$ and $\|y\| \leq \delta$, then

$$\|x + y\| + \|x - y\| \leq 2 + \varepsilon\|y\|.$$

The modulus of smoothness of a normed space X is the function $\rho_X(\cdot)$ defined, for every $t > 0$, by the formula

$$\rho_X(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}.$$

The normed space X is uniformly smooth if and only if $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$.

1.9.3 Strictly convex Banach spaces

Definition 1.9.5. A Banach space X is said to be *strictly convex* if for all $x, y \in \partial\mathbb{B}_1$, we have

$$\left\|\lambda x + (1 - \lambda)y\right\| < 1 \quad \text{for all } \lambda \in (0, 1).$$

We note that every uniformly convex space is strictly convex. This gives a large class of strictly convex spaces. However, some well-known Banach spaces are not strictly convex.

The space ℓ_1 is not strictly convex. Indeed, take $\varepsilon = 1$ and consider $x = (1, 0, 0, \ldots)$ and $y = (0, -1, 0, 0, \ldots)$. It is clear that $x, y \in \ell_1$, $\|x\|_{\ell_1} = \|y\|_{\ell_1} = 1$, $\|x - y\|_{\ell_1} = 2 > \varepsilon$. However, $\|\frac{1}{2}(x + y)\|_{\ell_1} = 1$, showing that ℓ_1 is not strictly convex.

The space ℓ_∞ is not strictly convex. Indeed, take $\varepsilon = 1$ and consider $x = (1, 1, 0, \ldots)$ and $y = (-1, 1, 0, 0, \ldots)$. Clearly, $x, y \in \ell_\infty$, $\|x\|_{\ell_\infty} = \|y\|_{\ell_\infty} = 1$, $\|x - y\|_{\ell_\infty} = 2 > \varepsilon$. However, $\|\frac{1}{2}(x + y)\|_{\ell_\infty} = 1$, which shows that ℓ_∞ is not strictly convex.

Consider $C[a, b]$, the space of real-valued continuous functions on the compact interval $[a, b]$, with the “sup norm.” Then $C[a, b]$ is not strictly convex. To see this, choose two functions f, g defined as follows:

$$f(t) = 1 \quad \text{for all } t \in [a, b], \quad g(t) = \frac{b - t}{b - a} \quad \text{for all } t \in [a, b].$$

Take $\varepsilon = \frac{1}{2}$. Clearly, $f, g \in C[a,b]$, $\|f\| = \|g\| = 1$, and $\|f - g\| = 1 > \varepsilon$. Also $\|\frac{1}{2}(f + g)\| = 1$ and so $C[a,b]$ is not strictly convex.

We also notice that the spaces L^1, L^∞, and c_0 are not strictly convex.

1.9.4 Smooth Banach spaces

Definition 1.9.6. A Banach space X is said to be smooth if for every $x \in X$ with $\|x\| = 1$, there exists a unique $x^* \in X^*$ such that $\|x^*\| = x^*(x) = 1$.

It is not difficult to prove that a Banach space X is smooth if and only if, for every $x, y \in X$ with $x \neq 0$, the following limit exists:

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = \psi_x(y). \tag{1.9}$$

This limit defines a functional $\psi_x \in X^*$ which is called the Gateaux derivative of the norm at x.

Definition 1.9.7. A Banach space X is called uniformly smooth if the limit (1.9) exists uniformly in the set $\{(x,y) : \|x\| = \|y\| = 1\}$; thus X is uniformly smooth if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for $|t| < \delta$ and all $x, y \in X$ with $\|x\| = \|y\| = 1$, we have

$$\big| \|x + ty\| - \|x\| - \psi_x(y) \big| < \varepsilon |t|.$$

If the limit (1.9) exists uniformly for $\|y\| = 1$ when x is fixed, then the norm of X is said to be Fréchet differentiable.

1.9.5 Banach spaces with Dunford–Pettis property

Definition 1.9.8. Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is completely continuous or a *Dunford–Pettis* operator if whenever W is a weakly compact subset of X then $T(W)$ is a norm-compact subset of Y.

Clearly, if an operator is compact then it is Dunford–Pettis. If X is reflexive then an operator $T : X \to Y$ is compact if and only if T is Dunford–Pettis.

Proposition 1.9.2. *Suppose that X and Y are Banach spaces. A linear operator $T : X \to Y$ is Dunford–Pettis if and only if T is weak-to-norm sequentially continuous, that is, whenever $(x_n)_{n\geq 1}$ converges to x weakly, $(Tx_n)_{n\geq 1}$ converges to Tx in norm.*

Proof. Let $T : X \to Y$ be Dunford–Pettis and suppose that there is a weakly null sequence $(x_n)_{n\geq 1}$ of X such that $\|Tx_n\| \geq \delta > 0$ for some positive δ. Since the subset $W = \{x_n : n \geq 1\} \cup \{0\}$ is weakly compact, its image under T is norm-compact, therefore it contains a subsequence $(Tx_{n_k})_{k\geq 1}$ that converges in norm to some $y \in Y$. From the fact that T, in particular, is weak-to-weak continuous, it follows that the sequence $(T_n x_n)_{n\geq 1}$ is weakly null, so y must be 0, which contradicts our assumption.

For the converse implication, suppose T is weak-to-norm sequentially continuous. Let W be a weakly compact subset of X and let $(y_n)_{n\geq 1}$ be a sequence in $T(W)$. Pick x_n in X so that $y_n = Tx_n$ for all n. By Eberlein–Šmulian theorem (Theorem 1.7.3), $(x_n)_{n\geq 1}$ contains a subsequence $(x_{n_k})_{k\geq 1}$ that converges weakly to some x in W. Hence $(y_{n_k})_{k\geq 1}$ converges in norm to Tx. We conclude that $T(W)$ is norm-compact. ☐

Definition 1.9.9. A bounded linear operator $T : X \to Y$, where X and Y are Banach spaces, is called weakly compact if, for each bounded subset U of X, $T(U)$ is a weakly compact set of Y.

A. Grothendieck [122] has introduced the following definition as an abstraction of ideas originally developed by N. Dunford and J. B. Pettis [88].

Definition 1.9.10. A Banach space X is said to have Dunford–Pettis property (or, in short, X has (DPP)) if every weakly compact operator T from X into a Banach space Y is Dunford–Pettis.

For example, c_0 has (DPP) because if Y is a Banach space and $T : c_0 \to Y$ is a weakly compact operator, then T is compact, hence Dunford–Pettis. Further, ℓ_1 has also (DPP) because ℓ_1 has Schur property (cf. [89, Corollary 14]). We say that a normed space X has Schur property if, whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $x \in X$ is such that $x_n \rightharpoonup x$, then $x_n \to x$ (see, for example, [173, Definition 2.5.25, p. 220]). On the other hand, no infinite-dimensional reflexive Banach space X has (DPP) since the identity operator $I : X \to X$ is weakly compact but cannot be a Dunford–Pettis operator because the closed unit ball of X is not compact.

Remark 1.9.2. We note that if X is a Banach space with Dunford–Pettis property, then every linear weakly compact operator on X is a Dunford–Pettis operator.

Proposition 1.9.3. *Suppose that X is a Banach space. Then X has (DPP) if and only if for every sequence $(x_n)_{n\geq 1}$ of X satisfying $x_n \rightharpoonup 0$ and for all sequence $(x_n^*)_{n\geq 1}$ in X^* such that $x_n^* \rightharpoonup 0$, the sequence of scalars $(x_n^*(x_n))_{n\geq 1}$ converges to* 0.

Proof. Let Y be a Banach space and $T : X \to Y$ a weakly compact operator. Let us suppose that T is not Dunford–Pettis. Then there is $(x_n)_{n\geq 1}$ in X such that $x_n \rightharpoonup 0$ but $\|Tx_n\| \geq \delta > 0$ for all n. Pick a sequence $(y_n^*)_{n\geq 1}$ of Y^* such that $y_n^*(Tx_n) = \|Tx_n\|$ and $\|y_n^*\| = 1$ for all n. By Gantmacher's theorem (Gantmacher's theorem says that the dual operator of a weakly compact operator is also weakly compact; see, for example, [89, p. 485] or [173, p. 343]), T^* is weakly compact hence $T^*(\mathbb{B}_1^{Y^*})$ is a relatively weakly compact subset of X^*. By Eberlein–Šmulian theorem, $(T^*(y_n^*))_{n\geq 1} \subset T^*(\mathbb{B}_1^{Y^*})$ can be assumed weakly convergent to some x^* in X^*. Then $(T^* y_n^* - x^*)_{n\geq 1}$ is weakly convergent to 0, which implies $(T^*y_n^* - x^*)(x_n) \to 0$. But since $x^n(x_n) \to 0$, it then follows that $(T^*y_n^*(x_n))_{n\geq 1} = (\|Tx_n\|)_{n\geq 1}$ must converge to 0, which is absurd.

For the converse, let $(x_n)_{n\geq 1}$ in X be such that $x_n \rightharpoonup 0$ and $(x_n^*)_{n\geq 1}$ in X^* be such that $x_n^* \rightharpoonup 0$. Consider the operator

$$T : X \to c_0, \quad Tx = (x_n^*(x))_{n\geq 1}.$$

The adjoint operator T^* of T satisfies $T^*e_k = x_k^*$ for all $k \geq 1$ where $(e_k)_{k\geq 1}$ denotes the canonical basis of ℓ_1. This implies that $T^*(\mathbb{B}_1^{\ell})$ is contained in the convex hull of the weakly null sequence $(x_n^*)_{n\geq 1}$. Therefore T^* is weakly compact, hence, by Gantmacher's theorem, so is T. As T is weakly compact, T is also Dunford–Pettis by the hypothesis. Then, by Proposition 1.9.2, $\|Tx_n\| \to 0$. Thus $(x^*(x_n))_{n\geq 1}$ converges to 0 since, for all $n \geq 1$, $|x_n(x_n)| \leq \max_k |x_k^*(x_n)| = \|Tx_n\|_\infty$. ☐

Definition 1.9.11. A bounded subset $\mathcal{F} \subset L^1(\mu)$ is called equiintegrable (or uniformly integrable) if for all $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ so that, for every set $E \subset \Omega$ with $\mu(E) < \delta$, we have $\sup_{f\in\mathcal{F}} \int_E |f|\,d\mu < \varepsilon$, i. e.,

$$\lim_{\mu(E)\to 0} \sup_{f\in\mathcal{F}} \int_E |f|\,d\mu = 0.$$

Lemma 1.9.1. *Suppose $\mathcal{F}$ is a bounded subset of $L^1(\sum, \mu)$. Then the following are equivalent:*

(a) *$\mathcal{F}$ is equiintegrable,*

(b) $\lim_{M\to+\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\,d\mu = 0.$

Proof. (a) $\Longrightarrow$ (b) Since $\mathcal{F}$ is bounded, there is a constant $A > 0$ such that $\sup_{f\in\mathcal{F}} \|f\|_1 \leq A$. Given $f \in \mathcal{F}$, by Markov's inequality,

$$\mu(\{|f| > M\}) \leq \frac{\|f\|_1}{M} \leq \frac{A}{M}.$$

Therefore, $\lim_{M\to+\infty} \mu(\{|f| > M\}) \leq \frac{\|f\|_1}{M} = 0$. Using the equiintegrability of $\mathcal{F}$, we conclude that

$$\lim_{M\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\,d\mu = 0.$$

(b) $\Longrightarrow$ (a) Given $f \in \mathcal{F}$ and $E \in \sum$, for any $M > 0$, we have

$$\begin{aligned}\int_E |f|\,d\mu &= \int_{E\cap\{|f|\leq M\}} |f|\,d\mu + \int_{E\cap\{|f|>M\}} |f|\,d\mu \\ &\leq M\mu(E) + \int_{E\cap\{|f|>M\}} |f|\,d\mu\end{aligned}$$

$$\leq M\mu(E) + \int_{\{|f|>M\}} |f|\, d\mu$$

$$\leq M\mu(E) + \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\, d\mu.$$

Hence

$$\sup_{f\in\mathcal{F}} \int_E |f|\, d\mu \leq M\mu(E) + \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\, d\mu.$$

Let $\varepsilon > 0$, and consider $M = M(\varepsilon)$ such that $\sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\, d\mu < \frac{\varepsilon}{2}$. Then, if $\mu(E) < \frac{\varepsilon}{2M}$, we obtain

$$\sup_{f\in\mathcal{F}} \int_E |f|\, d\mu \leq M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Let K be a compact Hausdorff space. We denote by $\mathcal{C}(K)$ the space of all continuous scalar functions on K.

Theorem 1.9.4.
(a) *If K is a compact Hausdorff space, then $\mathcal{C}(K)$ has (DPP).*
(b) *If μ is a σ-finite measure, then $L^1(\mu)$ has (DPP).*

The first assertion is due to A. Grothendieck [122]. The second one was establish by N. Dunford and J. B. Pettis [88].

Corollary 1.9.1. *Let $T : L^1(\mu) \to L^1(\mu)$ be a weakly compact operator. Then T^2 is compact.*

Proof. Let $\mathbb{B}_1^{L^1(\mu)}$ be the closed unit ball of $L^1(\mu)$. Since T is weakly compact, $T(\mathbb{B}_1^{L^1(\mu)})$ is weakly compact. Because $L^1(\mu)$ has (DPP), we infer that $T^2(\mathbb{B}_1^{L^1(\mu)})$ is relatively norm-compact. $\square$

The result of Corollary 1.9.1 remains valid if we replace $L^1(\mu)$ by $\mathcal{C}(K)$ where K is a compact Hausdorff space.

1.10 Duality mapping

Let X be a Banach space and let X^* be its topological dual space. We call the normalized duality map of X the map $J : X \to 2^{X^*}$ defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 \text{ with } \|x^*\| = \|x\|\}.$$

We also define the mapping $F : X \to 2^{X^*}$ by

$$F(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\| \text{ with } \|x^*\| \leq 1\} \quad \forall x \in X.$$

We note that $J(x) = \|x\|F(x)$. Evidently, $J(x)$ is closed, convex, and bounded in X^* for each $x \in X$. Moreover, a simple application of Hahn–Banach theorem guarantees that $J(x)$ is nonempty for every $x \in X$. In some specials cases, $J(x)$ is single-valued.

The next result will be very useful in the chapter dedicated to the accretive operators since, among other things, it allows us to introduce the concept of accretive operator without using the normalized duality map.

Lemma 1.10.1. *Let X be a Banach space and $x, y \in X$. The following assumptions are equivalent:*

(a) $\|x\| \leq \|x + \lambda y\| \; \forall \lambda > 0$.
(b) *There exists $x^* \in J(x)$ such that $\langle x^*, y\rangle \geq 0$.*
(c) *The function $t \to \|x + ty\|$ is increasing.*

The equivalence between the two first assertions is due to T. Kato [139, p. 509].

Proof. (b) $\Longrightarrow$ (a) Suppose that for $x, y \in X$ there exists $x^* \in J(x)$ with $\langle x^*, y\rangle \geq 0$. Then for $\lambda > 0$ we can write

$$\|x\|^2 = \langle x^*, x\rangle \leq \langle x^*, x\rangle + \lambda\langle x^*, y\rangle = \langle x^*, x + \lambda y\rangle \leq \|x^*\|\|x + \lambda y\|.$$

Therefore

$$\|x\| \leq \|x + \lambda y\| \quad \forall \lambda > 0.$$

(a) $\Longrightarrow$ (b) Let $\lambda > 0$. Take $x_\lambda^* \in J(x + \lambda y)$ and set $y_\lambda^* := \frac{x_\lambda^*}{\|x_\lambda^*\|}$. Then, we have

$$\begin{aligned}\|x\| \leq \|x + \lambda y\| &\leq \frac{1}{\|x + \lambda y\|}\langle x_\lambda^*, x + \lambda y\rangle = \langle y_\lambda^*, x + \lambda y\rangle \\ &= \langle y_\lambda^*, x\rangle + \lambda\langle y_\lambda^*, y\rangle \\ &\leq \|x\| + \lambda\langle y_\lambda^*, y\rangle.\end{aligned}$$

From the last expression we derive that, for all $\lambda > 0$, $\langle y_\lambda^*, y\rangle \geq 0$.

Let us see that from the net $(y_\lambda^*)_{\lambda>0}$ we may obtain $x^* \in J(x)$ satisfying (b). Indeed, since the unit closed ball of X^* is a weak* compact and $(y_\lambda^*)_{\lambda>0} \subset B_{X^*}$, there exists $y^* \in \overline{(y_\lambda^*)}^{w^*} \subset B_{X^*}$. On the one hand, $\|y^*\| \leq 1$ and, moreover, $\|x\| \leq \langle y_\lambda^*, x\rangle + \lambda\langle y_\lambda^*, y\rangle$, thus

$$\|x\| \leq \liminf_{\lambda \to 0^+}\langle y_\lambda^*, x\rangle + 0 \leq \langle y^*, x\rangle.$$

Using the fact that $\langle y_\lambda^*, y\rangle \geq 0$ for all $\lambda > 0$, we get

$$0 \le \liminf_{\lambda \to 0^+} \langle y_\lambda^*, y \rangle \le \langle y^*, y \rangle.$$

This means

$$\|x\| \le \langle y^*, x \rangle \le \|y^*\| \|x\| \le \|x\|,$$

i. e.,

$$\langle y^*, x \rangle = \|x\|,$$

hence $\|y^*\| = 1$. Take $x^* := y^* \|x\| \in J(x)$. Then

$$\langle x^*, y \rangle = \|x\| \langle y^*, y \rangle \ge 0.$$

(c) $\Longrightarrow$ (a) By (c), we have that $\|x + 0y\| \le \|x + \lambda y\|\ \forall \lambda \ge 0$.

(a) $\Longrightarrow$ (c) By contradiction, suppose that there exist $0 \le t_1 < t_2 \le 1$ such that $\|x + t_2 y\| < \|x + t_1 y\|$. Then, writing $u := x + t_1 y$, we obtain $x + t_2 y = u + hy$ for some $h > 0$. Hence $\|u\| > \|u + hy\|$, i. e., condition (a) is not fulfilled for u, y. If we use the equivalence between (a) and (b), we can guarantee that for every $j \in J(u)$ we have $\langle j, y \rangle < 0$. Consequently,

$$\|x + t_1 y\|^2 = \langle j, x + t_1 y \rangle = \langle j, x \rangle + t_1 \langle j, y \rangle < \langle j, x \rangle \le \|x + t_1 y\| \|x\|,$$

that is,

$$\|x + t_1 y\| < \|x\|,$$

which contradicts (a). □

Definition 1.10.1. Let X be a Banach space, we define the map $\langle \cdot, \cdot \rangle_+ : X \times X \to \mathbb{R}$ by

$$\langle y, x \rangle_+ = \sup\{\langle y, j \rangle : j \in J(x)\}.$$

Since $J(x)$ is a weak* compact subset, given $(x, y) \in X \times X$, we have $\langle y, x \rangle_+ < \infty$; moreover, such a supremum is achieved.

Proposition 1.10.1. *Let $x, y \in X$. Then*
(a) $\langle \alpha y, \beta x \rangle_+ = \alpha\beta \langle y, x \rangle_+$ *whenever* $\alpha, \beta \ge 0$.
(b) $\langle \alpha x + y, x \rangle_+ = \alpha \|x\|^2 + \langle y, x \rangle_+$ *whenever* $\alpha \in \mathbb{R}$.
(c) $\langle \cdot, \cdot \rangle_+ : X \times X \to \mathbb{R}$ *is upper semicontinuous.*

Proof. (a) It is not hard to see that if $x \in X$ and $\beta \ge 0$ then $J(\beta x) = \beta J(x)$. Hence

$$\langle \alpha y, \beta x \rangle_+ = \sup\{\langle \alpha y, \beta j \rangle : j \in J(x)\},$$

consequently,

$$\langle ay,\beta x\rangle_+ = \alpha \sup\{\beta\langle y,j\rangle : j \in J(x)\} = \alpha\beta\langle y,x\rangle_+.$$

(b) By definition, $\langle ax+y,x\rangle_+ = \sup\{\langle ax+y,j\rangle : j \in J(x)\}$. Therefore

$$\langle ax+y,x\rangle_+ = \sup\{a\|x\|^2 + \langle y,j\rangle \quad j \in J(x)\},$$

now the properties of the supremum yield

$$\langle ax+y,x\rangle_+ = a\|x\|^2 + \langle y,x\rangle_+.$$

(c) We have to show that the function $f : X \times X \to \mathbb{R}$ defined by

$$f(x,y) := \langle y,x\rangle_+$$

is upper semicontinuous. Since Jx is weak* compact, there exists $x_0^* \in Jx$ such that $f(x,y) = \langle x_0^*,y\rangle$.

Considering sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $y_n \to y$, it will be enough to see that $\sup_{n\in\mathbb{N}} f(x_n,y_n) \le f(x,y)$. Since $|f(x,y)| \le \|x\|\|y\|$, it is clear that $\sup_{n\in\mathbb{N}} f(x_n,y_n) < +\infty$ and therefore

$$\exists x_n^* \in J(x_n) : f(x_n,y_n) = \left\langle x_n^*,y_n\right\rangle \quad \forall n \in \mathbb{N}.$$

Take $x^* = w^* - \lim_{n\to+\infty} x_n^*$. Then

$$\left\langle x^*,x\right\rangle = \lim_{n\to+\infty} \left\langle x^*,x_n\right\rangle = \lim_{n\to+\infty} \|x_n\|^2 = \|x\|^2.$$

Now, by the weak* lower semicontinuity of the norm in X^*, we obtain $\|x^*\| \le \liminf_n \|x_n^*\| = \|x\|$. That is, $x^* \in J(x)$. Therefore,

$$\lim_{n\to\infty} f(x_n,y_n) = \lim_{n\to\infty} \left\langle x_n^*,y_n\right\rangle = \left\langle x^*,y\right\rangle \le f(x,y).$$

In conclusion, f is upper semicontinuous. □

We note that the properties of the normalized duality map are related to the subdifferentiability of the norm. Indeed, let X be a Banach space X. We define

$$[x,y]_s = \lim_{t\to0^+} \frac{\|x+ty\| - \|x\|}{t}, \quad x,y \in X.$$

The number $[x,y]_s$ is the right-hand side derivative of the norm at x. We now list some useful properties of the bracket $[\cdot,\cdot]_s$.

Proposition 1.10.2. *Let X be a Banach space. We have the following:*
(a) $[\cdot,\cdot]_s : X \times X \to \mathbb{R}$ *is upper semicontinuous.*
(b) $[ax,\beta y]_s = \beta[x,y]_s$, *for all* $\beta > 0$, $a \in \mathbb{R}$, $x,y \in X$.

(c) $[x, ax + y]_s = a\|x\| + [x, y]_s$ *if* $a \in \mathbb{R}^+$, $x \in X$.
(d) $[x, y]_s \leq \|y\|$, $[x, y + z]_s \leq [x, y]_s + [x, z]_s$, *for all* $x, y, z \in X$.
(e) $[x, y]_s = \max\{\langle x^*, y\rangle; x^* \in F(x)\}$.

In the next proposition we present a connection between $\langle \cdot, \cdot \rangle_+$ and $[\cdot, \cdot]_s$.

Proposition 1.10.3. *Let X be a Banach space. For every $x, y \in X$, we have*

$$\langle y, x\rangle_+ = \|x\|[x, y]_s \quad \textit{and the supremum is attained.}$$

Proof. If $x = 0$, then the equality is obvious. So let $x \neq 0$. To derive the equality, we use the continuity of the norm and the fact that $J(x) = \|x\|F(x)$. Recalling that J is a weak*-compact-valued mapping, we conclude that the supremum is attained. ☐

1.11 Convex function and its subdifferentials

1.11.1 Convex functions

Let X be a vector space, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex, if $\forall x, y \in X$, $\forall \lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

It is called concave, if $-f$ is convex. Also f is called affine if f is both convex and concave. The domain, epigraph, and level set of f are defined as follows:

$$\begin{aligned} \operatorname{dom}(f) &= \{x \in X \text{ such that } f(x) < +\infty\}, \\ \operatorname{epi}(f) &= \{(x, \lambda) \in X \times \mathbb{R} \text{ such that } f(x) \leq \lambda\}, \\ f_a &= \{x \in X \text{ such that } f(x) < a\} \quad \forall a \in \mathbb{R}. \end{aligned}$$

The function f is said to be proper if $f \not\equiv +\infty$. By definition,

$$\begin{aligned} f \text{ is convex} \iff & \forall x_1, \ldots, x_n, \quad \forall \lambda_1, \ldots, \lambda_n \geq 0, \quad \sum_{i=1}^n \lambda_i = 1, \quad f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i) \\ \iff & \operatorname{epi}(f) \text{ is convex.} \end{aligned}$$

Also f being convex implies that for all $a \in \mathbb{R}$, f_a is convex, but the converse is not true. A function f is called quasiconvex (quasiconcave) if f_a is convex (concave resp.) $\forall a \in \mathbb{R}$.

The following simple propositions hold:

(1) If f and g are convex, then for all $\alpha, \beta \geq 0$, $\alpha f + \beta g$ is convex.
(2) If $A : X \to Y$ is a linear mapping and if $f : Y \to \mathbb{R} \cup \{+\infty\}$ is convex, then $f \circ A : X \to \mathbb{R} \cup \{+\infty\}$ is convex.
(3) If $\{f_i : i \in I\}$ is a set of convex functions, then $\sup\{f_i : i \in I\}$ is convex.

1.11.2 Lower semicontinuous functions

Definition 1.11.1. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semicontinuous (l. s. c.), if $\forall \lambda \in \mathbb{R}$, the level set $f_\lambda = \{x \in X : f(x) \leq \lambda\}$ is closed. It is called sequentially lower semicontinuous (s. l. s. c.), if for any sequence $(x_n)_{n \in \mathbb{R}}$, with $x_n \to x \in X$, we have

$$\liminf_{n\to\infty} f(x_n) \geq f(x).$$

From the definition, it follows directly that

(1) f is lower semicontinuous $\iff$ epi(f) is closed in $\mathbb{R} \cup \{+\infty\}$.

(2) If $\{f_i : i \in I\}$ is a family of lower semicontinuous functions, then

$$f(x) = \sup\{f(x_i) : i \in I\}$$

is lower semicontinuous.

(3) If f, g are lower semicontinuous and $\lambda, \mu \geq 0$, then $\lambda f + \mu g$ is lower semicontinuous.

1.11.3 Subdifferentials

Convex functions on Banach spaces are not differentiable, in general. However, the notion of subdifferentials is introduced as a replacement of derivatives of differentiable functions.

Definition 1.11.2. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function on a vector space X. For all x_0 in dom(f), $x^* \in X^*$ is called a subgradient of f at x_0 if

$$\langle x^*, x - x_0 \rangle + f(x_0) \leq f(x) \quad \forall x \in X.$$

The set of all subgradients at x_0 is called the subdifferential of f at x_0, and is denoted by $\partial f(x_0)$.

Geometrically, $x^* \in \partial f(x_0)$ if and only if the hyperplane

$$y = \langle x^*, x - x_0 \rangle + f(x_0)$$

lies below the epigraph of f, i. e., it is a support of epi(f).

Obviously, $\partial f(x_0)$ may contain more than one point. If X is a Banach space, the following propositions hold:

(1) $\partial f(x_0)$ is a weak* closed convex set.

(2) If $x_0 \in$ int(dom(f)), then $\partial f(x_0) \neq \emptyset$. To see this, we apply Hahn–Banach separation theorem to the convex set epi(f): $\exists (x^*, \lambda) \in X^* \times \mathbb{R} \setminus \{(0,0)\}$ such that

$$\langle x^*, x_0 \rangle + \lambda f(x_0) \geq \langle x^*, x \rangle + \lambda t \quad \forall (x, t) \in \text{epi}(f).$$

Since $(x_0, f(x_0)+1) \in \operatorname{epi}(f)$, it follows that $\lambda \leq 0$. However, $\lambda \neq 0$. Otherwise, we would have $\langle x^*, x - x_0\rangle \leq 0\ \forall x \in \operatorname{dom}(f)$. From $x_0 \in \operatorname{int}(\operatorname{dom}(f))$, we conclude that $x^* = 0$. It contradicts the fact that $(x^*, \lambda) \neq (0,0)$. Setting $x_0^* = \frac{1}{-\lambda}x^*$, we obtain $x_0^* \in \partial f(x_0)$.

(3) $\forall \lambda \geq 0, \partial(\lambda f)(x_0) = \lambda \partial f(x_0)$.

(4) If $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are convex, then $\forall x_0 \in \operatorname{int}(\operatorname{dom}(f) \cap \operatorname{dom}(g))$ we have

$$\partial(f+g)(x_0) = \partial(f)(x_0) + \partial(g)(x_0).$$

Example 1.11.1 (Normalized duality map). Let X be a real Banach space, and let $f(x) = \frac{1}{2}\|x\|^2$. Then

$$\partial f(x) = J(x) := \{x^* \in X^* : \|x^*\| = \|x\|,\ \langle x^*, x\rangle = \|x\|^2\}.$$

It is called the normalized duality map (see Section 1.10). To see this we first prove the inclusion “$\subset$”. Indeed, for all $x^* \in \partial f(x)$, we have

$$\langle x^*, y - x\rangle \leq \frac{1}{2}(\|y\|^2 - \|x\|^2),$$

and then $\forall h \in X$,

$$\langle x^*, h\rangle \leq \frac{1}{2t}(\|x+th\|^2 - \|x\|^2) \leq \|x\|\|h\| + \frac{t}{2}\|h\|^2,$$

as $t > 0$. It follows that $\langle x^*, h\rangle \leq \|x\|\|h\|$, and then $\|x^*\| \leq \|x\|$. On the other hand, setting $y = \lambda x$, we have $(\lambda - 1)\langle x^*, x\rangle \leq \frac{1}{2}(\lambda^2 - 1)\|x\|^2$. Dividing by $(\lambda - 1)$ as $\lambda \in (0,1)$, and then letting $\lambda + 1$, we obtain $\langle x^*, x \geq \|x\|^2$; it follows that $\|x\| \leq \|x^*\|$. Thus $\|x\| = \|x^*\|$, and $\langle x^*, x\rangle = \|x\|^2$.

We now prove the inclusion “$\supset$”. For all $x^* \in F(x)$, one has

$$\begin{aligned}\langle x^*, y - x\rangle &= \langle x^*, y\rangle - \|x\|^2\\ &\leq \|x^*\|\|x\| - \|x\|^2\\ &\leq \frac{1}{2}(\|y\|^2 - \|x\|^2)\\ &= f(y) - f(x)\end{aligned}$$

for all $y \in X$, i. e., $x \in \partial f(x)$. □

1.12 Retracts of normed spaces

Definition 1.12.1. Let X be a normed space, K a subset of X, and $R : X \to K$ a continuous map. The map R is called a *retraction* if and only if $R(x) = x$ for all $x \in K$. In that case, the set K is called a retract of X.

Hence a retraction transforms the space X into K continuously, while leaving each point of K unmoved.

Let $\zeta > 0$. We call the radial retraction of a normed space X into $\mathbb{B}_\zeta$ the map $R_\zeta(\cdot)$ from X into $\mathbb{B}_\zeta$ defined by

$$R_\zeta(x) = \begin{cases} x & \text{if } \|x\| \le \zeta, \\ \zeta \frac{x}{\|x\|} & \text{if } \|x\| > \zeta. \end{cases}$$

We also recall the following useful estimate (see [81, p. 364] or [32, Chapter 4]):

$$\|R_\zeta(z_1) - R_\zeta(z_2)\| \le 2\|z_1 - z_2\| \quad \text{for all } z_1, z_2 \in Z. \tag{1.10}$$

As it is shown in the next result, the interest in retractions lies in their ability to reduce fixed point problems for complicated sets to fixed point problems for simpler subsets.

Theorem 1.12.1. *Every closed convex subset of a normed space X is a retract of X.*

The proof of Theorem 1.12.1 relies on the following extension result due to J. Dugundji (1951) (see also [244, Proposition 2.1, p. 49]).

Proposition 1.12.1. *Let $f : K \subset X \to Y$ be a continuous map on a closed, nonempty subset of a metric space (X, d) into a normed space Y. Then f has a continuous extension $\widetilde{f} : X \to \operatorname{co}(f(K))$.*

Proof of Theorem 1.12.1. Let $I : K \to K$ be the identity map. Because it is continuous, it follows from Proposition 1.12.1 that I can be extended continuously to the map $R : X \to \operatorname{co}(K)$. Now using the fact that $\operatorname{co}(K) = K$ completes the proof. □

1.13 Banach algebras

Let X be a vector space over a scalar field $\mathbb{K}$. We say that X is an algebra if it has a multiplication (or product) operation from $X \times X$ into X

$$(x, y) \mapsto xy \quad \forall x, y \in X,$$

which satisfies, for all $x, y, z \in X$ and every $\lambda \in \mathbb{K}$, the following axioms:

(a) $x(yz) = (xy)z$,

(b) $\lambda(xy) = (\lambda x)y = x(\lambda y)$,

(c) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.

An algebra is an algebraic structure that is both a ring and a vector space, where the addition of the ring is the same as the vector addition and multiplication by scalars relates to the ring multiplication by axiom (b) given above.

An algebra X is commutative if its ring multiplication is commutative, that is,

$$xy = yx \quad \text{for all } x, y \in X.$$

An algebra X has an identity element, say e or (e_X), provided that e satisfies $ex = xe = x$ for every x in X. It is evident that an identity element is unique whenever it exists. An algebra with an identity is called a unital algebra.

Let X and Y be two algebras. A map $\theta : X \to Y$ is a homomorphism if θ is a linear map such that

$$\theta(xy) = \theta(x)\theta(y) \quad \forall x, y \in X.$$

A homomorphism $\theta : X \to Y$ is an embedding or a monomorphism if it is injective, and then we often regard X as a subalgebra of Y. A bijective homomorphism is an isomorphism, and an isomorphism from X to itself is an automorphism. Two algebras X and Y are isomorphic, written $X \sim Y$, if there is an isomorphism from X onto Y. In the case where X and Y have identities e_X and e_Y, respectively, θ is a unital homomorphism whenever $\theta(e_X) = e_Y$.

Definition 1.13.1. Let X be an algebra over $\mathbb{K}$. By an algebra-norm on X we mean a mapping $x \mapsto \|x\|$ of X into $\mathbb{R}^+$ such that
(a) $(X, \|\cdot\|)$ is a normed space over $\mathbb{K}$,
(b) $\|xy\| \leq \|x\|\|y\|\ \forall x, y \in X$.

The normed algebra $(X, \|\cdot\|)$ is a Banach algebra if $(X, \|\cdot\|)$, as normed space, is a Banach space.

If $(X, \|\cdot\|)$ is a unital Banach algebra with identity e, then $\|e\| = 1$.

We shall now give some facts in Banach algebras involving the weak topology.

Proposition 1.13.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Banach algebra X such that $x_n \rightharpoonup x$, then $ax_n \rightharpoonup ax$, and $x_n a \rightharpoonup xa$, for any fixed $a \in X$.*

Proof. Let a be an arbitrary point of X. Consider the left-hand side multiplication operator $l_a(x) = ax$ and the right-hand side multiplication operator $r_a(x) = xa$. Clearly, l_a and r_a are continuous linear operators. Now the use of Theorem 1.7.1 ends the proof. □

Proposition 1.13.2. *Let X be a Banach algebra and let A, B be subsets of X.*
(a) *If A and B are compact, then AB is also compact.*
(b) *If A is weakly compact and B is compact, then AB is weakly compact.*

Proof. Let ϕ be the multiplication map from $X \times X$ into X. It is clearly continuous. Since B is compact, we deduce that $AB = \phi(A \times B)$ is compact.

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of A and let $(b_n)_{n\in\mathbb{N}}$ be a sequence of B. Keeping in mind the compactness of A and the weak compactness of B, and by extracting a subsequence if necessary, we may assume that $(a_n)_{n\in\mathbb{N}}$ converges strongly to some $a \in A$ and $(b_n)_{n\in\mathbb{N}}$ converges weakly to some $b \in B$. According to Proposition 1.13.1, we conclude that

$$a(b_n - b) \rightharpoonup 0. \tag{1.11}$$

Using the fact that weakly convergent sequences are norm bounded, we get

$$\|(a_n - a)b_n\| \leq \|a_n - a\|\|b_n\| \to 0. \tag{1.12}$$

Next, using the equality $a_nb_n - ab = (a_n - a)b_n + a(b_n - b)$, together with (1.11) and (1.12), we infer that $a_nb_n \rightharpoonup ab$. ☐

We shall now introduce the concept of WC-Banach algebras.

Definition 1.13.2. Let X be a Banach algebra. We say that X is a WC-Banach algebra if the product of any two weakly compact subsets of X is weakly compact.

Definition 1.13.3. Let X be a Banach algebra. We say that X satisfies property $(\mathcal{P})$ if, for any sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X such $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, for some $x, y \in X$, then $x_ny_n \rightharpoonup xy$.

Proposition 1.13.3. *If X is a Banach algebra satisfying condition $(\mathcal{P})$, then X is a WC-Banach algebra.*

Proof. Let K, K' be arbitrarily weakly compact subsets of X. Let $(z_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of KK'. By construction of $(z_n)_{n\in\mathbb{N}}$, for each fixed $n \in \mathbb{N}$, one can find $x_n \in K$ and $y_n \in K'$ such that $z_n = x_ny_n$. Consider the sequences $(x_n)_{n\in\mathbb{N}} \subset K$ $(y_n)_{n\in\mathbb{N}} \subset K'$. Since K is weakly compact, we can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of the sequence $(x_n)_{n\in\mathbb{N}}$ which is weakly convergent to some $x \in K$. Using again the weak compactness of K', one can extract a subsequence of $(y_{n_{k_j}})_{j\in\mathbb{N}}$ of $(y_{n_k})_{k\in\mathbb{N}}$ which is weakly convergent to some $y \in K'$. It is clear that $(x_{n_{k_j}})_{j\in\mathbb{N}}$ converges weakly to x. Since X satisfies condition $(\mathcal{P})$, we infer that $z_{n_{k_j}} = x_{n_{k_j}}y_{n_{k_j}} \rightharpoonup xy$, which ends the proof. ☐

1.14 Bochner integral

In this section we will try to obtain from a direct way the concept of Lebesgue integral for functions taking values in a Banach space (see, for example, [79]).

1.14.1 Measurable functions

We will introduce the notions of a measurable function and weakly measurable function in order to establish the concept of an integral. We start with a measure space with a finite measure (Ω, Σ, μ).

Definition 1.14.1. Let X be a Banach space. A function $f : \Omega \to X$ is said to be *simple or step function* if there exist $x_{1,}x_2, \dots, x_n \in X$ and $E_1, E_2, \dots, E_n \in \Sigma$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$, where $\chi_{E_i}(w) = 1$ if $w \in E_i$ and $\chi_{E_i}(w) = 0$ if $w \notin E_i$. A function $f : \Omega \to X$ is said to be μ-measurable if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} \|f_n - f\| = 0$ μ-a. e. A function $f : \Omega \to X$ is said to be *weakly μ-measurable* if for each $x^* \in X^*$ the function $\langle x^*, f\rangle$ is μ-measurable.

Theorem 1.14.1 (Egoroff). *Let $f_n : \Omega \to X$ be a μ-measurable function for every $n \in \mathbb{N}$. Suppose that $f_n \to f$ pointwise. Then given $\delta > 0$ there exists $E \in \Sigma$ with $\mu(E) < \delta$ such that $f_n \to f$ uniformly in* $\Omega\backslash E$.

We are going to establish a result about measurable functions which will give us tools sufficient to show the main properties of Bochner integral.

Theorem 1.14.2 (Pettis). *A function $f : \Omega \to X$ is μ-measurable if and only if*

(i) *there exists $E \in \Sigma$ with $\mu(E) = 0$ such that $f(\Omega\backslash E)$ is a (norm)-separable subset of X, and*

(ii) *f is weakly μ-measurable.*

Proof. Let $f : \Omega \to X$ be μ-measurable. By Egoroff's theorem, there is a sequence of simple functions $(f_k)_{k\in\mathbb{N}}$ such that for each positive integer $n \in \mathbb{N}$, there exists a set $E_n \in \Sigma$ with $\mu(E_n) < 1/n$ and $f_k \to f$ uniformly in $\Omega\backslash E_n$. Thus, $f_k \to f$ μ-a. e. uniformly.

We claim that $f(\Omega\backslash E_n)$ is precompact and then separable. Indeed, letting $\varepsilon > 0$, there is $p \in \mathbb{N}$ such that $\|f_p(x) - f(x)\| < \varepsilon/2$ for every $x \in \Omega\backslash E_n$. On the other hand, $f_p(\Omega\backslash E_n)$ is bounded, and since f_p has a finite range, we have that $f_p(\Omega\backslash E_n)$ is precompact (in X). This means that there exist $x_1, x_2, \dots, x_m \in X$ such that $f_p(\Omega\backslash E_n) \subseteq \bigcup_{i=1}^m B(x_i, \varepsilon/2)$. Thus $f(\Omega\backslash E_n) \subseteq \bigcup_{i=1}^m B(x_i, \varepsilon)$. Accordingly, $f(\bigcup_{i\geq 1}(\Omega\backslash E_n)) = \bigcup_{i\geq 1} f(\Omega\backslash E_n)$ is separable. Still more, $\Omega\backslash \bigcup_{i\geq 1}(\Omega\backslash E_n) = \bigcap_{i\geq 1} E_n$, which is null since $\mu(E_n) < 1/n$ for each n. This shows (i).

In order to see (ii), notice that, since $f_n(w) \to f(w)$ for almost every $w \in \Omega$, we have that, given $x^* \in X^*$, $x^*(f_n(w)) \to x^*(f(w))$ for almost every $w \in \Omega$. Moreover, if every f_n is simple, $x^*(f_n)$ is also simple. Therefore, $x^*(f)$ is measurable for every $x^* \in X^*$.

To obtain the converse, let $E \in \Sigma$ be such that $\mu(E) = 0$ and $f(\Omega\backslash E)$ is separable. Let $\{x_n\}$ be a dense countable subset of $f(\Omega\backslash E)$. By Hahn–Banach theorem, there exists a sequence $(x_n^*)_{n\in\mathbb{N}}$ in X^* with $x_n^*(x_n) = \|x_n\|$ and $\|x_n^*\| = 1$.

We claim that $\|f(w)\| = \sup_n |x_n(f(w))|$ for each $w \in \Omega\backslash E$. We only have to prove that $\|f(w)\| \leqslant \sup_n |x_n(f(w))|$. Letting $\varepsilon > 0$ and $w \in \Omega\backslash E$, there exists a positive integer n such that

$$\big|\|f(w)\| - \|x_n\|\big| \leqslant \|f(w) - x_n\| \leqslant \varepsilon,$$

hence

$$\big|x_n^*(f(w)) - x_n^*(x_n)\big| = \big|x_n^*(f(w)) - \|x_n\|\big| \leqslant \varepsilon$$

and, by the triangle inequality, we have

$$\big|\|f(w)\| - x_n^*(f(w))\big| \leqslant \big|\|f(w)\| - \|x_n\|\big| + \big|x_n^*(f(w)) - \|x_n\|\big| \leqslant 2\varepsilon,$$

as claimed.

The above argument says that the function $\|f(\cdot)\|$ is μ-measurable. Using the same argument, one sees that the functions g_n defined by $g_n(\cdot) = \|f(\cdot) - x_n\|$ are μ-measurable. Let $\varepsilon > 0$. We write $E_n = \{w \in \Omega : g_n(w) < \varepsilon\}$. We may assume that μ is a complete measure, hence $E_n \in \Sigma$. Define $g : \Omega \to X$ by

$$g(w) = \begin{cases} x_n & \text{if } w \in E_n \setminus \bigcup_{m<n} E_m, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, g is μ-measurable (it is the limit of simple functions) and also $\|g - f\| < \varepsilon$ μ-a. e. Indeed, $\|f(w) - x_n\| < \varepsilon$ if $w \in E_n$. Fix $w \in \bigcup_{n=1}^{+\infty} E_n$. Then there is an n such that $w \in E_n \setminus \bigcup_{m<n} E_m$ and therefore $g(w) = x_n$, thus $\|g(w) - f(w)\| = \|x_n - f(w)\| < \varepsilon$. We have to see that $\Omega \setminus \bigcup_{n=1}^{+\infty} E_n$ is μ-null. If $w \in \Omega \setminus E$, there exists an n such that $\|x_n - f(w)\| < \varepsilon$, hence $w \in E_n$. That is, $\Omega \setminus E \subseteq \bigcup_{n=1}^{+\infty} E_n$, and thus $\Omega \setminus \bigcup_{n=1}^{+\infty} E_n \subseteq E$ is μ-null.

Taking $\varepsilon = 1/n$, we generate a sequence $(g_n')_{n \in N}$ of *countably-valued* functions such that $\|g_n' - f\| < 1/n$ μ-a. e. for each n. Since every g_n' has the form $g_n' = \sum_{m=1}^{+\infty} x_{n,m} \chi_{E_{n,m}}$ with $E_{n,i} \cap E_{n,j} = \emptyset$ for $i \neq j$, and $E_{n,m} \in \Sigma$, defining

$$f_{n,m} = \sum_{k=1}^{m} x_{n,k} \chi_{E_{n,k}}$$

and reordering, we obtain a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions converging to f μ-a. e. ☐

Now, as a consequence of the above theorem, we may give a result which will be essential for the Bochner integral.

Corollary 1.14.1. *A function $f : \Omega \to X$ is μ-measurable if and only if f is a μ-a. e. uniform limit of a sequence of μ-measurable countably-valued functions.*

1.14.2 Bochner integrable functions

Definition 1.14.2. Let X be a Banach space. A μ-measurable function $f : \Omega \to X$ is said to be *Bochner integrable* if there exists a sequence of simple functions $(f_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to +\infty} \int_{\Omega} \|f_n - f\| \, d\mu = 0.$$

If f is Bochner integrable, then, for each $E \in \Sigma$, $\int_E f \, d\mu$ is defined by

$$\int_E f\,d\mu = \lim_{n\to+\infty} \int_E f_n\,d\mu,$$

where $\int_E f_n\,d\mu$ is defined in the usual way. That is, if $g = \sum_{i=1}^m x_i\chi_{E_i}$ is a simple function with $E_i \cap E_j = \emptyset$ for $i \neq j$, it is defined by $\int_E g\,d\mu = \sum_{i=1}^m x_i\mu(E\cap E_i)$.

For simple functions, if μ is a positive measure then

$$\left\|\int_E g\,d\mu\right\| = \left\|\sum_{i=1}^m x_i\mu(E\cap E_i)\right\| \leqslant \sum_{i=1}^m \|x_i\|\mu(E\cap E_i) = \int_E \|g\|\,d\mu.$$

From this we obtain that the last limit in Definition 1.14.2 exists and it does not depend on the choice of the sequence of simple functions $(f_n)_{n\in\mathbb{N}}$.

We are going to give a characterization of Bochner integrable functions.

Theorem 1.14.3. *A function μ-measurable $f : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f\|\,d\mu < +\infty$.*

Theorem 1.14.4 (Lebesgue dominated convergence theorem). *Let X be a Banach space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Bochner integrable functions from Ω into X. If*

$$\lim_n \mu\{w \in \Omega : \|f_n - f\| \geqslant \varepsilon\} = 0$$

for every $\varepsilon > 0$ and if there exists a function $g : \Omega \to R$, which is Lebesgue integrable and such that $\|f_n\| \leqslant g$, μ-a. e., then f is Bochner integrable and

$$\int_E f\,d\mu = \lim_{n\to+\infty} \int_E f_n\,d\mu$$

for every $E \in \Sigma$. In fact,

$$\lim_{n\to+\infty} \int_\Omega \|f_n - f\|\,d\mu = 0.$$

Proof. We only have to apply Lebesgue dominated convergence theorem for real functions to $\|f_n - f\|$ which is dominated by $2g$. □

Theorem 1.14.5. *If f is a Bochner integrable function with respect to the measure μ then*

(i) $\lim_{\mu(E)\to 0} \int_E f\,d\mu = 0$;

(ii) $\|\int_E f\,d\mu\| \leqslant \int_E \|f\|\,d\mu$, *for every $E \in \Sigma$;*

(iii) *if $(E_n)_{n\in\mathbb{N}}$ is a sequence in Σ such that $E_n \cap E_m = \emptyset$ whenever $n \neq m$ and $E = \bigcup_{n=1}^{+\infty} E_n$, then*

$$\int_E f\,d\mu = \sum_{n=1}^{+\infty} \int_{E_n} f\,d\mu,$$

where the series is absolutely convergent;

(iv) *if* $F(E) = \int_E f\,d\mu$, *then* F *is of bounded variation and*

$$|F|(E) = \int_E \|f\|\,d\mu \quad \textit{for each } E \in \Sigma.$$

Corollary 1.14.2. *If* f *and* g *are two Bochner integrable functions and* $\int_E f\,d\mu = \int_E g\,d\mu$ *for each* $E \in \Sigma$, *then* $f = g$ μ*-a. e.*

Proof. Let $F(E) = \int_E (f-g)\,d\mu$. Thus $F(E) = 0$ for every $E \in \Sigma$. Hence $|F|(E) = 0$, for each $E \in \Sigma$. In this case, $0 = |F|(E) = \int_E \|f-g\|\,d\mu$, and then $\|f-g\| = 0$ μ-a. e. This means that $f = g$ μ-a. e. □

1.14.3 Komura theorem

Definition 1.14.3. Let I be an interval of the real line and let X be a Banach space. A function $u : I \to X$ is said to be strongly differentiable at t_0 belonging to the interior of I if

$$\lim_{t\to 0} \frac{u(t_0+t) - u(t_0)}{t}$$

exists and it is an element of X. The above limit is denoted by $u'(t_0)$ and it is called (strong) derivative of u at t_0.

Definition 1.14.4. Let $u : [a,b] \to X$. The total variation of u on $[a,b]$ is defined by the expression

$$V(u,[a,b]) := \sup\left\{ \sum_{k=1}^{n} \|u(a_k) - u(a_{k-1})\| : a = a_0 < \cdots < a_n = b \right\}.$$

The function u is said to be of bounded variation in $[a,b]$ if $V(u,[a,b]) < +\infty$. The family of bounded variation functions in $[a,b]$ will be denoted by $BV([a,b])$.

Definition 1.14.5. Let $u : I \to X$. The function u is said to be strongly absolutely continuous if for every compact subinterval $[a,b] \subseteq I$ and for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\sum_{i=1}^{n} \|u(t_i) - u(t_{i-1})\| < \epsilon$$

whenever $]t_{i-1}, t_i[$ are subintervals of $[a,b]$ with $a \le t_1 < t_2 < \cdots < t_n \le b$ satisfying

$$\sum_{i=1}^{n}(t_i - t_{i-1}) < \delta.$$

Definition 1.14.6. A function $u : I \to X$ is said to be locally Lipschitz if for every compact subinterval $K \subseteq I$ there exists a constant $C_K > 0$ such that

$$\|u(t) - u(s)\| \leq C_K |t - s|$$

whenever $t, s \in K$.

Clearly, every locally Lipschitz function is strongly absolutely continuous and, if u is strongly absolutely continuous, by using the same proof as that in the scalar case, it is easy to see that it is of bounded variation.

Theorem 1.14.6 (Komura). *Let X be a reflexive Banach space and let $f : [0, t_0] \to X$ be a strongly absolutely continuous function. Then f is differentiable a. e. in $[0, t_0]$, f' is Bochner integrable in $[0, t_0]$, and*

$$f(t) = f(0) + \int_0^t f'(s)\, ds, \quad t \in [0, t_0].$$

Proof. Since f is strongly absolutely continuous, f is continuous and then

$$K := \{f(t) : t \in [0, t_0]\}$$

is a compact subset of X, which implies that K is a separable subset. Passing to the closed convex hull of K, without lost of generality, we may assume that X is separable. Since X is reflexive and separable, it is clear that X^* is also separable. On the other hand, since f is strongly absolutely continuous, f is of bounded variation. The function

$$v : [0, t_0] \to \mathbb{R}, \quad t \mapsto v(t) := V(f, [0, t])$$

is increasing, and then v is differentiable a. e. Moreover, we have

$$\|f(t) - f(s)\| \leq v(t) - v(s), \quad 0 \leq t \leq s \leq t_0. \tag{1.13}$$

Denote

$$f_h(t) := \frac{f(t+h) - f(t)}{h}, \quad \text{with } h \neq 0.$$

By (1.13), we have

$$\|f(t+h) - f(t)\| \leq v(t+h) - v(t).$$

Thus if $h \in \,]0, t_0[$ then

$$\begin{aligned}\int_0^{t_0-h}\|f(t+h)-f(t)\|\,dt &\le \int_0^{t_0-h}(v(t+h)-v(t))\,dt = \int_h^{t_0} v(t)\,dt - \int_0^{t_0-h} v(t)\,dt\\ &= \int_h^{t_0-h} v(t)\,dt + \int_{t_0-h}^{t_0} v(t)\,dt - \left(\int_0^h v(t)\,dt + \int_h^{t_0-h} v(t)\,dt\right)\\ &= \int_{t_0-h}^{t_0} v(t)\,dt - \int_0^h v(t)\,dt\\ &\le \int_{t_0-h}^{t_0} v(t)\,dt \le hV(f,[0,t_0]).\end{aligned}$$

This implies

$$\int_0^{t_0-h}\|f_h(t)\|\,dt \le V(f,[0,t_0]), \quad 0<h<t_0,$$

and consequently,

$$\limsup_{h\to 0^+}\|f_h(t)\| < +\infty \quad \text{a. e. in } [0,t_0].$$

Denote

$$S_0 = \Big\{t\in[0,t_0] : \limsup_{h\to 0^+}\|f_h(t)\| = +\infty\Big\}.$$

Let $\{x_n^*, n\in\mathbb{N}\}$ be a dense subset of X^*. For every $n\in\mathbb{N}$, the function $t\mapsto\langle x_n^*, f(t)\rangle$ is strongly absolutely continuous and thus it is differentiable a. e. in $[0,t_0]$. For each $n\in\mathbb{N}$, denote by S_n the subset of $[0,t_0]$ where the above function is not differentiable and put

$$S := \bigcup_{n=0}^{+\infty} S_n.$$

For each $t\in[0,t_0]\backslash S$, the limit $\lim_{h\to 0}\langle x_n^*, f_h(t)\rangle$ exists. Moreover, when $h\to 0^+$, we know that $(\|f_h\|)$ is bounded. Hence, given $x^*\in X^*$, we obtain that $\lim_{h\to 0}\langle x_n^*, f_h(t)\rangle$ exists a. e. in $[0,t_0]$. Since X is a Banach space, it is weakly complete. Therefore, there exists

$$w-\lim_{h\to 0} f_h(t)$$

a. e. in $[0,t_0]$. Denote this limit by $f'(t)$. Clearly, f' is weakly measurable, and, by Pettis' theorem, it is strongly measurable. Let us see that f' is Bochner integrable. Indeed, the function $t\mapsto\|f'(t)\|$ is integrable and

$$\limsup_{h\to 0^+} \int_0^{t_0} \|f_h(t)\|\, dt < \infty.$$

Applying Fatou's lemma, we have

$$\int_0^{t_0} \|f'\|\, dt < \infty.$$

This means that f' is Bochner integrable.

Let

$$g : [0, t_0] \to X, \quad t \mapsto g(t) := f(0) + \int_0^t f'(s)\, ds.$$

We know that g is strongly differentiable a. e. in $[0, t_0]$. Bearing in mind that $t \mapsto \langle x^*, f(t)\rangle$ is absolutely continuous, we infer that

$$\begin{aligned}\langle x^*, f(t)\rangle &= \langle x^*, f(0)\rangle + \int_0^t \frac{d}{ds}\langle x^*, f(s)\rangle\, ds \\ &= \langle x^*, f(0)\rangle + \int_0^t \langle x^*, f'(s)\rangle\, ds = \langle x^*, g(t)\rangle\end{aligned}$$

a. e. in $[0, t_0]$. Then, $\langle x^*, f(t)\rangle = \langle x^*, g(t)\rangle$ a. e. in $[0, t_0]$ for all $x^* \in X^*$. Accordingly, $f = g$ a. e. in $[0, t_0]$, which concludes the proof. ☐

Remark 1.14.1. Let $[a, b]$ be an interval of real numbers and X a Banach space; by $L^1(a, b, X)$ we denote the vector space of the Bochner integrable functions $f : [a, b] \to X$ with respect to Lebesgue's measure (i. e., the strongly measurable functions such that $\int_a^b \|f(t)\|\, dt < \infty$). By $L^1_{\mathrm{loc}}([a, b], X)$ we denote the space of functions $f : I \to X$ which are Bochner integrable on compact subsets of $[a, b]$. As in the case of real valued functions, if $f \in L^1(a, b, X)$ then, for almost every $t \in\,]a, b[$, the following equality holds:

$$\lim_{h\downarrow 0} \frac{1}{h} \int_{t-h}^{t+h} \|f(s) - f(t)\|\, ds = 0.$$

When the above equality holds, t is called a Lebesgue's point of the function f.

1.15 Superposition operators

1.15.1 Scalar-valued functions

Let Ω be a domain (an open and connected set) in $\mathbb{R}^n$. A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to satisfy Carathéodory conditions if

$$\begin{cases} f(x,u) & \text{is a continuous function of } u \text{ for almost all } x \in \Omega, \\ f(x,u) & \text{is a measurable function of } x \text{ for all } u \in \mathbb{R}. \end{cases}$$

A function satisfying Carathéodory conditions is said to be a *Carathéodory function.*

Given a function f satisfying Carathéodory conditions and a function $u : \Omega \to \mathbb{R}$, one can define a new function $\mathbf{N}_f$ from Ω into $\mathbb{R}$ by

$$(\mathcal{N}_f u)(x) = f(x, u(x))$$

for all $x \in \Omega$. The function $\mathbf{N}_f$ is called a Nemytskii (or superposition) operator generated by f. In L_p-spaces the Nemytskii operator has been extensively investigated (see, for example, [17, 19, 147] and the references therein). However, the following basic result for this class of operators is due to Krasnosel'skii (see, for example, [145, 146] or [147]).

Theorem 1.15.1. *Let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. The operator $\mathbf{N}_f$ is continuous from $L^p(\Omega, \mathbb{R})$ into $L^q(\Omega, \mathbb{R})$ with $1 \leq p, q < +\infty$ if and only if there exist $C \geq 0$ and a function $g \in L^q(\Omega, \mathbb{R})$ such that for all x and u we have*

$$|f(x,u)| \leq g(x) + C|u|^{p/q}.$$

We also have the following corollary, which is very useful in applications.

Corollary 1.15.1. *Assume that f is a Carathéodory function. If the operator $\mathbf{N}_f$ acts from $L^p(\Omega, \mathbb{R})$ into $L^q(\Omega, \mathbb{R})$ with $1 \leq p, q < +\infty$, then $\mathbf{N}_f$ is continuous and takes bounded sets into bounded sets.*

Remark 1.15.1. It should be noticed that Nemytskii operators are not weakly continuous on L^1-spaces. In fact, even in the scalar case, only linear functions generate weakly continuous Nemytskii operators on L^1-spaces (see, for example, [17, Theorem 2.6]).

1.15.2 Vector-valued functions

Let Ω be an open subset of $\mathbb{R}^n$ (bounded or not) and let X and Y be two separable Banach spaces. A function $f : \Omega \times X \to Y$ is said to be a Carathéodory function if

$$\begin{cases} f(x,u) & \text{is a continuous function of } u \text{ for almost all } x \in \Omega, \\ f(x,u) & \text{is a measurable function of } x \text{ for all } u \in X. \end{cases}$$

Let $\mathbf{m}(\Omega, X)$ be the set of all measurable functions $\psi : \Omega \to X$. If f is a Carathéodory function, then f defines a map $\mathbf{N}_f : \mathbf{m}(\Omega, X) \to \mathbf{m}(\Omega, Y)$ by $\mathbf{N}_f(\psi)(t) := f(t, \psi(t))$. The map $\mathbf{N}_f$ is called the superposition (or Nemytskii) operator generated by f (or associated to f). The following result due to Lucchetti and Patrone [167] is an extension of Theorem 1.15.1 to separable Banach spaces.

Theorem 1.15.2. *Let Ω be a domain of $\mathbb{R}^n$ and let X and Y be two separable Banach spaces. If f is a Carathéodory function, then the Nemytskii operator $\mathbf{N}_f$ maps $L^p(\Omega, X)$ into $L^q(\Omega, Y)$ with $1 \le p, q < +\infty$ if and only if there exist a constant $C > 0$ and a function $g \in L^q(\Omega, Y)$ such that*

$$\|f(x,u)\|_Y \le g(x) + C\|x\|_X^{p/q}.$$

1.15.3 The case of weighed Lebesgue spaces

Let Ω be a n-dimensional domain,

$$\Omega \subset \mathbb{R}^n,$$

i. e., Ω is an open, connected set of $\mathbb{R}^n$, bounded or unbounded, possibly even equal to the whole $\mathbb{R}^n$.

We will suppose that $\partial\Omega$ is sufficiently smooth (which means that $\partial\Omega$ is locally Lipschitz). This property will guarantee that the results given below are valid.

Let $u = u(x)$ be a real function defined and measurable a. e. in Ω, and let $w = w(x)$ be a positive measurable function a. e. in Ω. Letting $1 \le p < +\infty$, we define the weighted Lebesgue space $L^p(\Omega, w)$ by

$$L^p(\Omega, w) = \{u = u(x) : uw^{\frac{1}{p}} \in L^p(\Omega)\},$$

where $L^p(\Omega) = \{u = u(x) : \|u\|_p = (\int_\Omega |u(x)|^p \, dx)^{1/p} < +\infty\}$.

The space $L^p(\Omega, w)$, equipped with the norm

$$\|u\|_{p,w} = \left(\int\limits_\Omega |u(x)|^p w(x) \, dx \right)^{1/p},$$

is a Banach space (uniformly convex and hence reflexive if $p > 1$).

As above, let Ω be a bounded or unbounded domain of $\mathbb{R}^n$. Let w_1 and w_2 be two nonnegative measurable functions a. e. in Ω and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. The Nemytskii operator associated to g between the weighed Lebesgue spaces $L^{p_1}_{w_1}(\Omega)$ and $L^{p_2}_{w_2}(\Omega)$ is defined by $(\mathbf{N}_g u)(x) = g(x, u(x))$.

The following assertion is valid for bounded and unbounded domains Ω of $\mathbb{R}^n$ with sufficiently smooth boundaries (we refer to the book [85, pp. 17–19]).

Proposition 1.15.1. *The Nemytskii operator* $\mathbf{N}_g$ *maps continuously* $L^{p_1}_{w_1}(\Omega)$ *into* $L^{p_2}_{w_2}(\Omega)$ *if and only if g satisfies*

$$|g(x,\beta)| \le a(x) w_2^{-1/p_2}(x) + c w_2^{-1/p_2}(x) |\beta|^{\frac{p_1}{p_2}} w_1^{1/p_2}(x)$$

for a. e. $x \in \Omega$ *and every* $\beta \in \mathbb{R}$ *with a fixed nonnegative function* $a \in L^p(\Omega)$ *and a fixed nonnegative constant c.*

1.16 Sobolev functions

Let $[0, T]$ be a fixed real interval and let $\mathcal{D}([0, T])$ denote the space of all real-valued functions defined on $[0, T]$ which are infinitely differentiable on $[0, T]$ and have compact support in $]0, T[$. The space $\mathcal{D}([0, T])$ is topologized as the strict inductive limit of

$$\mathcal{D}_K([0,T]) = \{\varphi \in \mathcal{D}([0,T]) : \operatorname{supp}(\varphi) \subseteq K\},$$

where K ranges over all compact sets of $]0, T[$. If X is a Banach space, we denote by $\mathcal{D}'(0, T; X)$ the space of all linear continuous operators from $\mathcal{D}([0, T])$ to X. An element $u \in \mathcal{D}'(0, T; X)$ is called an X-valued distribution on $]0, T[$.

If $u \in \mathcal{D}'(0, T : X)$ then

$$u'(\varphi) := -u(\varphi') \quad \text{for all } \varphi \in \mathcal{D}([0,T]) \tag{1.14}$$

defines another vectorial distribution which is called the derivative of u. Notice that for each $u \in L^1(0, T, X)$ we may associate a distribution (denoted by u) as follows:

$$L^1(0,T,X) \ni u \longmapsto u(\varphi) = \int_0^T u(t)\varphi(t)\,dt, \quad \forall \varphi \in \mathcal{D}([0,T]).$$

Thus $L^1(0, T, X)$ may be considered as a linear subspace of $\mathcal{D}'(0, T; X)$ and therefore we may identify the elements of $L^1(0, T, X)$ with their corresponding vectorial distributions.

We define $W^{1,1}(0, T; X) := \{u \in \mathcal{D}'(0, T; X) : u' \in L^1(0, T; X)\}$, where u' is defined by (1.14).

Theorem 1.16.1. *Let X be a Banach space and let* $u \in L^1(0, T; X)$. *The following conditions are equivalent:*

(a) $u \in W^{1,1}(0,T;X)$.
(b) *There is an absolutely continuous function* $v : [0,T] \to X$ *whose derivative* v' *(defined almost everywhere) belongs to* $L^1(0,T,X)$ *such that* $u = v$ *a. e. on* $[0,T]$.

If X is a reflexive Banach space, both Komura's theorem (Theorem 1.14.6) and Theorem 1.16.1 say that $u \in W^{1,1}(0,T;X)$ if and only if u is absolutely continuous on $[0,T]$.

1.17 Sobolev spaces

1.17.1 Sobolev space in dimension one

Let $I = (a,b)$ be an open interval, possibly unbounded, and let $p \in \mathbb{R}$ be such that $1 \le p \le +\infty$.

Definition 1.17.1. The Sobolev space $W^{1,p}(I)$ is defined by

$$W^{1,p}(I) = \left\{ u \in L^p(I) : \exists g \in L^p(I) \text{ such that } \int_I u\varphi' = -\int_I g\varphi \; \forall \varphi \in C_c^1(I) \right\}$$

where, as usual, $C_c^1(I)$ represents the set of functions with continuous derivative and compact support on I. □

The Sobolev space $W^{1,2}(I)$ is usually denoted by $H^1(I)$. On the other hand, for $u \in W^{1,p}(I)$, we denote $u' = g$ (g is the derivative of u in the sense of distributions (see, for example, [214])).

Remark 1.17.1. In the above definition, the function φ is called a test function. As a test function we can take either functions in $C_c^1(I)$ or functions in $C_c^\infty(I)$. It is clear that if $u \in C^1(I) \cap L^p(I)$ and if $u' \in L^p(I)$ (here u' means the usual derivative of u), then $u \in W^{1,p}(I)$ and, moreover, such a usual derivative coincides with the derivative in the sense of $W^{1,p}(I)$ (the derivative of u in the sense of distributions). In particular, if I is bounded, we have that $C^1(\bar{I}) \subset W^{1,p}(I)$ for every $1 \le p \le +\infty$.

Notation

The space $W^{1,p}(I)$ can be endowed with the norm

$$\|u\|_{1,p} = \|u\|_p + \|u'\|_p$$

(or sometimes with the equivalent norm $(\|u\|_p^p + \|u'\|_p^p)^{\frac{1}{p}}$), whereas $H^1(I)$ is endowed with the scalar product

$$\langle u,v\rangle_1 = \langle u,v\rangle_{L^2} + \langle u',v'\rangle_{L^2}$$

and the norm associated to this scalar product

$$\|u\|_1 = \sqrt{\langle u, u\rangle_1}.$$

Proposition 1.17.1. *For* $1 \le p \le +\infty$, $(W^{1,p}(I), \|\cdot\|_{1,p})$ *is a Banach space. Moreover,*

(a) $W^{1,p}(I)$ *is reflexive whenever* $1 < p < +\infty$ *and it is separable for* $1 \le p < +\infty$.

(b) $H^1(I)$ *is a separable Hilbert space.*

Proposition 1.17.2. *Let* $u \in L^p$, *with* $1 < p \le \infty$. *The following properties are equivalent:*

(a) $u \in W^{1,p}$.

(b) *There exists a constant C such that*

$$\left|\int_I u\varphi'\right| \le C\|\varphi\|_{L^q(I)} \quad \forall \varphi \in C_c^\infty(I),$$

where q is the conjugate of p.

Remark 1.17.2. If $p = 1$, (a) ⇒ (b) in the above proposition remains true. If I is bounded, then functions satisfying (a) are the absolutely continuous functions, while functions satisfying (b) are the functions of bounded variation.

Theorem 1.17.1. *There exists a constant C (which depends only on the measure of I) such that*

$$\|u\|_{L^\infty(I)} \le C\|u\|_{W^{1,p}(I)} \quad \forall u \in W^{1,p}(I), \quad 1 \le p \le +\infty,$$

that is, $W^{1,p}(I) \subset L^\infty(I)$ *with continuous embedding for every* $1 \le p \le +\infty$. *Moreover, if I is bounded, then*

(a) *the embedding* $W^{1,p}(I) \subset C(\bar{I})$ *is compact whenever* $1 < p \le +\infty$,

(b) *the embedding* $W^{1,1}(I) \subset L^q(I)$ *is compact for* $1 \le q < +\infty$.

Corollary 1.17.1. *Let* $u, v \in W^{1,p}(I)$ *with* $1 \le p \le \infty$. *Then* $uv \in W^{1,p}(I)$ *and*

$$(uv)' = u'v + uv'.$$

Moreover, the following formula:

$$\int_y^x u'v = u(x)v(x) - u(y)v(y) - \int_y^x uv' \quad \forall x, y \in \bar{I}$$

holds.

Definition 1.17.2. Given $1 \le p < \infty$, we denote by $W_0^{1,p}(I)$ the closure of $C_c^1(I)$ in $W^{1,p}(I)$.

A very useful characterization of $W_0^{1,p}(I)$ is obtained in the next result.

Theorem 1.17.2. *Let $u \in W^{1,p}(I)$, then $u \in W_0^{1,p}(I)$ if and only if $u = 0$ on ∂I (the boundary of I).*

1.17.2 Sobolev spaces in higher dimension

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p \in \mathbb{R}$ be such that $1 \le p \le +\infty$.

Definition 1.17.3. The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \exists g_1, g_2, \dots, g_n \in L^p(\Omega) \text{ such that}$$
$$\int_\Omega u \frac{\partial}{\partial x_i}\varphi = -\int_\Omega g_i \varphi \ \forall \varphi \in C_c^\infty(\Omega)\ \forall i = 1, \dots, n \Big\}.$$

For each $u \in W^{1,p}(\Omega)$, we write

$$\frac{\partial u}{\partial x_i} = g_i \quad \text{and} \quad \nabla u = \left(\frac{\partial}{\partial x_1} u, \dots, \frac{\partial}{\partial x_n} u \right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_p,$$

or sometimes with the equivalent norm $(\|u\|_p^p + \sum_{i=1}^n \| \frac{\partial u}{\partial x_i} \|_p^p)^{\frac{1}{p}}$ if $1 \le p < +\infty$.

For $p = 2$, the space $W^{1,2}(\Omega)$ is usually denoted by $H^1(\Omega)$. The space $H^1(\Omega)$ is equipped with the scalar product

$$\langle u, v \rangle_1 = \langle u, v \rangle_{L^2} + \sum_{i=1}^n \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle_{L^2}.$$

The associated norm

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}^2 \right)^{1/2}$$

is equivalent to the $W^{1,2}(\Omega)$ norm.

Proposition 1.17.3. *The space $W^{1,p}(\Omega)$ is a Banach space for $1 \le p \le +\infty$. Moreover,*
(a) *$W^{1,p}(\Omega)$ is reflexive whenever $1 < p < +\infty$ and it is separable for $1 \le p < \infty$.*
(b) *$H^1(\Omega)$ is a separable Hilbert space.*

Theorem 1.17.3 (Rellich–Kondrachov). *Suppose that* Ω *is bounded with smooth boundary.*

(a) *If* $p < n$, *then* $W^{1,p}(\Omega) \subset L^q(\Omega)$ *for all* $q \in [1, p*[$ *where* $\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}$.

(b) *If* $p = n$, *then* $W^{1,p}(\Omega) \subset L^q(\Omega)$ *for all* $q \in [1, +\infty[$.

(c) *If* $p > n$ *then* $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

The above embeddings are compact. In particular, $W^{1,p}(\Omega) \subset L^p(\Omega)$ *with compact embedding for every* p.

Definition 1.17.4. Let $1 \le p < \infty$. We denote by $W_0^{1,p}(\Omega)$ the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$. For $p = 2$, the space $W_0^{1,2}(\Omega)$ is denoted by $H_0^1(\Omega)$.

Suppose that Ω is bounded where $\partial\Omega$ is C^1. The functions of $W_0^{1,p}(\Omega)$ are, in fact, the functions of $W^{1,p}(\Omega)$ taking the value zero on the boundary of Ω. Nevertheless, to give a sense to the above comment is not easy since, if a function belongs to $W^{1,p}(\Omega)$, it is only defined a. e. and $\partial\Omega$ is a null set in $\mathbb{R}^n$, thus such functions do not have a continuous representative. By using the trace theory, it is possible to give a meaning to this fact.

Theorem 1.17.4 (Trace theorem). *Let* $1 \le p < +\infty$. *Suppose that* Ω *is bounded and* $\partial\Omega$ *is* C^1. *Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega) \tag{1.15}$$

such that for each $u \in W^{1,p}(\Omega)$:

1. $T(u) = u$ *in* $\partial\Omega$, *if* $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$,
2. $\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$,

where the constant C *depends only on* p *and* Ω.

Theorem 1.17.5. *Let* $1 \le p < +\infty$, *Suppose that* Ω *is bounded and* $\partial\Omega$ *is* C^1. *If* $u \in W^{1,p}(\Omega)$ *then*

$$u \in W_0^{1,p}(\Omega) \quad \text{if and only if} \quad T(u) = 0 \text{ on } \partial\Omega. \tag{1.16}$$

Theorem 1.17.6 (Poincaré's inequality). *Suppose that* Ω *is a bounded open subset of* $\mathbb{R}^n$. *Then there exists a constant* C *(depending on both* Ω *and* p*) such that*

$$\|u\|_p \le C\|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega),\ 1 \le p < +\infty.$$

If Ω is a regular domain, by using the trace theory, the following Green's formula is obtained. Given $m \in \mathbb{N}$ with $m \ge 2$ and $1 \le p \le +\infty$, we can define the space

$$W^{m,p}(\Omega) := \left\{ u \in W^{m-1,p}(\Omega) : \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega), \quad i = 1, \dots, n \right\}.$$

Theorem 1.17.7 (Green's formula). *Let* Ω *be a bounded domain with smooth boundary* $\partial\Omega$. *Given* $1 < p < +\infty$, *if* $u, v \in W^{2,p}(\Omega)$ *then*

$$-\int_{\Omega} \Delta u \cdot v = \int_{\Omega} \langle \nabla u, \nabla v \rangle - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma,$$

where $\frac{\partial u}{\partial n} := \langle \nabla u, n \rangle$ *and* σ *denotes the surface measure on* $\partial\Omega$. *By* $n(x)$ *we denote the outer unit normal at each* $x \in \partial\Omega$.

1.18 Bibliographical remarks

Most of the material presented in Sections 1.1–1.6 is classical and may be found in the books by H. Brezis [46], J. Dugundji [86], N. Dunford and J. T. Schwartz [89], R. E. Edwards [92], M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler [95], J. Horváth [125], J. L. Kelley [140], E. R. Megginson [173], W. Rudin [206], V. Runde [207], H. H. Schaefer and M. P. Wolff [211], M. Schechter [213], F. Trèves [229], and K. Yosida [243].

Concerning the results about the weak and weak* topologies (Sections 1.7–1.8), we refer to the books by H. Brezis [46], N. Dunford and J. T. Schwartz [89], M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler [95], E. R. Megginson [173], and K. Yosida [243]. Theorem 1.7.6 is Theorem 4 in [80].

The material collected in Section 1.9 may be found, for example, in the books by H. Brezis [46], I. Cioranescu [64], C. Chidume [62], J. Diestel and J. J. Uhl [79], M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler [95].

The material presented in Section 1.10 concerning the duality mapping may be found in the books by V. Barbu [31, 32], Ph. Bénilan, M. G. Crandall, and A. Pazy [37], H. Brezis [45], I. Cioranescu [64], and K. Deimling [77].

For retraction mappings (Section 1.12) our references are essentially the paper by D. De Figueiredo and L. Karlovitz [81] and the book by E. Zeidler [244].

In Section 1.13 we gather some elementary facts concerning Banach algebras which may be found in the books by G. R. Allan [16] and H. G. Dales [72]. We refer also the paper by J. Banaś and M. A. Taoudi [30].

The material collected in Section 1.14 may be found, for example, in the book by J. Diestel and J. J. Uhl [79]. Theorem 1.14.6 is the lemma in the appendix of the paper by Y. Komura [144].

The material presented in Section 1.15 may be found in the papers by J. Appell [17], as well as the books by J. Appell and P. P. Zabrejko [19] and S. N. Chow and J. K. Hale [63]. Theorem 1.15.2 is Theorem 3.1 in the paper by R. Lucchetti and F. Patrone [167], while Proposition 1.15.1 is Theorem 1.1 in the book by P. Drábek, A. Kufner, and F. Nicolosi [85].

The basic references of Section 1.17 are the books by R. A. Adams [2] and H. Brezis [46].

2 A short introduction to semigroups of linear operators

For the reader convenience, we present in this chapter a short introduction of those parts of the theory of linear bounded operator semigroups that will be needed in the main text.

2.1 Strongly continuous semigroups

Definition 2.1.1. A family $(T(t))_{t\geq 0}$ of bounded linear operators from a Banach space X into itself is called a semigroup if

(a) $T(0) = I$,

(b) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$.

Definition 2.1.2. Let $(T(t))_{t\geq 0}$ be a semigroup of bounded linear operators on a Banach space X. We say that $(T(t))_{t\geq 0}$ is strongly continuous if, for each $x \in X$, we have

$$\lim_{t\searrow 0^+} T(t)x = x. \tag{2.1}$$

A strongly continuous semigroup of bounded linear operators on X will also be called a semigroup of class C_0 or a C_0-semigroup.

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. The infinitesimal generator of $(T(t))_{t\geq 0}$ is the operator $A : D(A) \subseteq X \to X$ defined by

$$D(A) := \left\{ x \in X : \lim_{h\searrow 0} \frac{1}{h}\big(T(h)x - x\big) \text{ exists} \right\}$$

and

$$Ax := \lim_{h\searrow 0} \frac{1}{h}\big(T(h)x - x\big).$$

If $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a semigroup of linear operators, then $D(A)$ is a vector subspace of X and A is a possibly an unbounded linear operator.

Proposition 2.1.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup, then there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq Me^{\omega t} \tag{2.2}$$

for each $t \geq 0$.

Proof. We show first that there exists $\eta > 0$ such that $\|T(t)\|$ is bounded for $0 \leq t \leq \eta$. If this fails, then there is a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \geq 0$ such that $t_n \to 0$ as $n \to \infty$ and

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$\|T(t_n)\| \geq n$. From the uniform boundedness theorem (Theorem 1.5.1) it follows that, for some $x \in X$, the sequence $(\|T(t_n)x\|)_{n\in\mathbb{N}}$ is unbounded, contrary to (2.1). Thus, $\|T(t)\| \leq M$ for all $t \in [0,\eta]$. Since $\|T(0)\| = 1$, $M \geq 1$. Let $\omega = \eta^{-1}\ln M \geq 0$. Given $t \geq \eta$, we have that $t = n\eta + \delta$ where $0 \leq \delta < \eta$ and therefore

$$\|T(t)\| = \|T(\delta)T(\eta)^n\| \leq M^{n+1} \leq MM^{\frac{t}{\eta}} = Me^{\omega t}. \qquad \square$$

A C_0-semigroup, $(T(t))_{t\geq 0}$, is called of type (M,ω) with $M \geq 1$ and $\omega \in \mathbb{R}$, if for each $t \geq 0$, we have

$$\|T(t)\| \leq Me^{\omega t}.$$

A C_0-semigroup $(T(t))_{t\geq 0}$ is called a semigroup of contractions, or of nonexpansive operators, if it is of type $(1,0)$, that is,

$$\|T(t)\| \leq 1 \quad \text{for each } t \geq 0.$$

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. We define

$$\omega(T) := \inf\{\omega \in \mathbb{R} : \exists M \geq 1\ \forall t \geq 0\ \|T(t)\| \leq Me^{\omega t}\}. \tag{2.3}$$

The real number $\omega(T)$ is called the type, or uniform growth bound, of the semigroup $(T(t))_{t\geq 0}$. It is clear that $\omega(T) < +\infty$, but it may be $-\infty$. Observe that if X is equipped with a different but equivalent norm, then the type of the semigroup $(T(t))_{t\geq 0}$ does not change.

The type can also defined in another way.

Proposition 2.1.2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Then*

$$\omega(T) = \inf_{t>0} \frac{1}{t}\ln\|T(t)\| = \lim_{t\to+\infty} \frac{1}{t}\ln\|T(t)\|.$$

For an elegant proof of the above result using subadditive mappings, see, for example, [89, p. 619].

Proposition 2.1.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Then the mapping $(t,x) \mapsto T(t)x$ is jointly continuous from $[0,+\infty) \times X$ into X.*

Proof. Let $x, y \in X$, $t \geq 0$ and $h \neq 0$ with $t + h \geq 0$. We distinguish between two cases, namely $h > 0$ and $h < 0$. By Proposition 2.1.1, when $h > 0$, we have

$$\begin{aligned}\|T(t+h)y - T(t)x\| &\leq \|T(t+h)y - T(t+h)x\| + \|T(t+h)x - T(t)x\| \\ &\leq \|T(t+h)\|\|y - x\| + \|T(t+h)x - T(t)x\| \\ &\leq Me^{(t+h)\omega}\|x - y\| + \|T(t)\|\|T(h)x - x\|.\end{aligned}$$

This implies that

$$\lim_{(s,y)\to(t^+,x)} T(s)y = T(t)x.$$

If $h < 0$, by Proposition 2.1.1, we can write

$$\begin{aligned}\|T(t+h)y - T(t)x\| &\le \|T(t+h)y - T(t+h)x\| + \|T(t+h)x - T(t+h)T(-h)x\|\\ &\le \|T(t+h)\|\|y - x\| + \|x - T(-h)x\|\\ &\le Me^{(t+h)\omega}(\|x-y\| + \|T(-h)x - x\|).\end{aligned}$$

Since the semigroup is strongly continuous, we conclude that

$$\lim_{(s,y)\to(t^-,x)} T(s)y = T(t)x$$

which ends the proof. □

In the following proposition we gather some basic properties of C_0-semigroups (see, for example, [195, p. 4]).

Proposition 2.1.4. *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\ge0}$. Then*

(a) *for each $x \in X$ and each $t \ge 0$, we have*

$$\lim_{h\to0} \frac{1}{h} \int_t^{t+h} T(s)x\,ds = T(t)x,$$

(b) *for each $x \in X$ and each $t \ge 0$, we have*

$$\int_0^t T(s)x\,ds \in D(A) \quad \text{and} \quad A\left(\int_0^t T(s)x\,ds\right) = T(t)x - x,$$

(c) *for each $x \in D(A)$ and each $t \ge 0$, we have $T(t)x \in D(A)$. In addition, the mapping $t \mapsto T(t)x$ is of class C^1 on $[0,+\infty)$, and satisfies*

$$\frac{d}{dt}(T(t)x) = AT(t)x = T(t)Ax,$$

(d) *for each $x \in D(A)$ and each $0 \le s \le t < +\infty$, we have*

$$\int_s^t AT(\tau)x\,d\tau = \int_s^t T(\tau)Ax\,d\tau = T(t)x - T(s)x.$$

Proof. (a) Note that, for all $t \geq 0$ and $x \in X$, we have

$$\left\| \frac{1}{h} \int_t^{t+h} T(s)x\, ds - T(t)x \right\| \leq \frac{1}{h} \int_0^{t+h} \|T(s)x - T(t)x\|\, ds.$$

The result follows from Proposition 2.1.3.

(b) Let $x \in X$, $t > 0$ and $h > 0$. Let us first observe that

$$\frac{1}{h}(T(h) - I) \int_0^t T(s)x\, ds = \frac{1}{h} \int_0^t T(s+h)x\, ds - \frac{1}{h} \int_0^t T(s)x\, ds.$$

The change of variable $s + h = \tau$ in the first integral on the right-hand side yields

$$\begin{aligned} \frac{1}{h}(T(h) - I) \int_0^t T(s)x\, ds &= \frac{1}{h} \int_h^{t+h} T(\tau)x\, d\tau - \frac{1}{h} \int_0^t T(\tau)x\, d\tau \\ &= -\frac{1}{h} \int_0^h T(\tau)x\, d\tau + \frac{1}{h} \int_t^{t+h} T(\tau)x\, d\tau. \end{aligned}$$

Now using (a) yields

$$\lim_{h \to 0} \frac{1}{h}(T(h) - I) \int_0^t T(\tau)\, d\tau = T(t)x - x,$$

which proves (b).

(c) Let $x \in D(A)$, $t \geq 0$ and $h > 0$. We have

$$\left\| \frac{1}{h}(T(t+h)x - T(t)x) - T(t)Ax \right\| \leq \|T(t)\| \left\| \frac{1}{h}(T(h)x - x) - Ax \right\|,$$

which proves that $T(t)x \in D(A)$, $t \mapsto T(t)x$ is differentiable from the right, and that

$$\frac{d^+}{dt}(T(t)x) = AT(t)x = T(t)Ax. \tag{2.4}$$

On the other hand, for each $t > 0$ and $h < 0$, with $t + h > 0$, we have

$$\begin{aligned} \left\| \frac{1}{h}(T(t+h)x - T(t)x) - T(t)Ax \right\| &\leq \|T(t+h)\| \left\| \frac{1}{h}(x - T(-h)x) - T(-h)Ax \right\| \\ &\leq \|T(t+h)\| \\ &\quad \times \left(\left\| -\frac{1}{h}(T(-h)x - x) - Ax \right\| + \|T(-h)Ax - Ax\| \right). \end{aligned}$$

This shows that $t \mapsto T(t)x$ is differentiable from the left as well. Now using (2.4) together with the continuity of the function $t \mapsto T(t)Ax$ on $[0,+\infty)$, we deduce that $t \mapsto T(t)x$ is of class C^1 on $[0,+\infty)$, which ends the proof.

Assertion (d) follows from (c) by integrating from s to t both sides in (2.4). □

2.2 Uniformly continuous semigroups

Let X be a Banach space and $(T(t))_{t\geq 0}$ a semigroup of bounded linear operators on X. We recall that $(T(t))_{t\geq 0}$ is called *uniformly continuous* if it is continuous at 0 for the uniform operator topology, that is, if

$$\lim_{t\searrow 0}\|T(t)-I\| = 0. \tag{2.5}$$

It is clear that, if $(T(t))_{t\geq 0}$ is uniformly continuous, then

$$\lim_{s\to t}\|T(s)-T(t)\| = 0.$$

A first significant example of uniformly continuous semigroup is given by $t \mapsto e^{tA}$ where e^{tA} is the exponential of the matrix tA. Namely, let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ (the space of all square $n \times n$ matrices with entries in $\mathbb{R}$) and let $T(t) = e^{tA}$ for each $t \geq 0$, where

$$e^{tA} = \sum_{n=0}^{+\infty}\frac{t^n}{n!}A^n.$$

We can easily see that $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup of linear operators. Moreover, straightforward computations show that $t \mapsto T(t)$ is of class C^1 from $[0,+\infty)$ into $\mathcal{L}(X)$ and satisfies the first-order differential equation

$$\frac{d}{dt}(T(t)) = AT(t) = T(t)A, \tag{2.6}$$

for each $t \geq 0$. Actually, $(T(t))_{t\geq 0}$ is nothing else than the fundamental matrix of the first-order vector differential equation

$$\psi' = A\psi,$$

which satisfies $T(0) = I$. In fact, as the following result shows, all uniformly continuous semigroups are of the form $(e^{tA})_{t\geq 0}$, with $A \in \mathcal{L}(X)$, and satisfy (2.6).

Theorem 2.2.1. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup on X if and only if $A \in \mathcal{L}(X)$.*

Proof. Let A be a bounded linear operator on X and set

$$T(t) = e^{tA} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} A^n. \tag{2.7}$$

It is clear that, for every $t \geq 0$, the right-hand side of (2.7) converges in the operator norm and defines a bounded linear operator $T(t)$. It is clear that $T(0) = I$, and, since A commutes with itself, we get $T(t+s) = T(t)T(s)$. Next, using (2.7), we derive the estimate $\|T(t) - I\| \leq t\|A\|e^{t\|A\|}$ which implies that $(T(t))_{t\geq 0}$ is uniformly continuous. The fact that A is its generator follows from

$$\left\| \frac{T(t) - I}{t} - A \right\| \leq \|A\| \max_{0\leq s\leq t} \|T(s) - I\|.$$

Conversely, let $(T(t))_{t\geq 0}$ be a uniformly continuous semigroup of bounded linear operators on X. Fix $\tau > 0$ small enough, one has $\|I - \tau^{-1} \int_0^\tau T(s)\,ds\| < 1$. This shows that $\tau^{-1} \int_0^\tau T(s)\,ds$ is invertible and therefore $\int_0^\tau T(s)\,ds$ is invertible. For $h > 0$, we have

$$\begin{aligned} \frac{T(h) - I}{h} \int_0^\tau T(s)\,ds &= \frac{1}{h}\left(\int_0^\tau T(s+h)\,ds - \int_0^\tau T(s)\,ds \right) \\ &= \frac{1}{h}\left(\int_\tau^{\tau+h} T(s)\,ds - \int_0^h T(s)\,ds \right) \end{aligned}$$

and therefore

$$\frac{T(h) - I}{h} = \left(\frac{1}{h} \int_\tau^{\tau+h} T(s)\,ds - \frac{1}{h} \int_0^h T(s)\,ds \right) \left(\int_0^\tau T(s)\,ds \right)^{-1}. \tag{2.8}$$

Letting $h \downarrow 0$ in (2.8) shows that $\frac{T(h)-I}{h}$ converges in norm and therefore strongly to the bounded linear operator $(T(\tau) - I)(\int_0^\tau T(s)\,ds)^{-1}$ which is the infinitesimal generator of $(T(t))_{t\geq 0}$. □

Proposition 2.2.1. *If $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup of linear operators then, for each $t \geq 0$, $T(t)$ is invertible.*

Proof. Since $\lim_{t\searrow 0} T(t) = I$ in the norm topology of $\mathcal{L}(X)$, there exists $\delta > 0$ such that

$$\|T(t) - I\|_{\mathcal{L}(X)} < 1$$

for each $t \in (0, \delta]$. Thus, for each $t \in (0, \delta]$, the operator $T(t)$ is invertible. If $t > \delta$, then there exist $n \in \mathbb{N}^*$ and $\tau \in [0, \delta)$ such that $t = n\delta + \tau$. Therefore, $T(t) = T(\delta)^n T(\tau)$, and so $T(t)$ is invertible. This ends the proof. □

2.3 The infinitesimal generator of a semigroup

In this section we shall give some basic properties of the generator of a strongly continuous semigroup. As above, X will denote a Banach space.

Proposition 2.3.1. *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then $D(A)$ is dense in X, and A is a closed operator.*

Proof. For every $x \in X$ and $\varepsilon > 0$, set $x_\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon T(s)x\,ds$. By Proposition 2.1.4(b), $x_\varepsilon \in D(A)$ and, by Proposition 2.1.4(a), $\lim_{\varepsilon\to 0} x_\varepsilon = x$. This shows that $D(A)$ is dense in X. Next, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(A)$ such that $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} Ax_n = y$. It follows from Proposition 2.1.4(d) that, for each $h > 0$ and $n \in \mathbb{N}^*$,

$$T(h)x_n - x_n = \int_0^h T(s)Ax_n\,ds.$$

Passing to the limit in this equality, we obtain

$$T(h)x - x = \int_0^h T(s)y\,ds.$$

Now using Proposition 2.1.4(a), we infer that

$$\lim_{h\to 0} \frac{T(h)x - x}{h} = \lim_{h\to 0} \frac{1}{h}\int_0^h T(s)y\,ds = y.$$

This proves that $x \in D(A)$ and $Ax = y$, which completes the proof. □

Proposition 2.3.2. *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of two C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$. Then $T(t) = S(t)$ for each $t \geq 0$.*

Proof. Let $x \in D(A)$. It follows from Proposition 2.1.4(d) that the function $f : [0, t] \to X$, $s \mapsto T(t-s)S(s)x$, is differentiable on $[0, t]$ and that

$$\begin{aligned} f'(s) &= -AT(t-s)S(s)x + T(t-s)AS(s)x \\ &= -AT(t-s)S(s)x + AT(t-s)T(s)x = 0 \end{aligned}$$

for each $s \in [0, t]$. Therefore, f is constant. Hence we have $f(0) = f(t)$, or equivalently, $T(t)x = S(t)x$ for each $x \in D(A)$. Since $D(A)$ is dense in X, and $T(t)$ and $S(t)$ are bounded, we infer that $T(t)x = S(t)x$ for each $x \in X$. □

Let $A : D(A) \subset X \to X$ be a linear operator and $n \in \mathbb{N}$. We define the nth-order power of A by

$$\begin{cases} D(A^0) = X, \\ A^0 = I, \end{cases} \qquad \begin{cases} D(A^1) = D(A), \\ A^1 = A, \end{cases}$$

and

$$\begin{cases} D(A^n) = \{x \in D(A^{n-1}) : A^{n-1}x \in D(A)\}, \\ A^n = AA^{n-1}, \end{cases}$$

for $n \geq 2$.

We know from Proposition 2.3.1 that if A is the generator of a strongly continuous semigroup, then $\overline{D(A)} = X$. Actually, a much stronger result is true. Indeed, we have

Proposition 2.3.3. *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. If $D(A^n)$ is the domain of A^n, then $D(A^n)$ is dense in X.*

Proof. Note that, for each $n \in \mathbb{N}$, $D(A^n)$ is a vector subspace in X. Accordingly, $\bigcap_{n\geq 0} D(A^n)$ is also a vector subspace in X. Let $x \in X$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$ a C^∞-function for which there exists an interval $[a, b] \subset (0, +\infty)$ such that $\varphi(t) = 0$ for each $t \notin [a, b]$. We define

$$x(\varphi) = \int_0^{+\infty} \varphi(t)T(t)x\,dt.$$

We note that

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h}(T(h) - I)x(\varphi) &= \lim_{h\to 0} \frac{1}{h}\left(\int_0^{+\infty} \varphi(t)T(t+h)x\,dt - \int_0^{+\infty} \varphi(t)T(t)x\,dt \right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\int_h^{+\infty} \varphi(t-h)T(t)x\,dt - \int_0^{+\infty} \varphi(t)T(t)x\,dt \right). \end{aligned}$$

For $s \in (-\infty, a)$, we have $\varphi(s) = 0$, and consequently,

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h}(T(h) - I)x(\varphi) &= \lim_{h\to 0} \frac{1}{h}\left(\int_0^{+\infty} \varphi(t-h)T(t)x\,dt - \int_0^{+\infty} \varphi(t)T(t)x\,dt \right) \\ &= \int_0^{+\infty} \lim_{h\to 0} \frac{\varphi(t-h) - \varphi(t)}{h} T(t)x\,dt \\ &= -x(\varphi'). \end{aligned}$$

Therefore $x(\varphi) \in D(A)$ and $Ax(\varphi) = -x(\varphi')$. Repeating the above arguments, one may prove by mathematical induction that, for each $n \in \mathbb{N}$, $x(\varphi) \in D(A^n)$ and $A^n(x(\varphi)) = (-1)^n x(\varphi^{(n)})$. Accordingly, $x(\varphi) \in \bigcap_{n\geq 0} D(A^n)$.

Next, let φ be a function as above such that

$$\int_0^{+\infty} \varphi(t)\,dt = 1.$$

Let $\varepsilon > 0$ and define the function $\varphi_\varepsilon : \mathbb{R} \to (0, +\infty)$ by $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$. It is clear that φ_ε is of class C^∞, $\varphi_\varepsilon(t) = 0$ for $t \notin [\varepsilon a, \varepsilon b]$, and

$$\int_0^{+\infty} \varphi_\varepsilon(t)\,dt = 1.$$

Let us remark that

$$\begin{aligned}\|x(\varphi_\varepsilon) - x\| &= \left\| \frac{1}{\varepsilon} \int_0^{+\infty} \varphi\left(\frac{t}{\varepsilon}\right)(T(t)x - x)\,dt \right\| \le \frac{1}{\varepsilon} \int_{\varepsilon a}^{\varepsilon b} \varphi\left(\frac{t}{\varepsilon}\right)\|T(t)x - x\|\,dt \\ &\le \sup_{t \in [\varepsilon a, \varepsilon b]} \|T(t)x - x\|.\end{aligned}$$

This proves that

$$\lim_{\varepsilon \to 0} x(\varphi_\varepsilon) = x$$

and, therefore, $x \in \overline{\bigcap_{n\ge 0} D(A^n)}$. This ends the proof. ☐

2.4 Generation results

The goal of this section is to prove the fundamental result within the theory of C_0-semigroups, namely Hille–Yosida generation theorem. Further, we derive Lumer–Phillips theorem for dissipative operators.

Recall that, if A is a linear, not necessary bounded, operator in X, then the resolvent set $\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}$, that is, for $\lambda \in \rho(A)$, $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family of bounded linear operators $R(\lambda, A) := (\lambda I - A)^{-1}$ with $\lambda \in \rho(A)$ is called the resolvent of A. Moreover, it is not hard to show that, if $\lambda \in \rho(A)$, then

$$\left(I - \frac{1}{\lambda}A\right)^{-1} = \lambda R(\lambda, A). \tag{2.9}$$

2.4.1 Hille–Yosida theorem

Let X be a Banach space and let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. We say that $(T(t))_{t\geq 0}$ is a C_0-semigroup of contractions if and only if $\|T(t)\| \leq 1$ for all $t \geq 0$.

Theorem 2.4.1 (Hille–Yosida). *A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup of contractions if and only if*

(a) *A is closed, densely defined, and*

(b) *$(0, +\infty) \subset \rho(A)$ and, for each $\lambda > 0$, we have*

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

Remark 2.4.1. Since $(\lambda - A)^{-1} = \lambda^{-1}(I - \lambda^{-1}A)^{-1}$ whenever at least one of the two sides of the equality is well-defined, it follows that (b) is equivalent to:

(b′) for each $\lambda > 0$, we have $(\lambda - A)^{-1} \in \mathcal{L}(X)$ and

$$\|(I - \lambda A)^{-1}\|_{\mathcal{L}(X)} \leq 1.$$

Proof (Necessity). Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geq 0}$. According to Proposition 2.3.1, A is densely defined and closed. Thus (a) holds. In order to prove (b), let $\lambda > 0$, $x \in X$, and define

$$R(\lambda)x = \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt. \tag{2.10}$$

To see that the integral on the right-hand side of the equality above is convergent, consider $a, b > 0$, $a < b$. Then we have

$$\left\| \int_a^b e^{-\lambda t} T(t)x \, dt \right\| \leq \int_a^b e^{-\lambda t} \|T(t)\|_{\mathcal{L}(X)} \|x\| \, dt \leq \int_a^b e^{-\lambda t} \|x\| \, dt = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda} \|x\|.$$

Hence, we are under the hypotheses of the Cauchy test, and therefore the integral is convergent. It is clear that $R(\lambda) \in \mathcal{L}(X)$ and

$$\|R(\lambda)x\| \leq \left\| \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt \right\| \leq \int_0^{+\infty} e^{-\lambda t} \|T(t)\|_{\mathcal{L}(X)} \|x\| \, dt \leq \frac{1}{\lambda} \|x\|,$$

which implies that

$$\|R(\lambda)\| \leq \frac{1}{\lambda}.$$

To establish that $R(\lambda)$ coincides with $R(\lambda, A)$, we have just to prove that $R(\lambda)$ is both the right and the left inverse of the operator $\lambda I - A$. Let $x \in X$, $\lambda > 0$, and $h > 0$. We have

$$\begin{aligned}\frac{1}{h}(T(h) - I)R(\lambda)x &= \frac{1}{h}\int_0^{+\infty} e^{-\lambda t}T(t+h)x\,dt - \frac{1}{h}\int_0^{+\infty} e^{-\lambda t}T(t)x\,dt \\ &= \frac{e^{\lambda h} - 1}{h}\int_0^{+\infty} e^{-\lambda t}T(t)x\,dt - \frac{e^{\lambda h}}{h}\int_0^{h} e^{-\lambda t}T(t)x\,dt.\end{aligned}$$

Since the right-hand side of the above equality converges to $\lambda R(\lambda)x - x$, it follows that $R(\lambda)x \in D(A)$ and $AR(\lambda) = \lambda R(\lambda) - I$, which shows that

$$(\lambda I - A)R(\lambda) = I.$$

Hence $R(\lambda)$ is the right inverse of $(\lambda I - A)$. Now, let $x \in D(A)$ and observe that

$$\begin{aligned}R(\lambda)Ax = \int_0^{+\infty} e^{-\lambda t}T(t)Ax\,dt &= \int_0^{+\infty} e^{-\lambda t}\frac{d}{dt}(T(t)x)\,dt \\ &= \lim_{t\to+\infty} e^{-\lambda t}T(t)x - x + \lambda\int_0^{+\infty} e^{-\lambda t}T(t)x\,dt \\ &= \lambda R(\lambda)x - x.\end{aligned}$$

This equality may be equivalently rewritten as $R(\lambda)(\lambda I - A) = I$, which shows that $R(\lambda)$ is the left inverse of $(\lambda I - A)$, and this completes the proof of the necessity. □

Formula (2.10) is called the integral representation of the resolvent of the operator A; it will be denoted by $R(\lambda, A)$. Of course, the integral has to be understood as an improper Riemann integral, i. e.,

$$R(\lambda, A)x = \lim_{t\to+\infty}\int_0^{t} e^{-\lambda s}T(s)x\,ds, \tag{2.11}$$

for all $x \in X$. Bearing in mind (2.9), we can write

$$(I - \lambda A)^{-1}x = \frac{1}{\lambda}\int_0^{\infty} e^{-\frac{s}{\lambda}}T(s)x\,ds, \quad \text{for all } \lambda > 0.$$

If the semigroup is of type (M, ω), then, for each $\lambda \in \rho(A)$ such that $\mathrm{Re}\lambda > \omega$, the resolvent of the operator A can be written in the form

$$R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda t}T(t)x\,dt. \tag{2.12}$$

Equation (2.10) shows that $(\lambda - A)^{-1}$, the resolvent of the operator A, is given by the Laplace transform of the semigroup $(T(t))_{t\geq 0}$. Hence the sufficiency part of Hille–Yosida theorem can be regarded as an inversion theorem for the Laplace transform in an infinite-dimensional setting.

To prove that conditions (a) and (b) are sufficient for A to be the infinitesimal generator of a C_0-semigroup of contractions, some preliminaries are needed.

Let $A : D(A) \subset X \to X$ be a linear operator satisfying conditions (a) and (b) in Theorem 2.4.1, and let $\lambda > 0$. The operator $A_\lambda : X \to X$ defined by $A_\lambda = \lambda AR(\lambda, A)$ is called the *Yosida approximant* of A. Notice that

$$A_\lambda = \lambda^2 R(\lambda, A) - \lambda I = \frac{(I - \frac{1}{\lambda}A)^{-1} - I}{\frac{1}{\lambda}}. \tag{2.13}$$

Proof of Theorem 2.4.1 *(continued).* Let $\lambda > 0$. Since $A_\lambda \in \mathcal{L}(X)$, by Theorem 2.2.1, it follows that it is the infinitesimal generator of a uniformly continuous semigroup $(e^{tA_\lambda})_{t\geq 0}$. Note also that

$$\|e^{tA_\lambda}\| = \|e^{t\lambda^2 R(\lambda,A)-t\lambda I}\| \leq \|e^{t\lambda^2 R(\lambda,A)}\|\|e^{-t\lambda I}\| \leq e^{t\lambda^2\|R(\lambda,A)\|}e^{-t\lambda} \leq e^{t\lambda}e^{-t\lambda} = 1.$$

Thus, $(e^{tA_\lambda})_{t\geq 0}$ is a semigroup of contractions. Next, for any x in $D(A)$, we can write

$$\begin{aligned}\|\lambda R(\lambda, A)x - x\| &= \|AR(\lambda, A)x\| = \|R(\lambda, A)Ax\| \\ &\leq \frac{1}{\lambda}\|Ax\| \to 0 \quad \text{as } \lambda \to +\infty.\end{aligned}$$

Since $D(A)$ is dense in X and $\|\lambda R(\lambda, A)\| \leq 1$, it follows from the latter estimate that

$$\lim_{\lambda\to+\infty} \lambda R(\lambda, A)x = x \quad \text{for each } x \in X. \tag{2.14}$$

Now, using (2.14) gives

$$\lim_{\lambda\to+\infty} A_\lambda x = \lim_{\lambda\to+\infty} \lambda AR(\lambda, A)x = \lim_{\lambda\to+\infty} \lambda R(\lambda, A)Ax = Ax. \tag{2.15}$$

Let $\lambda, \mu > 0$. It is clear that e^{tA_λ}, e^{tA_μ}, A_λ, and A_μ commute with each other. Hence

$$\begin{aligned}\|e^{tA_\lambda}x - e^{tA_\mu}x\| &= \left\|\int_0^1 \frac{d}{ds}\left(e^{stA_\lambda}e^{(1-s)tA_\mu}x\right) ds\right\| \\ &\leq \int_0^1 t\|e^{stA_\lambda}e^{(1-s)tA_\mu}(A_\lambda x - A_\mu x)\| \, ds \\ &\leq t\|A_\lambda x - A_\mu x\|\end{aligned}$$

for each $x \in D(A)$. So, for t fixed, we have

$$\left\|e^{tA_\lambda} - e^{tA_\mu}\right\| \leq t\|A_\lambda x - A_\mu x\| \to 0 \quad \text{as } \lambda, \mu \to +\infty,$$

for each $x \in D(A)$. Define

$$T(t)x = \lim_{\lambda \to +\infty} e^{tA_\lambda}x, \quad x \in D(A).$$

For each $t > 0$, $\|T(t)\| \leq 1$ because $(e^{tA_\lambda})_{t\geq 0}$ is a semigroup of contractions on X. In addition, for each $t > 0$ and $x, y \in X$, we have

$$\begin{aligned}\left\|T(t)x - x\right\| &\leq \left\|T(t)x - T(t)y\right\| + \left\|T(t)y - e^{tA_\lambda}y\right\| + \left\|e^{tA_\lambda}y - y\right\| + \|y - x\| \\ &\leq \left\|T(t)y - e^{tA_\lambda}y\right\| + \left\|e^{tA_\lambda}y - y\right\| + 2\|x - y\|.\end{aligned}$$

Let $S > 0$ and $\varepsilon > 0$. Fix $y = x_\varepsilon \in D(A)$, with $\|x - x_\varepsilon\| \leq \varepsilon$, and a sufficiently large λ such that

$$\left\|T(t)x_\varepsilon - e^{tA_\lambda}x_\varepsilon\right\| \leq \varepsilon$$

for each $t \in [0, S]$. From this inequality, we deduce

$$\left\|T(t)x - x\right\| \leq 3\varepsilon + \left\|e^{tA_\lambda}x_\varepsilon - x_\varepsilon\right\|. \tag{2.16}$$

Since $(e^{tA_\lambda})_{t\geq 0}$ is a uniformly continuous semigroup, for the same $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that $\|e^{tA_\lambda} - I\|_{\mathcal{L}(X)} \leq \varepsilon$ for each $t \in (0, \delta(\varepsilon))$. Hence, for each $t \in (0, \delta(\varepsilon))$, we have

$$\left\|e^{tA_\lambda}x_\varepsilon - x_\varepsilon\right\| \leq \left\|e^{tA_\lambda} - I\right\|_{\mathcal{L}(X)}\|x_\varepsilon\| \leq \varepsilon\|x_\varepsilon\|.$$

Since the set $\{x_\varepsilon : \varepsilon > 0\}$ is bounded, this inequality, along with (2.16), shows that $(T(t))_{t\geq 0}$ is a semigroup of class C_0. In order to end the proof, we have to show that the infinitesimal generator, $B : D(B) \subseteq X \to X$ of this semigroup coincides with the operator $A : D(A) \subseteq X \to X$. To this end, let $x \in D(A)$ and $h > 0$. We have

$$\lim_{\lambda \to +\infty} e^{tA_\lambda}A_\lambda x = T(t)Ax$$

uniformly on compact subsets in $\mathbb{R}^+$. Indeed,

$$\begin{aligned}\left\|e^{tA_\lambda}A_\lambda x - T(t)Ax\right\| &\leq \left\|e^{tA_\lambda}A_\lambda x - e^{tA_\lambda}Ax\right\| + \left\|e^{tA_\lambda}Ax - T(t)Ax\right\| \\ &\leq \left\|e^{tA_\lambda}\right\|\|A_\lambda x - Ax\| + \left\|e^{tA_\lambda}Ax - T(t)Ax\right\|.\end{aligned}$$

But this relation, along with (2.15) and with the partial conclusions above, proves that

$$T(h)x - x = \lim_{\lambda \to +\infty} (e^{hA_\lambda}x - x) = \lim_{\lambda \to +\infty} \int_0^h e^{tA_\lambda} A_\lambda x \, dt = \int_0^h T(s)Ax \, ds.$$

Dividing both sides of this equality by h and letting h tend to 0 from the right, we deduce that $X \in D(B)$ and $Bx = Ax$. It remains to show that $D(B) = D(A)$. Since B is the infinitesimal generator of a C_0-semigroup of contractions, from the necessity it follows that $1 \in \rho(B)$. Accordingly $I-B$ is invertible and $(I-B)^{-1}X = D(B)$. As $(I-B)^{-1}D(A) = (I-A)D(A)$ and by (b), $(I-A)D(A) = X$, thus it follows that $(I-B)D(A) = X$, that is, $(I-B)^{-1}X = D(A)$. This shows that $D(A) = D(B)$ and completes the proof of Theorem 2.4.1. ☐

2.4.2 Lumer–Phillips theorem

Let $(X, \|\cdot\|)$ be a real Banach space and X^* be its dual space. We remind (see Chapter 1, Section 1.10) that the normalized duality mapping $J : X \to 2^{X^*}$ is defined, for each $x \in X$, by

$$J(x) := \left\{x^* \in X^*;\ \langle x, x^* \rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}$$

Definition 2.4.1. A linear operator $A : D(A) \subset X \to X$ is dissipative if, for every $x \in D(A)$, there is $x^* \in J(x)$ such that $\langle Ax, x^* \rangle \leq 0$.

A dissipative operator A is called m-dissipative provided $\rho(A) \cap (0, \infty) \neq \emptyset$. An operator B is accretive or m-accretive if $A = -B$ is dissipative or m-dissipative.

Proposition 2.4.1. *A linear operator $A : D(A) \subset X \to X$ is dissipative if and only if, for each $x \in D(A)$ and $\lambda > 0$, we have*

$$\left\|(\lambda I - A)x\right\| \geq \lambda\|x\|. \tag{2.17}$$

Proof. If A is dissipative, then, for each $x \in D(A)$ and $\lambda > 0$, there exists $x^* \in J(x)$ such that $\langle Ax, x^* \rangle \leq 0$. Therefore,

$$\lambda\|x\|^2 \leq \langle \lambda x - Ax, x^* \rangle \leq \left|\langle \lambda x - Ax, x^* \rangle\right| \leq \|\lambda x - Ax\|\|x\|.$$

This implies (2.17). Conversely, let $x \in D(A)$ and assume that (2.17) holds true for all $\lambda > 0$. Then

$$\|x\| \leq \|x - \eta Ax\| = \left\|x + \eta(-Ax)\right\| \quad \text{for all } \eta > 0, \tag{2.18}$$

thus, by Lemma 1.10.1, we conclude that there exists $x^* \in J(x)$ such that $\langle x^*, Ax \rangle \leq 0$. ☐

Theorem 2.4.2 (Hille–Yosida theorem: Lumer–Philips form). *Let A be a linear operator with dense domain $D(A)$ in X. Then A generates a C_0-semigroup of contractions on X if and only if*

(a) *A is dissipative and*
(b) *there is a $\lambda > 0$ such that $\lambda I - A$ is surjective.*

Moreover, if A generates a C_0-semigroup of contractions, then $\lambda I - A$ is surjective for any $\lambda > 0$, and we have $\langle Ax, x^* \rangle \leq 0$ for each $x \in D(A)$ and each $x^* \in J(x)$.

Proof. (Necessity). If A is the infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t \geq 0}$, by Hille–Yosida Theorem 2.4.1, we have $(0, +\infty) \subset \rho(A)$ and therefore $\lambda I - A$ is surjective for all $\lambda > 0$. Next, if $x \in D(A)$ and $x^* \in J(x)$, we have

$$\left|\left\langle T(t)x, x^* \right\rangle\right| \leq \|x^*\| \|T(t)x\| \leq \|x\|^2,$$

and then

$$\left\langle T(t)x - x, x^* \right\rangle \leq \left\langle T(t)x, x^* \right\rangle - \|x\|^2 \leq 0.$$

Dividing by $t > 0$ and letting $t \to 0$, we deduce $\langle Ax, x^* \rangle \leq 0$, which completes the proof of the necessity.

(*Sufficiency*) Since A is dissipative, for each $\lambda > 0$ and $x \in D(A)$, we have

$$\|(\lambda I - A)x\| \geq \lambda \|x\|. \tag{2.19}$$

Since $\lambda I - A$ is surjective for some λ, say λ_0, it follows from (2.19) that $\lambda_0 I - A$ has a bounded inverse, and thus it is closed. Hence A is closed. Next we shall prove that $\lambda_0 I - A$ is surjective for each $\lambda > 0$. To see this, let us consider the set

$$\Xi = \{\lambda \in \mathbb{R} : \lambda > 0 \text{ and } \lambda I - A \text{ is surjective}\}.$$

So, it suffices to check that

$$\Xi = (0, +\infty) \tag{2.20}$$

By (2.19), it follows that $\Xi \subset \rho(A)$. Since $\rho(A)$ is open, it follows that, for each $\lambda \in \Xi$, there exists an open neighborhood of λ, $V \subset \mathbb{C}$, which is contained in $\rho(A)$. The intersection of V with the real line is clearly included in Ξ, and accordingly V is open. To complete the proof of (2.20), let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in Ξ with $\lim_{n \to +\infty} \lambda_n = \lambda > 0$. Then, for each $n \in \mathbb{N}$ and $y \in X$, there exists $x_n \in D(A)$ so that

$$\lambda_n x_n - Ax_n = y. \tag{2.21}$$

It follows from (2.19) that there exists $M > 0$ such that $\|x_n\| \leq \frac{1}{\lambda}\|y\| \leq M$. Next, using again (2.19), we get

$$\begin{aligned} \lambda_m \|x_n - x_m\| &\leq \|\lambda_m (x_n - x_m) - A(x_n - x_m)\| = |\lambda_n - \lambda_m| \|x_n\| \\ &\leq M |\lambda_n - \lambda_m|. \end{aligned}$$

Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $x = \lim_{n\to+\infty} x_n$ and note that, by (2.21), we have $\lim_{n\to+\infty} Ax_n = \lambda x - y$. Since A is closed, if follows that $x \in D(A)$ and $\lambda x - Ax = y$. This proves that $\lambda \in \Xi$, and so Ξ is both open and closed and, since $\lambda_0 \in \Xi$ by assumption, $\Xi \neq \emptyset$ and therefore $\Xi = (0, +\infty)$, which completes the proof. ☐

2.5 Hille–Yosida exponential formula

We have established in Theorem 2.2.1 that if A is a bounded linear operator, then A generates a strongly continuous semigroup (in fact, a uniformly continuous semigroup) $(T(t))_{t\geq 0}$ where $T(t) = e^{tA}$ for each $t \geq 0$. In the case where A is unbounded, in the proof of Theorem 2.4.1 we have seen that $T(t)x = \lim_{\lambda\to+\infty} e^{tA_\lambda}x$. In this section we shall give one more result of the same nature.

Theorem 2.5.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of contractions on a Banach space X. If A is the infinitesimal generator of $(T(t))_{t\geq 0}$, then*

$$T(t)x = \lim_{n\to+\infty}\left(I - \frac{t}{n}A\right)^{-n} x = \lim_{n\to+\infty}\left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^{n} x, \quad \text{for } x \in X,$$

and the limit is uniform in t on bounded intervals.

Proof. Assume that $\|T(t)\| \leq 1$. We already know (cf. (2.12)) that, for $\lambda > 0$,

$$R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda s}T(s)x\,ds, \quad \text{for } x \in X. \tag{2.22}$$

Differentiating (2.22) n times with respect to λ, putting $s = vt$, and taking $\lambda = \frac{n}{t}$, we obtain

$$R\left(\frac{n}{t}, A\right)^{(n)} x = (-1)^n t^{n+1} \int_0^{+\infty} (ve^{-v})^n T(tv)x\,dv.$$

But

$$R(\lambda, A)^{(n)} = (-1)^n n! R(\lambda, A)^{n+1}$$

and then

$$\left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^{n+1} x = \frac{n^{n+1}}{n!} \int_0^{+\infty} (ve^{-v})^n T(tv)x\,dv.$$

Using the fact that

$$\frac{n^{n+1}}{n!}\int_0^{+\infty}(ve^{-v})^n\,dv = 1,$$

we obtain

$$\left(\frac{n}{t}R\left(\frac{n}{t},A\right)\right)^{n+1}x - T(t)x = \frac{n^{n+1}}{n!}\int_0^{+\infty}(ve^{-v})^n[T(vt)x - T(t)x]\,dv. \tag{2.23}$$

By Proposition 2.1.3, for each $x \in X$, the function $t \mapsto T(t)x$ is continuous, so, for $t_0 > 0$, the latter function is uniformly continuous in $[0,t_0]$. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that, if $|t-s| \le \delta$ with $t,s \in [0,t_0]$, then $\|T(s)x - T(t)x\| < \varepsilon$. In particular, if $v \in \mathbb{R}$ is such that $|v-1| \le \frac{\delta}{t_0}$, then $\|T(tv)x - T(t)x\| \le \varepsilon$. Hence, there exist $a,b \in \mathbb{R}$ with $0 < a < 1 < b$ such that if $v \in [a,b]$, then

$$\|T(tv)x - T(t)x\| \le \varepsilon.$$

So, we break the integral on the right-hand side of (2.23) into three integrals I_1, I_2, and I_3 on the intervals $[0,a]$, $[a,b]$, and $[b,\infty)$, respectively. We obtain

$$\|I_1\| \le \frac{n^{n+1}}{n!}(ae^{-a})^n\int_0^a\|T(vt)x - T(t)x\|\,dv,$$

$$\|I_2\| \le \varepsilon\frac{n^{n+1}}{n!}\int_a^b(ve^{-v})^n\,dv < \varepsilon,$$

and

$$\|I_3\| = \left\|\frac{n^{n+1}}{n!}\int_b^{+\infty}(ve^{-v})^n(T(tv)x - T(t)x)\,dv\right\|.$$

Here we used the fact that $ve^{-v} \ge 0$ is monotonically nondecreasing for $0 \le v \le 1$ and nonincreasing on $v \ge 1$. Since, moreover, $ve^{-v} < e^{-1}$ for $v \ne 1$, $\|I_1\| \to 0$ uniformly in $t \in [0,t_0]$ as $n \to +\infty$. Taking $n \ge 1$ in I_3, it is easy to see that $\|I_3\| \to 0$ uniformly in $t \in [0,t_0]$ as $n \to +\infty$. Consequently,

$$\limsup_{n\to+\infty}\left\|\left(\frac{n}{t}R\left(\frac{n}{t},A\right)\right)^{n+1}x - T(t)x\right\| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{n\to+\infty}\left(\frac{n}{t}R\left(\frac{n}{t},A\right)\right)^{n+1}x = T(t)x.$$

Using (2.14), we get

$$\lim_{n\to+\infty} \frac{n}{t} R\left(\frac{n}{t}, A\right)x = x,$$

and thus we obtain the result. □

2.6 Bibliographical remarks

Extensive and deep treatment of linear strongly continuous semigroups may be found in the books by K. J. Engel and R. Nagel [94], J. A. Goldstein [119], E. Hille and R. S. Phillips [123], as well as A. Pazy [195].

3 Accretive operators in Banach spaces

3.1 Introduction

The evolution with respect to the time of a physical system is usually described through an initial value problem of a differential equation. Suppose that $u(t)$ describes the state of the physical system at time t and assume that the change of the system is given by a certain function A of the state $u(t)$. Then, if the initial state comes given by $u(0) = f$, the system will be governed by the equation

$$\begin{cases} u'(t) = A(u(t)), & t \geq 0, \\ u(0) = f, \end{cases} \tag{3.1}$$

where X is a Banach space, $u : [0, \infty[\ \to X$ is a function, and A is an operator from $D(A) \subset X$ into X.

The equation $u'(t) = A(u(t))$ means that, for all $t \geq 0$, $u(t) \in D(A)$ and

$$\lim_{h \to 0} \left\| \frac{u(t+h) - u(t)}{h} - A\big(u(t)\big) \right\| = 0.$$

3.1.1 Linear case

Example 3.1.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator on Ω. Consider the initial boundary value problem (the heat equation),

$$\begin{cases} \frac{\partial w}{\partial t} = \Delta w & \text{in } [0, \infty[\times \Omega, \\ w(0, x) = f(x), & \\ w(t, x) = 0 & \forall t \geq 0, \ x \in \partial\Omega. \end{cases} \tag{3.2}$$

We look for a function $w : [0, \infty[\times \bar{\Omega} \to \mathbb{R}$ satisfying (3.2).

Take the Banach space $X := \mathcal{C}(\bar{\Omega})$, call $u(t) := w(t, \cdot)$ and define the operator A by

$$\begin{cases} A : D(A) \to \mathcal{C}(\bar{\Omega}), \\ \quad v \mapsto Av := \Delta v, \\ D(A) = \{v \in X : v \text{ is twice differentiable, } \Delta v \in X, \text{ and } v(x) = 0 \ \forall x \in \partial\Omega\}. \end{cases}$$

Thus, we have transformed problem (3.2) into a problem of type (3.1), that is,

$$\begin{cases} u'(t) = A(u(t)), & t \geq 0, \\ u(0) = f. \end{cases}$$

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Definition 3.1.1. The abstract Cauchy problem (3.1) is said to be *well-posed* if
1. (*Existence*) it has a solution;
2. (*Uniqueness*) the solution is unique;
3. (*Stability*) the solution depends continuously on the initial data.

If a problem is not well-posed, then it is called ill-posed.

Suppose that problem (3.1) is well-posed. If $T(t)$ sends the solution $u(s)$ at time s to the solution $u(s+t)$ at time $t+s$, to say that A does not depend on the time means that $T(t)$ does not depend on time s hence $u(s+t)$ can be computed either as $T(t+s)f$ or we can take $u(s) = T(s)f$ as an initial data and solve problem (3.1). Thus we get the identity $u(s+t) = T(s)(T(t)f)$. The uniqueness of the solution gives the semigroup conditions (cf. Chapter 2):
- $T(t+s) = T(t)T(s)$ $\forall t, s \geq 0$,
- $T(0) = I$, where I denotes the identity operator.

Theorem 3.1.1. *Suppose that $A : D(A) \subseteq X \to X$ is a closed, densely defined linear operator with $\rho(A) \neq \emptyset$. Then, problem* (3.1) *is well-posed for every initial data in $D(A)$ if and only if A is the generator of a C_0-semigroup, $(T(t))_{t\geq 0}$. Moreover, in this case for every initial data $f \in D(A)$ the unique solution of problem* (3.1) *is given by $u(t) := T(t)f$.*

Proof. Assume that A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Given $f \in D(A)$, Proposition 2.1.4(c) shows that $t \mapsto u(t) := T(t)f$ is a solution to problem (3.1). Now, if we suppose that there exists $x_0 \in D(A)$ such that problem

$$\begin{cases} u'(t) = A(u(t)), & t \geq 0, \\ u(0) = x_0 \end{cases} \tag{3.3}$$

admits another solution, say $v_{x_0}(\cdot)$. We may argue by contradiction as follows. Define the operator $S : [0, +\infty[\times X \to X$ by

$$S(t,y) := \begin{cases} T(t)y, & y \neq x_0, \\ v_{x_0}(t), & y = x_0, \end{cases} \tag{3.4}$$

It is not hard to show that $(S(t))_{t\geq 0}$ is a C_0-semigroup and since

$$\lim_{t\to 0^+} \frac{S(t,x_0) - x_0}{t} = \lim_{t\to 0^+} \frac{v_{x_0}(t) - v_{x_0}(0)}{t} = v'_{x_0}(0) = Ax_0,$$

it is clear that A is its generator. Next, with the help of Proposition 2.3.2, we see that both semigroups are the same and therefore problem (3.3) has a unique solution.

On the other hand, if problem (3.1) has a unique solution for every initial data in $D(A)$, we will see that A is the generator of a C_0-semigroup. We call $u(t,x)$ the unique solution of problem (3.1) with initial data $x \in D(A)$.

For $x \in D(A)$, we consider the graph norm of x defined by $\|x\|_A := \|x\| + \|Ax\|$. Since $\rho(A) \neq \emptyset$, A is closed and hence $(D(A), \|\cdot\|_A)$ is a Banach space. Let X_{t_0} be the Banach space

$$(C([0,t_0];(D(A),\|\cdot\|_A)),\|\cdot\|_\infty).$$

Let $S : D(A) \to X_{t_0}$ be the mappings defined by $S(x) : [0,t_0] \to D(A)$ with $S(x)(t) := u(t,x)$. From the linearity of problem (3.1) and the uniqueness of the solution, S is clearly a linear operator defined on the whole $D(A)$.

We claim that S is closed. Indeed, if $x_n \to x$ in $(D(A), \|\cdot\|_A)$ and $Sx_n \to v$ in X_{t_0}, then it follows from the closedness of A and the equation

$$u(t,x_n) = x_n + \int_0^t Au(s,x_n)\,ds$$

that, as $n \to +\infty$, we have

$$v(t) = x + \int_0^t Av(s)\,ds.$$

This yields that $v(t) = u(t,x)$ for all $t \in [0,t_0]$, which shows the closedness of S, as claimed.

Next, by the closed graph theorem (Theorem 1.5.4), S is bounded, and there exists a constant $C \geq 1$ such that

$$\sup\{\|u(t,x)\|_A : 0 \leq t \leq t_0\} \leq C\|x\|_A. \tag{3.5}$$

The above argument allows us to define a mapping $T : [0,\infty) \times D(A) \to D(A)$ by $T(t,x) := T(t)x := u(t,x)$. The uniqueness of the solution says that T enjoys the semigroup properties. By (3.5), we know that $T(t)$ is uniformly bounded whenever $0 \leq t \leq t_0$. Consider the real number $\omega = t_0^{-1}\log(C)$. For any $t > t_0$, there exist $n \in \mathbb{N}$ and $0 \leq \delta < t_0$ such that $t = nt_0 + \delta$, where $n \in \mathbb{N}$ and $0 \leq \delta < t_0$, and, therefore, by the semigroup property, we have $T(t) = T(nt_0 + \delta) = T(\delta)T^n(t_0)$. Hence

$$\|T(t)x\|_A = \|T(\delta)T^n(t_0)x\|_A \leq C^{n+1}\|x\|_A \leq CC^{\frac{t}{t_0}}\|x\|_A = Ce^{\omega t}\|x\|_A.$$

Now we claim that

$$T(t)Ay = AT(t)y \quad \text{for } y \in D(A^2). \tag{3.6}$$

Writing $w(t) = y + \int_0^t u(s,Ay)\,ds$, we derive that

$$w'(t) = u(t, Ay) = Ay + \int_0^t u'(s, Ay)\,ds = A\left(y + \int_0^t u(s, Ay)\,ds\right) = Aw(t).$$

Since $w(0) = y$, we obtain by the uniqueness of the solution that $w(t) = u(t, y)$, and therefore $Au(t, y) = w'(t) = u(t, Ay)$, which proves our claim.

Now, since $D(A)$ is dense in X and by hypothesis $\rho(A) \neq \emptyset$, $D(A^2)$ is also dense in X. Let $\lambda_0 \in \rho(A)$ be fixed and let $y \in D(A^2)$. If $x = (\lambda_0 I - A)(y)$, by (3.6), $T(t)x = (\lambda_0 - A)T(t)y$ and then

$$\|T(t)x\| = \|(\lambda_0 I - A)T(t)y\| \leq \lambda_0\|T(t)y\| + \|AT(t)y\| \leq K\|T(t)y\|_A \leq KCe^{\omega t}\|y\|_A,$$

but, since $x = (\lambda_0 I - A)(y)$, we have that $\|y\|_A = \|y\| + \|Ay\| \leq K_1\|x\|$. Hence,

$$\|T(t)x\| \leq KCK_1 e^{\omega t}\|x\|.$$

Therefore, $T(t)$ can be extended, by continuity, to the whole space X. It is clear that such an extension is a C_0-semigroup.

In order to finish the proof, we only have to show that A is the generator of $(T(t))$. Let A_1 be the generator of the semigroup $(T(t))$. If $x \in D(A)$, by the definition of $T(t)$, we have that $T(t)x = u(t, x)$ and hence

$$T'(t)x = AT(t)x, \quad \text{for } t \geq 0,$$

which implies that $T'(0)x = Ax$ and then $A \subseteq A_1$.

Let $\operatorname{Re}\lambda > w$ and $y \in D(A^2)$. From (3.6) and since $A \subseteq A_1$, we infer that

$$e^{-\lambda t}AT(t)y = e^{-\lambda t}T(t)Ay = e^{-\lambda t}T(t)A_1(y).$$

Integrating the above equality from 0 to ∞, we obtain (see the proof of Theorem 2.4.1)

$$AR(\lambda, A_1)y = R(\lambda, A_1)A_1 y.$$

But $A_1 R(\lambda, A_1)y = R(\lambda, A_1)A_1 y$ and thus $AR(\lambda, A_1)y = A_1 R(\lambda, A_1)y$ for every $y \in D(A^2)$. Since $A_1 R(\lambda, A_1)$ are uniformly bounded, A is closed and $D(A^2)$ is dense in X, we have that $AR(\lambda, A_1)y = A_1 R(\lambda, A_1)y$ for every $y \in X$. This implies that $D(A_1) = \mathrm{R}(R(\lambda, A_1)) \subseteq D(A)$ and $A_1 \subseteq A$. Then $A = A_1$. ☐

At this point we can consider the following question: What conditions does an operator A have to fulfill to be a generator of a C_0-semigroup of contractions? This question was answered in Theorems 2.4.1 and 2.4.2.

Theorem 3.1.2 (Hille–Yosida: Lumer–Philips form). *A linear operator A is the generator of a contraction C_0-semigroup in X (that is, $\|T(t)\| \leq 1$ for each $t \geq 0$) if and only if $D(A)$ is dense in X and the following conditions hold:*

(i) $(I - \alpha A)(D(A)) = X$, *for all* $\alpha > 0$,
(ii) $\|(I - \alpha A)^{-1}x\| \leq \|x\|$, *for all* $x \in X$ *and* $\alpha > 0$.

Evidently, under the conditions of Theorem 3.1.2, the associated semigroup to the operator A can be built by using several methods; one of them is the *Hille–Yoshida exponential formula* (cf. Theorem 2.5.1)

$$T(t)f = \lim_{n\to+\infty}\left(I - \frac{t}{n}A\right)^{-n} f.$$

3.1.2 Nonlinear case

Example 3.1.2. Consider the one-dimensional Hamilton–Jacobi equation

$$\begin{cases} \frac{\partial w}{\partial t} + H(\frac{\partial w}{\partial x}) = 0, & t \geq 0,\ x \in \mathbb{R}, \\ w(0,x) = f(x), \end{cases} \tag{3.7}$$

where $H : \mathbb{R} \to \mathbb{R}$ is a smooth function.

Let X be the space of continuous functions, denote $u(t) := w(t,\cdot)$ and define the operator

$$\begin{cases} A : D(A) \subset X \to X, \\ \quad v \mapsto Av := -H(v'), \\ D(A) := \{v \in X : v' \in X\}. \end{cases}$$

The above argument allows us to transform the 1-dimensional Hamilton–Jacobi equation into a problem of type (3.1). ☐

In 1971, M. G. Crandall and T. Liggett [70] noticed that in the theory of contraction C_0-semigroups, the linearity is not relevant in the sense that it is possible to establish a theorem of Hille–Yosida type for nonlinear operators. In this case, the main thing is the concept of a generator.

Let $A : D(A) \subseteq X \to X$ be an operator on a Banach space X and define the function $\|\cdot\|_{\text{lip}}$ which assigns to each operator the real number

$$\|A\|_{\text{lip}} := \sup\left\{\frac{\|Af - Ag\|}{\|f - g\|} : f \neq g, \quad f, g \in D(A)\right\}.$$

It is clear that if A is a bounded linear operator, then $\|A\| = \|A\|_{\text{lip}}$.

If we consider the abstract Cauchy problem (3.1) where A is a nonlinear operator, then a way to solve problem (3.1) consists in replacing $u'(t)$ by $\frac{u(t+s)-u(t)}{t}$ and so we obtain the difference equation

$$\begin{cases} \frac{1}{\varepsilon}(u_\varepsilon(t) - u_\varepsilon(t-\varepsilon)) = A(u_\varepsilon(t)), \\ u_\varepsilon(s) = f, \quad -\varepsilon \le s \le 0. \end{cases} \tag{3.8}$$

This equation can be easily solved, and its solution is given by

$$u_\varepsilon(t) = (I - \varepsilon A)^{-1}\big(u_\varepsilon(t-\varepsilon)\big).$$

Taking $\varepsilon = \frac{t}{n}$, we obtain

$$\begin{aligned} u_\varepsilon(t) &= (I - \varepsilon A)^{-1}\big(u_\varepsilon(t-\varepsilon)\big) = (I - \varepsilon A)^{-2}\big(u_\varepsilon(t-2\varepsilon)\big) \\ &= (I - \varepsilon A)^{-n}\big(u_\varepsilon(0)\big) \\ &= \left(I - \frac{t}{n}A\right)^{-n} f. \end{aligned}$$

Moreover, we wish that

$$\lim_{\varepsilon \to 0^+} u_\varepsilon(t) = \lim_{n \to +\infty} \left(I - \frac{t}{n}A\right)^{-n} f.$$

Therefore, the abstract Cauchy problem (3.1) will be well-posed if

(1) $(I - \alpha A)D(A) = X, \forall \alpha > 0$;

(2) $\|(I - \alpha A)^{-1}\|_{\mathrm{lip}} \le 1, \forall \alpha > 0$.

Notice that in the linear case the above two conditions are just to ensure that A is the generator of a C_0-semigroup of contractions.

Remark 3.1.1. If the equation can be written in the form

$$\begin{cases} u'(t) + A(u(t)) = 0, \\ u(0) = f, \end{cases} \tag{3.9}$$

then the above argument leads us to the following conditions:

(1′) $(I + \alpha A)D(A) = X, \forall \alpha > 0$;

(2′) $\|(I + \alpha A)^{-1}\|_{\mathrm{lip}} \le 1, \forall \alpha > 0$.

In the nonlinear case we often have to work with multivalued operators since if, for example, we consider the operator

$$A : \mathbb{R} \to \mathbb{R}, \quad r \mapsto A(r) = \begin{cases} 1, & r > 0, \\ 0, & r = 0, \\ -1, & r < 0, \end{cases}$$

problem $u'(t) = A(u(t))$ does not have any solution. Indeed, if it were to admit a solution, then it could not be continuous. Nevertheless, if we take the multivalued operator

$$\tilde{A} : \mathbb{R} \to 2^{\mathbb{R}}, \quad r \mapsto \tilde{A}(r) = \begin{cases} 1, & r > 0, \\ I, & r = 0, \\ -1, & r < 0, \end{cases}$$

where $I = [0,1]$, then problem $u'(t) = \tilde{A}(u(t))$, as we shall see later, admits a solution.

3.2 Accretive operators

In this section we introduce the most important properties of accretive operators defined on a Banach space in order to study the generation of semigroups of nonexpansive mappings. The concept of an accretive operator was introduced in 1967 by F.E. Browder [51] and T. Kato [139] independently and it has a very simple origin. If D is a subset of $\mathbb{R}$ and $\phi : D \to \mathbb{R}$ is a function, then ϕ is monotone increasing if and only if

$$(s - t)\big(\phi(s) - \phi(t)\big) \geq 0.$$

This concept can be generalized to a Banach spaces as follows.

Definition 3.2.1. A mapping $T : D \subset X \to X^*$ is said to be monotone if, for each $u, v \in D$,

$$\langle u - v, Tu - Tv\rangle \geq 0.$$

This type of mappings was introduced by G. Minty [177] in 1962. A natural analogue of Definition 3.2.1 for mappings taking values in a Banach space X is the following.

Definition 3.2.2. A mapping $T : D \subset X \to X$ is said to be accretive if, for all $u, v \in D$, there exists some $j \in J(u - v)$ such that

$$\langle Tu - Tv, j\rangle \geq 0.$$

As we mentioned in the introduction, in the nonlinear case it is interesting to work with multivalued operators.

Definition 3.2.3. A mapping A from X into 2^X is called a multivalued operator.

We shall identify operators with graphs, i. e., if $A \subset X \times X$, A will be considered as the operator

$$Af := \{g \in X : (f, g) \in A\}$$

and, reciprocally, if $A : X \to 2^X$ is an operator, it will be identified with its graph

$$A := \{(f, g) : g \in Af\} = \operatorname{Gr}(A).$$

When A is a multivalued operator, we define:

(a) the effective domain of A,

$$D(A) := \{f \in X : Af \neq \emptyset\};$$

(b) the range of A,

$$\mathrm{R}(A) := \{Af : f \in D(A)\};$$

(c) given two multivalued operators A, B,

$$\begin{aligned} A + B &:= \{(f, g + h) : (f, g) \in A, (f, h) \in B\}, \\ \lambda A &:= \{(f, \lambda g) : (f, g) \in A\}, \\ A^{-1} &:= \{(g, f) : (f, g) \in A\}. \end{aligned}$$

Definition 3.2.4. An operator $A : D(A) \subset X \to 2^X$ is said to be:

(a) *accretive* if for each $(x, u), (y, v) \in A$ there exists $x^* \in J(x-y)$ such that $\langle x^*, u-v\rangle \geq 0$. Equivalently, $\langle u - v, x - y\rangle_+ \geq 0$;
(b) *maximal accretive* if A is accretive and the inclusion $A \subset B$, with B being accretive, implies that $A = B$;
(c) *m-accretive* if A is accretive and $\mathrm{R}(A + I) = X$;
(d) *dissipative (m-dissipative)* if $-A$ is accretive (m-accretive).

Many examples will be given below and in Chapter 5 where the accretiveness property of an operator will be used in several equivalent forms. In the following remark, we give two formulations of the accretiveness property of an operator equivalent to that given above.

Remark 3.2.1. According to Proposition 1.10.3 and Definition 3.2.4(a), an equivalent way to define the accretiveness of an operator $A : D(A) \subset X \to 2^X$ is

$$[x - x', y - y']_s \geq 0, \quad \forall (x, y), (x', y') \in A. \tag{3.10}$$

An equivalent metric formulation of the accretiveness will be given in Theorem 3.4.1.

3.3 Examples of accretive operators

Example 3.3.1. If H is a Hilbert space, it is well known that $H = H^*$, and, moreover, the normalized duality mapping is the identity mapping. So, the class of monotone operators and the class of accretive operators coincide (see, for example, [45, 64]). Thus, every monotone increasing function of real variable will be an accretive operator.

Example 3.3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function. We can define a multivalued operator $A : \mathbb{R} \to 2^{\mathbb{R}}$ as

$$Ax := [f(x^-), f(x^+)],$$

where $f(x^-) := \lim_{y\to x^-} f(y)$ and $f(x^+) := \lim_{y\to x^+} f(y)$.

Let us see that A is an accretive operator. Bearing in mind the form of the discontinuities of an increasing function, we know

$$f(x^-) \le f(x) \le f(x^+).$$

Given $x, y \in \mathbb{R}$ with $x < y$ and taking $u \in Ax$, $v \in Ay$, we have

$$u \in [f(x^-), f(x^+)] \quad \text{and} \quad v \in [f(y^-), f(y^+)].$$

Since f is increasing, we have $u \le f(x^+) \le f(y^-) \le v \le f(y^+)$ and therefore $u \le v$. Accordingly,

$$(v - u)(y - x) \ge 0.$$

This proves that A is accretive.

Let H be a Hilbert space and let $\varphi : H \to]-\infty, +\infty]$ be a proper convex function, that is, $D(\varphi) := \{x \in H : \varphi(x) < +\infty\} \ne \emptyset$. Then $D(\varphi)$ is the effective domain of φ and, as a consequence of the convexity, it is always a convex set. The subdifferential of φ is, in general, a multivalued operator $\partial\varphi : D(\varphi) \to 2^H$ defined as (cf. Section 1.11)

$$\partial\varphi(x) := \{y \in H : \langle y, z - x\rangle \le \varphi(z) - \varphi(x) \quad \forall z \in H\}, \quad \forall x \in D(\phi).$$

Example 3.3.3. The subdifferential $\partial\varphi$ is an accretive operator. To see this, take $x_i \in H$, $i = 1, 2$ and consider $y_i \in \partial\varphi(x_i)$, $i = 1, 2$. By the definition of subdifferential, we have

$$\langle y_i, z - x_i\rangle \le \varphi(z) - \varphi(x_i) \quad i = 1, 2, \quad \forall z \in H.$$

In particular,

$$\langle y_1, x_2 - x_1\rangle \le \varphi(x_2) - \varphi(x_1),$$
$$\langle y_2, x_1 - x_2\rangle \le \varphi(x_1) - \varphi(x_2).$$

Adding up the above two expressions, we obtain

$$\langle y_1, x_2 - x_1\rangle + \langle y_2, x_1 - x_2\rangle \le 0.$$

Hence,

$$\langle y_2 - y_1, x_2 - x_1\rangle \ge 0.$$

Example 3.3.4. A linear mapping $A : X \to X$ is an accretive operator if and only if $\forall x \in D(A)$ there exists $x^* \in J(x)$ such that $\langle x^*, Ax\rangle \geq 0$.

($\Longrightarrow$) Since A is accretive, given $x, y \in D(A)$, there exists $j \in J(x-y)$ such that $\langle Ax - Ay, j\rangle \geq 0$. The linearity of A says that $Ax - Ay = A(x-y)$ and then

$$\langle A(x-y), j\rangle \geq 0.$$

Since $D(A)$ is a subspace of X and if we take $x \in D(A)$ and $y = 0$, there is $x^* \in J(x)$ with $\langle x^*, Ax\rangle \geq 0$.

($\Longleftarrow$) Consider $x, y \in D(A)$ and let $z = x - y$. Then there exists $z^* \in J(z)$ such that $\langle z^*, A(z)\rangle \geq 0$, and, by the linearity of A, we derive that

$$\langle z^*, Ax - Ay\rangle \geq 0.$$

This shows that A is accretive.

Example 3.3.5. Let Ω be an open bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and consider the space $L^p(\Omega)$ where $1 < p < +\infty$. According to Theorem 1.9.2, $L^p(\Omega)$ is a uniformly convex and also uniformly smooth Banach space. Moreover, its normalized duality mapping is given by

$$J : L^p(\Omega) \to L^q(\Omega), \quad J(u)(x) = \frac{u(x)|u(x)|^{p-2}}{\|u\|_p^{p-2}}, \qquad \text{a.e. } x \in \Omega,$$

where q is the conjugate exponent of p (for the duality mapping for L^p-spaces, see, for example, [32, p. 4] or [47]).

Define on $L^p(\Omega)$ the following operator:

$$D(A) := W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega).$$
$$Au = -\Delta u, \quad u \in D(A).$$

Let us see that A is an accretive operator. We just have to show that

$$\int_\Omega (\Delta v(x) - \Delta u(x))(u(x) - v(x))\big|u(x) - v(x)\big|^{p-2}\,dx \geq 0. \tag{3.11}$$

This is a consequence of Green's formula (see Theorem 1.17.7). If N denotes the outer unit normal on $\partial\Omega$ and we denote $\frac{\partial}{\partial n}u = \langle \nabla u, N\rangle$, then the left-hand side of (3.11) can be written as

$$\int_{\partial\Omega} (u-v)|u-v|^{p-2}\frac{\partial}{\partial n}\nabla(v-u)\,dA - \int_\Omega \langle \nabla(v-u), \nabla((u-v)|u-v|^{p-2})\rangle\,dx.$$

The first summand of the above expression is zero since $u, v \in W_0^{1,p}(\Omega)$. Thus, we only have to study the sign of the second summand, which can be written as follows:

$$-\int_{\Omega} -\left\|\nabla(v-u)\right\|^2 (p-1)|u-v|^{p-2}\,dx = \int_{\Omega} \left\|\nabla(v-u)\right\|^2 (p-1)|u-v|^{p-2}\,dx \geq 0.$$

3.4 Properties of accretive operators

In the following results we present a characterization of the accretivity which does not depend of the normalized duality map.

Theorem 3.4.1. *Let X be a Banach space. A multivalued operator* $A : D(A) \to 2^X$ *is accretive if and only if*

$$\left\|x-y+\lambda(u-v)\right\| \geq \|x-y\|$$

for all $\lambda > 0$, $(x,u), (y,v) \in A$.

Proof. ($\Longrightarrow$) If A is an accretive operator, we know that there exists $j \in J(x-y)$ such that $\langle u-v, j\rangle \geq 0$. Hence, applying Lemma 1.10.1, we get

$$\|x-y\| \leq \left\|x-y+\lambda(u-v)\right\| \quad \forall \lambda > 0.$$

($\Longleftarrow$) If $\|x-y\| \leq \|x-y+\lambda(u-v)\|$ $\forall\lambda > 0$, applying again Lemma 1.10.1, we derive the result. ☐

Notations

Let X be a Banach space. For a multivalued operator $A : D(A) \to 2^X$ and $\lambda > 0$, we denote:

- $J_\lambda^A := (I+\lambda A)^{-1}$, *the resolvent of* A (when there is no confusion, the resolvent will be denoted by J_λ);
- $A_\lambda := \frac{1}{\lambda}(I - J_\lambda^A)$, *the Yosida approximant of* A*;*
- $D_\lambda := \mathrm{R}(I+\lambda A)$;
- $|Ax| := \inf\{\|u\| : u \in Ax\}$.

Corollary 3.4.1. *Let X be a Banach space, and let* $A : D(A) \to 2^X$ *be an accretive operator. The following conditions are equivalent:*

(a) *A is accretive.*

(b) $J_\lambda : D_\lambda \to X$ *is a nonexpansive mapping.*

Proof. (a) $\Longrightarrow$ (b) Let us see first that J_λ is a single-valued mapping. Take $x, y \in J_\lambda(x_0)$, where $x_0 \in D_\lambda$. This means

$$x_0 \in x + \lambda Ax, \quad x_0 \in y + \lambda Ay,$$

that is, there exist $u \in Ax$, $v \in Ay$ such that $x_0 = x + \lambda u = y + \lambda v$. By Theorem 3.4.1, we have

$$\|x - y\| \leq \|x - y + \lambda(u - v)\| = \|x + \lambda u - (y + \lambda v)\| = \|x_0 - x_0\| = 0,$$

hence, $x = y$.

Now, we are going to prove that J_λ is nonexpansive. Consider $x_0, x_1 \in D_\lambda$ and denote $y = J_\lambda(x_0)$, $z = J_\lambda(x_1)$. This means that there exist $u \in Ay$ and $v \in Az$ such that

$$x_0 = y + \lambda u \quad \text{and} \quad x_1 = z + \lambda v.$$

Applying Theorem 3.4.1, we obtain

$$\|J_\lambda(x_0) - J_\lambda(x_1)\| = \|y - z\| \leq \|y - z + \lambda(u - v)\| = \|y + \lambda u - (z + \lambda v)\| = \|x_0 - x_1\|.$$

(b) $\Longrightarrow$ (a) Given $x \in D(A)$ and $u \in Ax$, we have that $J_\lambda(x + \lambda u) = x$. Then,

$$\|x - y + \lambda(u - v)\| \geq \|J_\lambda(x + \lambda u) - J_\lambda(y + \lambda v)\| = \|x - y\|,$$

and this holds whenever $(y, v) \in A$, because in this case $J_\lambda(y + \lambda v) = y$. Consequently, by Theorem 3.4.1, we derive the result. ☐

Corollary 3.4.2. *Let X be a Banach space. If $A : D(A) \to 2^X$ is an accretive operator, then its Yosida approximants are also accretive operators.*

Proof. Fix a $\lambda > 0$. Let us show that the Yosida approximant $A_\lambda : D_\lambda \to X$ is accretive. To this end, it is enough to apply Theorem 3.4.1. Indeed, if $x, y \in D_\lambda$, we have

$$\begin{aligned}\|x - y + r(A_\lambda x - A_\lambda y)\| &= \left\|x - y + \frac{r}{\lambda}((x - J_\lambda x) - (y - J_\lambda y))\right\| \\ &= \left\|\left(1 + \frac{r}{\lambda}\right)(x - y) + \frac{r}{\lambda}(J_\lambda y - J_\lambda x)\right\| \\ &\geq \left(1 + \frac{r}{\lambda}\right)\|x - y\| - \frac{r}{\lambda}\|J_\lambda y - J_\lambda x\|\end{aligned}$$

Since A is accretive, it follows from Corollary 3.4.1 that J_λ is nonexpansive, which yields

$$\|x - y + r(A_\lambda x - A_\lambda y)\| \geq \left(1 + \frac{r}{\lambda}\right)\|x - y\| - \frac{r}{\lambda}\|y - x\| \geq \|x - y\|.$$ ☐

Proposition 3.4.1. *Let X be a Banach space. If $A : D(A) \to 2^X$ is an accretive operator, then:*

(a) *For all $\lambda > 0$, A_λ is accretive and $A_\lambda x \in AJ_\lambda x$, $\forall x \in D_\lambda$.*

(b) $\|A_\lambda x - A_\lambda y\| \leq \frac{2}{\lambda}\|x - y\|, \forall x, y \in D_\lambda$.

(c) $\|A_\lambda x\| \leq |Ax|, \forall x \in D_\lambda \cap D(A)$.

(d) $\lim_{\lambda\to 0^+} J_\lambda x = x$, $\forall x \in D(A) \cap (\bigcap_{\lambda>0} D_\lambda)$.

(e) *If* $x \in D_\lambda$, *then, for each* $\gamma > 0$, *we have*

$$\frac{\gamma}{\lambda}x + \left(1 - \frac{\gamma}{\lambda}\right)J_\lambda x \in D_\gamma.$$

Moreover,

$$J_\lambda x = J_\gamma\left(\frac{\gamma}{\lambda}x + \left(1 - \frac{\gamma}{\lambda}\right)J_\lambda x\right).$$

(f) *Given* $x \in \bigcap_{\lambda>0} D_\lambda$, *the function* $\lambda \to J_\lambda x$ *is continuous.*

Proof. (a) The first part of this assertion follows from Corollary 3.4.2. Thus, we are going to prove that if $x \in D_\lambda$, then $A_\lambda x \in AJ_\lambda x$. Let $y = J_\lambda x$, hence $x = y + \lambda u$ with $u \in Ay$. By the definition of Yosida approximant, we have $A_\lambda x = \frac{x - J_\lambda x}{\lambda} = \frac{x - x + \lambda u}{\lambda} = u \in Ay$. Because $y = J_\lambda x$, we conclude that $A_\lambda x \in AJ_\lambda x$.

(b) Let $x, y \in D_\lambda$. Since the resolvents of an accretive operator are nonexpansive mappings, it is clear that

$$\begin{aligned}\|A_\lambda x - A_\lambda y\| &= \frac{1}{\lambda}\|x - y + J_\lambda y - J_\lambda x\| \le \frac{1}{\lambda}(\|x - y\| + \|J_\lambda y - J_\lambda x\|) \\ &\le \frac{1}{\lambda}(\|x - y\| + \|x - y\|) \\ &= \frac{2}{\lambda}\|x - y\|.\end{aligned}$$

(c) Let $x \in D_\lambda \cap D(A)$. It is clear that $Ax \neq \emptyset$, and then, by the definition of J_λ, for every $u \in Ax$, we have $J_\lambda(x + \lambda u) = x$. Next, using the fact that J_λ is nonexpansive, for all $\lambda > 0$, we find

$$\|A_\lambda x\| = \frac{1}{\lambda}\|x - J_\lambda x\| = \frac{1}{\lambda}\big\|J_\lambda(x + \lambda u) - J_\lambda x\big\| \le \frac{1}{\lambda}\|\lambda u\| = \|u\|.$$

This proves the required estimate for all $\lambda > 0$.

(d) Let $x \in D(A) \cap (\bigcap_{\lambda>0} D_\lambda)$. Using the definition of A_λ and the result of (c), one can write

$$\|x - J_\lambda x\| = \lambda\|A_\lambda\| \le \lambda|Ax|.$$

We get the result by letting λ tend towards 0 from the right.

(e) Let $x \in D_\lambda = R(I + \lambda A)$. Then there exists $(x_0, u_0) \in A$ such that $x = x_0 + \lambda u_0$, and thus $J_\lambda x = x_0$. Therefore,

$$\frac{\gamma}{\lambda}x + \left(1 - \frac{\gamma}{\lambda}\right)J_\lambda x = \frac{\gamma}{\lambda}(x_0 + \lambda u_0) + \left(1 - \frac{\gamma}{\lambda}\right)x_0 = x_0 + \gamma u_0 \in D_\gamma.$$

Moreover,

$$J_\gamma\left(\frac{\gamma}{\lambda}x + \left(1 - \frac{\gamma}{\lambda}\right)J_\lambda x\right) = J_\gamma(x_0 + \gamma u_0) = x_0 = J_\lambda x.$$

(f) Considering $\lambda, \gamma > 0$, we have to study what happens with $\|J_\lambda x - J_\gamma x\|$. Applying (e), we derive that

$$\|J_\lambda x - J_\gamma x\| = \left\|J_\gamma\left(\frac{\gamma}{\lambda}x + \left(1 - \frac{\gamma}{\lambda}\right)J_\lambda x\right) - J_\gamma x\right\|.$$

Next, using the nonexpansivity of J_γ, we get

$$\|J_\lambda x - J_\gamma x\| \le \left\|\frac{\gamma}{\lambda}x + \left(1 - \frac{\gamma}{\lambda}\right)J_\lambda x - x\right\| = \left(1 - \frac{\gamma}{\lambda}\right)\|x - J_\lambda x\|,$$

which allows us to achieve the result. ☐

3.5 *m*-accretive operators

Remark 3.5.1. Let X be a Banach space and let $A \subseteq X \times X$ be a maximal accretive operator. If $(x_0, u) \in X \times X$ and for all $(y, v) \in A$ there exists $x^* \in J(x_0 - y)$ such that $\langle u - v, x^*\rangle \ge 0$, then $(x_0, u) \in A$. Indeed, consider the following operator $B : D(A) \cup \{x_0\} \to 2^X$ defined by

$$B(x) = \begin{cases} Ax, & x \in D(A), \\ u, & x = x_0. \end{cases}$$

It is clear that B is accretive and $A \subset B$. Since A is a maximal accretive operator, we conclude that $A = B$.

Proposition 3.5.1. *Let X be a Banach space and let $A : D(A) \to 2^X$ be an m-accretive operator. Then A is maximal accretive and* $\mathrm{R}(I + \lambda A) = X$ *for all* $\lambda > 0$.

Proof. First we shall show that A is maximal accretive. In order to do it, consider an accretive operator B such that $A \subset B$. Now, we can take $(x, u) \in B$ and, since A is m-accretive, there exits $y \in D(A) \subset D(B)$ such that $x + u \in (I + A)y$, hence $x + u = y + v$ for some $v \in Ay \subset By$. The fact that B is accretive, along with Theorem 3.4.1, implies that

$$\|x - y\| \le \|x - y + \lambda(u - v)\| \quad \text{for all } \lambda > 0.$$

Using the fact that $x + u = y + v$ and taking $\lambda = 1$, we get

$$\|x - y\| \le \|x - y + (u - v)\| = 0,$$

i. e., $x = y \in D(A)$. This yields that $u = v \in Ay$, and consequently $(x, u) \in A$. Therefore $A = B$, i. e., A is maximal accretive.

Now, we shall prove that, for all $\lambda > 0$, $\mathrm{R}(I + \lambda A) = X$. Letting $\lambda > 0$ and $u \in X$, we have to show that $u \in \mathrm{R}(I + \lambda A)$. We claim that this is equivalent to

$$\exists x_0 \in D(A): \quad x_0 = J_1\left(\frac{u}{\lambda} - \frac{1-\lambda}{\lambda}x_0\right).$$

Indeed, $u \in \mathrm{R}(I + \lambda A)$ if and only if there exists $y \in D(A)$ such that $u \in (I + \lambda A)y$; which is equivalent to $\exists y \in D(A)$ such that $u \in \frac{\lambda}{\lambda}y + \lambda Ay$. This means that there exists a unique $y \in D(A)$, such that $\frac{u}{\lambda} \in \frac{y}{\lambda} + Ay$ or, after an elementary transformation, $\frac{u}{\lambda} \in y + \frac{1-\lambda}{\lambda}y + Ay$. The latter equation may be written in the form

$$\frac{u}{\lambda} - \frac{1-\lambda}{\lambda}y \in y + Ay,$$

and therefore

$$y = (I + A)^{-1}\left(\frac{u}{\lambda} - \frac{1-\lambda}{\lambda}y\right).$$

This proves our claim.

Next, let B denote the operator defined by

$$Bx := (I + A)^{-1}\left(\frac{u}{\lambda} - \frac{1-\lambda}{\lambda}x\right).$$

Since A is m-accretive, we have $D(B) = X$. Moreover, by Corollary 3.4.1, we have

$$\|Bx - By\| = \left\|(I + A)^{-1}\left(\frac{u}{\lambda} - \frac{1-\lambda}{\lambda}x\right) - (I + A)^{-1}\left(\frac{u}{\lambda} - \frac{1-\lambda}{\lambda}y\right)\right\| \le \left|\frac{1-\lambda}{\lambda}\right| \|x - y\|.$$

Suppose that $\lambda > \frac{1}{2}$. In this case $|\frac{1-\lambda}{\lambda}| < 1$, so, by Banach contraction principle (cf. Theorem 6.2.1), there exists a unique point $x_0 \in X$ such that $Bx_0 = x_0$. In order to obtain the property when $0 < \lambda \le \frac{1}{2}$, we argue as follows: choose $\lambda > \frac{1}{2}$ and define the operator

$$\tilde{A}x = \lambda Ax.$$

By the above characterization, we obtain $\mathrm{R}(I+\gamma\tilde{A}) = X \ \forall \gamma > \frac{1}{2}$. This proves that $\mathrm{R}(I+\lambda A) = X$, $\forall \lambda > \frac{1}{4}$. Repeating this argument for all $n \ge 3$ and $\lambda > \frac{1}{2^n}$, we obtain $\mathrm{R}(I + \lambda A) = X$. This ends the proof. □

Remark 3.5.2. Note that m-accretive operators are just the operators which satisfy both conditions given in Remark 3.1.1. Recall that conditions of Remark 3.1.1 are necessary in order to see that, from a heuristic point of view, the corresponding abstract Cauchy problem (3.9) is well-posed.

As we have seen in Proposition 3.5.1 that every m-accretive operator is, in fact, maximal accretive. Now we will show that in Hilbert spaces these two concepts are equivalent.

Theorem 3.5.1. *Let H be a Hilbert space and let $A : D(A) \subseteq H \to 2^H$ be an accretive operator. Then, for every $y \in H$, there exists $x \in H$ such that*

$$\langle z + x, v - x \rangle \geq \langle y, v - x \rangle, \quad \forall (v, z) \in A.$$

The proof of Theorem 3.5.1 uses the following classical result (its proof is omitted).

Theorem 3.5.2 (minimax). *Let A, B be closed, convex, and bounded subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and $f : A \times B \to \mathbb{R}$. Assume that, for every $y \in B$, $x \mapsto f(x, y)$ is convex and lower semicontinuous and, for every $x \in A$, $y \mapsto f(x, y)$ is concave and upper semicontinuous. Then there exists a point $(a, b) \in A \times B$ such that*

$$f(a, y) \leq f(a, b) \leq f(x, b), \quad \forall (x, y) \in A \times B. \tag{3.12}$$

Proof of Theorem 3.5.1. For each $(v, z) \in A$, define $H(v, z) := \{x \in H : \langle z + x, v - x \rangle \geq \langle y, v - x \rangle\}$. We can notice that $H(v, z)$ is a bounded, closed, and convex subset of H. In order to establish the result, we must show that

$$\bigcap_{(v,z) \in A} H(v, z)$$

is nonempty. Since for each $(v, z) \in A$ the set $H(v, z)$ is weakly compact, it is enough to show that, for any finite family $(v_i, z_i) \in A$, $i = 1, \dots, N$, we can obtain

$$\bigcap_{i=1}^{N} H(v_i, z_i) \neq \emptyset.$$

Let

$$K := \left\{ \lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N : \lambda_i \geq 0,\ i = 1, \dots, N, \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

Clearly, K is a convex, closed, and bounded subset of $\mathbb{R}^N$. Thus, we can define the following mapping:

$$f : K \times K \to \mathbb{R}, \quad f(\lambda, \mu) := \sum_{i=1}^{N} \mu_i \langle x(\lambda) + z_i - y, x(\lambda) - v_i \rangle, \quad \text{where } x(\lambda) = \sum_{i=1}^{N} \lambda_i v_i.$$

It is not difficult to see that f satisfies the hypotheses of Theorem 3.5.2 and so there exists $\lambda_0 \in K$ such that

$$f(\lambda_0, y) \leq \max\{f(\lambda, \lambda) : \lambda \in K\} \quad \text{for all } y \in K.$$

This implies that $x(\lambda_0) \in \bigcap_{i=1}^{N} H(v_i, z_i)$. ☐

Corollary 3.5.1. *Let H be a Hilbert space and let $A : D(A) \to 2^H$ be a multivalued operator. The following conditions are equivalent:*
(a) *A is maximal monotone,*
(b) *A is m-accretive.*

Proof. (b) $\Longrightarrow$ (a) This is a consequence of Proposition 3.5.1, since the concepts of a maximal monotone and maximal accretive operator are the same in Hilbert spaces.

(a) $\Longrightarrow$ (b) In order to obtain the proof, it will be enough to see that $R(I + A) = H$. Letting $y \in H$, by Theorem 3.5.1, we know that there exists $x \in H$ with

$$\langle z + x, v - x\rangle \geq \langle y, v - x\rangle, \quad \forall (v, z) \in A.$$

Since A is maximal monotone, we can argue as follows:

$$\langle z + x - y, v - x\rangle \geq 0, \quad \forall (v, z) \in A.$$

Now the maximality of A implies

$$\big(x, -(x - y)\big) \in A,$$

that is, $-(x - y) \in Ax$. This shows that $y \in x + Ax$, or again $y \in (I + A)x$. Consequently, $y \in R(I + A)$. ☐

3.6 Examples of *m*-accretive operators

Example 3.6.1. Consider the operator

$$A : \mathbb{R} \to 2^{\mathbb{R}}, \quad x \mapsto A(x) = \begin{cases} 1, & x > 0, \\ I, & x = 0, \\ -1, & x < 0, \end{cases}$$

where $I = [-1, 1]$. Example 3.3.2 shows that A is an accretive operator. Taking $y \in \mathbb{R}$, we have to prove that there exists $x_0 \in \mathbb{R}$ such that $y \in x_0 + Ax_0$. Indeed, if $-1 \leq y \leq 1$, then $y \in (I + A)0 = [-1, 1]$.

If $y > 1$, we can take $x_0 = y - 1 > 0$, hence $(I + A)(x_0) = y - 1 + 1 = y$.

Finally, if $y < -1$, we can take $x_0 = y + 1 < 0$, and then $(I + A)(x_0) = y + 1 - 1 = y$.

Example 3.6.2. Let X be a Banach space. The operator $A : X \to X$ defined by $Ax = \frac{x}{1+\|x\|}$ is m-accretive. First, let us see that A is accretive. Given $x, y \in X$ and $j \in J(x - y)$, we have

$$\begin{aligned}\left\langle \frac{x}{1+\|x\|} - \frac{y}{1+\|y\|}, j\right\rangle &= \frac{1}{(1+\|x\|)(1+\|y\|)}\langle (x-y) + x\|y\| - y\|x\|, j\rangle \\ &= \frac{1}{(1+\|x\|)(1+\|y\|)}(\langle x-y, j\rangle + \|y\|\langle x-y, j\rangle + \langle y(\|y\| - \|x\|), j\rangle \\ &= \frac{1}{(1+\|x\|)(1+\|y\|)}(\|x-y\|^2 + \|y\|\|x-y\|^2 + \langle y(\|y\| - \|x\|), j\rangle) \\ &\geq \frac{\|x-y\|^2}{(1+\|x\|)(1+\|y\|)} \geq 0.\end{aligned}$$

Now, we are going to show that $\mathrm{R}(I + A) = X$. Consider the function $f : [0, +\infty[\to \mathbb{R}$ defined by

$$f(\lambda) = \frac{\lambda(2+\lambda)}{1+\lambda}.$$

Easy calculations show that f is a continuous, increasing, one-to-one function with $f(0) = 0$, and $\lim_{\lambda\to+\infty} f(\lambda) = +\infty$. Hence, for any $y \in X$, there exists $\lambda_0 \in [0, +\infty[$ such that $f(\lambda_0) = \|y\|$. Therefore, if we take $x_0 = \frac{1+\lambda_0}{2+\lambda_0} y$, we obtain that $y = x_0 + Ax_0$.

Example 3.6.3. Let H be a Hilbert space. If $\varphi : H \to]-\infty, +\infty]$ is a proper lower semicontinuous convex function, then its subdifferential $\partial\varphi$ is a maximal monotone operator in H. Therefore, by Corollary 3.5.1, it is an m-accretive operator.

As we have seen in Example 3.3.3, $\partial\varphi$ is an accretive operator. Next, we will show that if we add that φ is lower semicontinuous function, then $\mathrm{R}(I + \partial\varphi) = H$.

Lemma 3.6.1. *Let H be a Hilbert space, and $\varphi : H \to]-\infty, +\infty]$ a proper convex function The function $\phi : H \to]-\infty, +\infty]$ defined by $\phi(x) = \varphi(x) + \frac{\alpha}{2}\|x-y\|^2$ achieves a minimum at x_0 if and only if $\alpha(y - x_0) \in \partial\varphi(x_0)$.*

Proof. ($\Longleftarrow$) Suppose that $\alpha(y - x_0) \in \partial\varphi(x_0)$. By the definition of $\partial\varphi$, we know that

$$\langle \alpha(y - x_0), z - x_0\rangle \leq \varphi(z) - \varphi(x_0), \quad \forall z \in H. \tag{3.13}$$

The scalar product yields

$$\begin{aligned}\alpha\langle y - x_0, z - x_0\rangle &= \alpha\langle y - x_0, z - y + y - x_0\rangle \\ &= \alpha(\|y - x_0\|^2 - \|y - z\|^2 + \langle z - x_0, z - y\rangle) \\ &= \alpha(\|y - x_0\|^2 - \|y - z\|^2 + \|z - x_0\|^2 + \langle x_0 - y, z - x_0\rangle),\end{aligned}$$

and consequently,

$$\alpha\langle y - x_0, z - x_0\rangle - \alpha\langle x_0 - y, z - x_0\rangle = \alpha(\|y - x_0\|^2 - \|y - z\|^2 + \|z - x_0\|^2).$$

Therefore

$$\alpha\langle y - x_0, z - x_0\rangle = \frac{\alpha}{2}(\|y - x_0\|^2 - \|y - z\|^2 + \|z - x_0\|^2). \tag{3.14}$$

Taking into account (3.13) and (3.14), we infer that

$$\varphi(z) - \varphi(x_0) \geq \frac{\alpha}{2}(\|y - x_0\|^2 - \|y - z\|^2)$$

Now, by the definition of ϕ, we obtain

$$\phi(z) \geq \phi(x_0), \quad \forall z \in H.$$

This means that x_0 is a minimum of ϕ.

($\Longrightarrow$) In this case, our goal is to show that $\alpha(y - x_0) \in \partial\varphi(x_0)$, knowing that x_0 is a minimum of ϕ. To this end, we will see that

$$\langle \alpha(y - x_0), z - x_0 \rangle \leq \varphi(z) - \varphi(x_0), \quad \forall z \in H.$$

Let $z \in H$ and take $v := (1-t)x_0 + tz$ with $t \in (0, 1)$. Since, by hypothesis, x_0 is a minimum of ϕ, we derive that

$$\varphi(v) + \frac{\alpha}{2}\|v - y\|^2 \geq \varphi(x_0) + \frac{\alpha}{2}\|x_0 - y\|^2,$$

and then

$$\varphi(v) - \varphi(x_0) \geq \frac{\alpha}{2}(\|x_0 - y\|^2 - \|v - y\|^2).$$

The convexity of φ yields

$$\varphi(v) \leq (1 - t)\varphi(x_0) + t\varphi(z),$$

and therefore

$$t(\varphi(z) - \varphi(x_0)) \geq \varphi(v) - \varphi(x_0) \geq \frac{\alpha}{2}(\|x_0 - y\|^2 - \|v - y\|^2). \tag{3.15}$$

Next, using (3.14), we obtain

$$\frac{\alpha}{2}(\|x_0 - y\|^2 - \|v - y\|^2) = \alpha\langle y - x_0, v - x_0\rangle - \frac{\alpha}{2}\|v - x_0\|^2.$$

Since $v = (1 - t)x_0 + tz$,

$$\alpha t\langle y - x_0, z - x_0\rangle - \frac{\alpha}{2}\|v - x_0\|^2 = \alpha t\langle y - x_0, z - x_0\rangle - \frac{\alpha}{2}t^2\|z - x_0\|^2.$$

Now, dividing expression (3.15) by t, we deduce

$$\varphi(z) - \varphi(x_0) \geq \alpha\langle y - x_0, z - x_0\rangle - \frac{\alpha}{2}t\|z - x_0\|^2.$$

Finally, letting t go to 0 from the right in the above expression, we obtain the desired result. ☐

Remark 3.6.1. Let H be a Hilbert space. If $\varphi : H \to]-\infty, +\infty]$ is a proper lower semicontinuous convex function, then φ will be lower bounded on the unit ball of H, $\mathbb{B}_1^H$. Otherwise, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in B_H such that $\lim_{n\to+\infty} \varphi(x_n) = -\infty$. Since H is reflexive, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which is weakly convergent to $x_0 \in \mathbb{B}_1^H$. Since φ is lower semicontinuous and convex, we have

$$\varphi(x_0) \le \lim_{k\to+\infty} \varphi(x_{n_k}) = -\infty,$$

which is a contradiction.

Proposition 3.6.1. *Let H be a real Hilbert space and let $\varphi : H \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then its subdifferential $\partial\varphi$ is a maximal monotone operator in H. In fact, it is an m-accretive operator.*

Proof. Lemma 3.6.1, along with Remark 3.6.1, says us that we just have to see that $\mathrm{R}(I + \partial\varphi) = X$. Let $y \in H$ and consider the function $\phi : H \to]-\infty, +\infty]$ defined by $\phi(x) = \varphi(x) + \frac{1}{2}\|x - y\|^2$. Since φ is convex and lower semicontinuous, the same is true for ϕ.

Let us show that ϕ has a minimum because if there exists $x_0 \in H$ such that x_0 is a minimum of ϕ, then, by Lemma 3.6.1, we know that $y - x_0 \in \partial\varphi(x_0)$. This means $y \in x_0 + \partial\varphi(x_0) = (I + \partial\varphi)(x_0)$, that is, $y \in R(I + \partial\varphi)$ for all $y \in H$.

Thus, our goal now is to check that ϕ has a minimum. Since ϕ is a lower semicontinuous convex function, in order to show that it has a minimum, it will be enough to see that

$$\lim_{\|x\|\to+\infty} \phi(x) = +\infty \tag{3.16}$$

because condition (3.16) implies that ϕ is lower bounded.

Define $M := \inf\{\phi(x) : x \in H\}$ and consider a sequence $(x_n)_{n\in\mathbb{N}}$ of points of H such that $\phi(x_n) \to M$. Since it is assumed that $\lim_{\|x\|\to+\infty} \phi(x) = +\infty$, we have that $(x_n)_{n\in\mathbb{N}}$ is bounded and, since H is reflexive, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges weakly to x_0. Therefore

$$M \le \phi(x_0) \le \liminf_{k\to+\infty} \phi(x_{n_k}) = M.$$

Consequently, $M = \phi(x_0)$. Hence, x_0 is a minimum of ϕ.

We claim that $\lim_{\|x\|\to+\infty} \phi(x) = +\infty$. We will prove this by contradiction. Suppose, to the contrary, that

$$\exists (x_n)_{n\in\mathbb{N}} \subset H : \quad \|x_n\| = t_n \to +\infty \quad \text{and} \quad \phi(x_n) \le k, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\varphi(x_n) + \frac{1}{2}\|x_n - y\|^2 \leq k, \quad \forall n \in \mathbb{N}.$$

Because $\|x_n - y\|^2 = \langle x_n - y, x_n - y\rangle$, we get

$$\varphi(x_n) + \frac{1}{2}(\|x_n\|^2 + \|y\|^2 - 2\langle x_n, y\rangle) \leq k, \quad \forall n \in \mathbb{N}.$$

Take $y_n := \frac{x_n}{t_n}$. Easy calculations give

$$\varphi(y_n) - \langle y_n, y\rangle = \varphi\left(\frac{1}{t_n}x_n + \left(1 - \frac{1}{t_n}\right)0\right) - \left\langle \frac{x_n}{t_n}, y\right\rangle.$$

Since φ is convex, the above expression will be bounded above by

$$\begin{aligned}\frac{1}{t_n}\varphi(x_n) + \left(1 - \frac{1}{t_n}\right)\varphi(0) - \frac{1}{t_n}\langle x_n, y\rangle &= \frac{1}{t_n}(\varphi(x_n) - \langle x_n, y\rangle) + \left(1 - \frac{1}{t_n}\right)\varphi(0)\\ &\leq \frac{1}{t_n}\left(k - \frac{1}{2}t_n^2\right) + \left(1 - \frac{1}{t_n}\right)\varphi(0).\end{aligned}$$

The latter expression goes to $-\infty$ as $n \to +\infty$. This is a contradiction, since the map $x \mapsto \varphi(x) - \langle x, y\rangle$ is convex lower semicontinuous and then, by Remark 3.6.1, it must be lower bounded on the unit ball of H. However, we have seen that $\varphi(y_n) - \langle y_n, y\rangle \to -\infty$. ☐

Example 3.6.4 ([32]). If Ω is an open bounded and convex subset of $\mathbb{R}^n$, then the operator $A : L^2(\Omega) \to L^2(\Omega)$ defined by

$$Ay = -\Delta y, \quad \forall y \in D(A) := W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$$

is m-accretive in $L^2(\Omega)$ because it can be expressed as the subdifferential of the following proper semicontinuous convex function $\varphi : L^2(\Omega) \to]-\infty, +\infty]$ given by

$$\varphi(y) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla y|^2\,dx, & y \in W_0^{1,2}(\Omega),\\ +\infty, & y \notin W_0^{1,2}.\end{cases}$$

Example 3.6.5 ([32]). Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2. Consider a proper semi-continuous convex function $j : \mathbb{R} \to]-\infty, +\infty]$ and let $\beta := \partial j$ be its subdifferential. Define the function $\varphi : L^2(\Omega) \to]-\infty, +\infty]$ by

$$\varphi(y) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla y|^2\,dx + \int_{\partial\Omega} j(u)\,d\sigma y \in W^{1,2}(\Omega), & j(u) \in L^1(\partial\Omega),\\ +\infty, & \text{otherwise}.\end{cases}$$

In this case, the subdifferential of φ is

$$\partial\varphi = -\Delta u, \quad \forall u \in D(\partial\varphi) = \left\{u \in W^{2,2}(\Omega) : -\frac{\partial u}{\partial n} \in \beta(u), \text{ a.e. on } \partial\Omega\right\}.$$

3.7 Properties of *m*-accretive operators

Definition 3.7.1. Let X be a Banach space. An operator $A : D(A) \to 2^X$ is said to be closed (respectively demiclosed) if, given a sequence $(x_n, u_n) \in A$ with $x_n \to x_0$, $u_n \to u_0$ (respectively $u_n \rightharpoonup u_0$), we have $(x_0, u_0) \in A$.

Proposition 3.7.1. *Let X be a Banach space and let $A : D(A) \subset X \to 2^X$ be a maximal accretive operator. Then:*

(a) *A is always closed.*

(b) *If X^* is reflexive with Kadec–Klee property (cf. Definition* 1.9.4*), then A is demiclosed.*

Proof. (a) Let $(x_n, u_n)_{n\in\mathbb{N}} \in A$ be a sequence such that $x_n \to x_0$ and $u_n \to u_0$. By Theorem 3.4.1, we know that

$$\|x_n - x\| \le \|x_n - x + \lambda(u_n - u)\|, \quad \text{for all } \lambda > 0 \text{ and } (x,u) \in A.$$

Taking the limit as $n \to +\infty$, we obtain

$$\|x_0 - x\| \le \|x_0 - x + \lambda(u_0 - u)\|, \quad \text{for all } \lambda > 0 \text{ and } (x,u) \in A.$$

By Lemma 1.10.1, there exists $j \in J(x_0 - x)$ such that $\langle j, u_0 - u\rangle \ge 0$. Now, since A is a maximal accretive operator, in view of Remark 3.5.1, one can conclude that $(x_0, u_0) \in A$.

(b) Suppose that X is a reflexive Banach space and that X^* enjoys Kadec–Klee property. We have to show that if $(x_n, u_n) \in A$ with $x_n \to x_0$ and $u_n \rightharpoonup u_0$, then $(x_0, u_0) \in A$. By Remark 3.5.1, we only have to see that for $(x,y) \in A$ there exists $j_0 \in J(x_0 - x)$ such that $\langle j_0, u_0 - y\rangle \ge 0$. Since A is accretive and $(x_n, u_n) \in A$, it is clear that there exists $j_n \in J(x_n - x)$ such that $\langle j_n, u_n - y\rangle \ge 0$ $\forall n \in \mathbb{N}$. On the other hand, since $(x_n)_{n\in\mathbb{N}}$ is bounded, the sequence $(j_n)_{n\in\mathbb{N}} \subset X^*$ is bounded, too. Because X is reflexive, the dual space X^* is also reflexive. Hence, there exists a weakly convergent subsequence $(j_{n_k})_{k\in\mathbb{N}}$, i. e., $j_{n_k} \rightharpoonup j_0 \in X^*$. Since $\|\cdot\|$ in X^* is weakly lower semicontinuous, we have

$$\|j_0\| \le \liminf_{k\to+\infty} \|j_{n_k}\| = \liminf_{k\to+\infty} \|x_{n_k} - x\| = \|x_0 - x\|. \tag{3.17}$$

Moreover, $\langle j_{n_k}, x_0 - x\rangle \to \langle j_0, x_0 - x\rangle$. Consequently,

$$\left|\langle j_{n_k}, x_{n_k} - x\rangle - \langle j_{n_k}, x_0 - x\rangle\right| = \left|\langle j_{n_k}, x_{n_k} - x_0\rangle\right| \le \|j_{n_k}\|\|x_{n_k} - x_0\| \to 0 \quad \text{as } k \to +\infty.$$

Then,

$$j_{n_k}(x_{n_k} - x) \to j_0(x_0 - x).$$

Therefore,

$$j_0(x_0 - x) = \lim_{k\to+\infty} \langle j_{n_k}, x_{n_k} - x\rangle = \lim_{k\to+\infty} \|x_{n_k} - x\|^2 = \|x_0 - x\|^2. \tag{3.18}$$

It follows from (3.17) and (3.18) that $j_0 \in J(x_0 - x)$. Since X^* has Kadec–Klee property, we have

$$j_{n_k} \to j_0. \tag{3.19}$$

Finally, we will show that $\langle j_0, u_0 - y\rangle \geq 0$. Since $u_{n_k} \rightharpoonup u_0$, we obtain $\langle j_0, u_{n_k} - y\rangle \to \langle j_0, u_0 - y\rangle$. Thus,

$$\begin{aligned}\langle j_0, u_{n_k} - y\rangle &= \langle j_0 - j_{n_k} + j_{n_k}, u_{n_k} - y\rangle\\ &= \langle j_0 - j_{n_k}, u_{n_k} - y\rangle + \langle j_{n_k}, u_{n_k} - y\rangle\\ &\geq \langle j_{n_k}, u_{n_k} - y\rangle - \|j_0 - j_{n_k}\|\|u_{n_k} - y\|,\end{aligned}$$

and consequently,

$$\begin{aligned}\langle j_0, u_0 - y\rangle &= \lim_{k\to+\infty} \langle j_0, u_{n_k} - y\rangle\\ &\geq \lim_{k\to+\infty} \langle j_{n_k}, u_{n_k} - y\rangle - \lim_{k\to+\infty} \|j_0 - j_{n_k}\|\|u_{n_k} - y\| \geq 0.\end{aligned}$$

Now, applying (3.19), we get $\lim_{k\to+\infty} \|j_0 - j_{n_k}\|\|u_{n_k} - y\| = 0$. This ends the proof. □

Corollary 3.7.1. *Let X be a Banach space and let $A : D(A) \to 2^X$ an m-accretive operator. Let $(x_\lambda)_{\lambda>0} \subset D(A)$ be such that $\lim_{\lambda\to0^+} x_\lambda = x_0$. Then:*

(a) *If $\lim_{\lambda\to0^+} A_\lambda x_\lambda = u_0$, then $(x_0, u_0) \in A$.*

(b) *Suppose that X is reflexive and X^* has Kadec–Klee property. If $(A_\lambda x_\lambda)_{\lambda>0}$ is bounded, $x_0 \in D(A)$, and $A_\lambda x_\lambda \rightharpoonup u_0$ as $\lambda \to 0^+$, then $(x_0, u_0) \in A$.*

Proof. (a) By Proposition 3.4.1(a), $A_\lambda x_\lambda \in AJ_\lambda x_\lambda$. Hence, it will be enough to prove that

$$\lim_{\lambda\to0^+} J_\lambda x_\lambda = x_0. \tag{3.20}$$

Since A is m-accretive, taking into account Proposition 3.5.1, we conclude that A is maximal accretive. Next, applying Proposition 3.7.1, we conclude that A is closed and, thus, $(x_0, u_0) \in A$.

We will show that (3.20) holds. Since $(A_\lambda x_\lambda)_{\lambda>0}$ is convergent, $(A_\lambda x_\lambda)_{\lambda>0}$ is bounded. Recall that, by the definition of A_λ, we have

$$A_\lambda x_\lambda = \frac{x_\lambda - J_\lambda x_\lambda}{\lambda},$$

and consequently,

$$\lim_{\lambda\to0^+} x_\lambda - J_\lambda x_\lambda = \lim_{\lambda\to0^+} \lambda A_\lambda x_\lambda = 0. \tag{3.21}$$

By hypothesis, $\lim_{\lambda\to 0^+} x_\lambda = x_0$, so, by (3.21), we have $\lim_{\lambda\to 0^+} J_\lambda x_\lambda = x_0$. Since we know that $(J_\lambda x_\lambda, A_\lambda x_\lambda) \in A$, the above facts imply that $(x_0, u_0) \in A$.

(b) This assertion is a trivial consequence of the fact that $(A_\lambda x_\lambda)$ is bounded and Proposition 3.7.1. ☐

Proposition 3.7.2. *Let X be a Banach space and let $A : D(A) \to 2^X$ be a maximal accretive operator. If X is a smooth Banach space, then, for every $x \in D(A)$, Ax is a closed and convex subset.*

Proof. Since A is maximal accretive and X is smooth, we have that

$$Ax = \bigcap_{(y,v)\in A} \{u \in X : \langle u - v, (x-y)^* \rangle \geq 0\}.$$

Since every subset in the above intersection is closed and convex, Ax is also closed and convex. ☐

Definition 3.7.2. Let X be a Banach space and let $A : D(A) \to 2^X$ be an accretive operator. We can define a new operator A^0 as follows:

$$A^0 : D(A^0) \to 2^X, \quad A^0 x = \{u \in Ax : \|u\| = |Ax|\}.$$

Clearly, $D(A^0) \subset D(A)$.

Proposition 3.7.3. *Let X be a Banach space and let $A : D(A) \to 2^X$ be a maximal accretive operator. Then:*

(a) *$D(A^0) = D(A)$ whenever X is a reflexive smooth Banach space.*

(b) *If, moreover, X is a strictly convex Banach space, then A^0 is single-valued.*

Proof. (a) Consider $u_n \in Ax$, $\forall n \in \mathbb{N}$ such that $\|u_n\| \to |Ax|$. Since X is reflexive, taking subsequences, we know that $u_{n_k} \rightharpoonup u_0$ for some $u_0 \in X$. By Proposition 3.7.2, for each $x \in D(A)$, Ax is a closed and convex subset of X. Moreover, using Proposition 1.7.2, we conclude that Ax is weakly closed and then $u_0 \in Ax$. Now the fact that the norm in X is weakly lower semicontinuous gives the result.

(b) Suppose $u, v \in A^0 x$, which means that

$$\|u\| = \|v\| = |Ax|. \tag{3.22}$$

Since Ax is convex and X is strictly convex, if $u \neq v$, then $\|\lambda u + (1-\lambda)v\| < |Ax|$, $\forall \lambda \in\,]0,1[$. This is a contradiction and then $u = v$. ☐

Theorem 3.7.1. *Let X be a Banach space and let $A : D(A) \to 2^X$ be an m-accretive operator. Then the following assertions hold true:*

(a) *If X is a strictly convex, smooth, reflexive Banach space, then $\|A_\lambda x\| \leq \|A^0 x\|$, $\forall x \in D(A)$.*

(b) *The function* $\lambda \to \|A_\lambda x\|$ *is decreasing.*

(c) *If* X *is uniformly convex and smooth, and* X^* *has Kadec–Klee property, then*

$$\lim_{\lambda\to 0^+} A_\lambda x = A^0 x, \quad \lim_{\lambda\to 0^+} A^0 J_\lambda x = A^0 x.$$

Proof. (a) Since A is m-accretive, $D_\lambda = X$ and $D(A) \cap D_\lambda = D(A)$. Applying Proposition 3.4.1(c), we get

$$\|A_\lambda x\| \le |Ax|, \quad \forall x \in D(A).$$

By Proposition 3.7.3, we have

$$\|A_\lambda x\| \le \|A^0 x\|, \quad \forall x \in D(A).$$

(b) Denote $x(\lambda) := J_\lambda x$ and $u(\lambda) := A_\lambda x$. It follows from Proposition 3.4.1(a) that

$$\big(x(\lambda), u(\lambda)\big) \in A, \quad \forall x \in D_\lambda = X, \ \forall \lambda > 0.$$

Now consider $\lambda, \gamma > 0$. If $j \in J(\lambda u(\lambda) - \gamma u(\gamma))$ (in fact, $j \in J(x(\gamma) - x(\lambda))$ because $\lambda u(\lambda) - \gamma u(\gamma) = \lambda \frac{1}{\lambda}(I - J_\lambda) - \gamma \frac{1}{\gamma}(I - J_\gamma) = J_\gamma - J_\lambda$), then we have

$$\begin{aligned} \|\lambda u(\lambda) - \gamma u(\gamma)\|^2 &= \langle j, \lambda u(\lambda) - \gamma u(\gamma)\rangle \\ &= \langle j, \lambda\big(u(\lambda) - u(\gamma)\big) + (\lambda - \gamma)u(\gamma)\rangle \\ &= \lambda\langle j, u(\lambda) - u(\gamma)\rangle + (\lambda - \gamma)\langle j, u(\gamma)\rangle. \end{aligned} \tag{3.23}$$

Since A is accretive, due to Proposition 3.4.1(a), $u(\lambda) \in Ax(\lambda)$, $u(\gamma) \in Ax(\gamma)$, so there exists $j_0 \in J(x(\gamma) - x(\lambda))$ such that

$$\langle j_0, u(\gamma) - u(\lambda)\rangle \ge 0. \tag{3.24}$$

Following the same argument like in (3.23), for $j_0 \in J(x(\gamma) - x(\lambda))$, we obtain

$$\begin{aligned} \|\lambda u(\lambda) - \gamma u(\gamma)\|^2 &= \langle j_0, \lambda u(\lambda) - \gamma u(\gamma)\rangle \\ &= \lambda\langle j_0, u(\lambda) - u(\gamma)\rangle + (\lambda - \gamma)\langle j_0, u(\gamma)\rangle. \end{aligned} \tag{3.25}$$

We know from (3.24) that $\langle j_0, u(\lambda) - u(\gamma)\rangle \le 0$, so using (3.25) we can write

$$\begin{aligned} \|\lambda u(\lambda) - \gamma u(\gamma)\|^2 &\le (\lambda - \gamma)\langle j_0, u(\gamma)\rangle \\ &\le |\lambda - \gamma| \|x(\gamma) - x(\lambda)\| \|u(\gamma)\| \\ &= |\lambda - \gamma| \|\lambda u(\lambda) - \gamma u(\gamma)\| \|u(\gamma)\|. \end{aligned}$$

Accordingly,

$$\|\lambda u(\lambda) - \gamma u(\gamma)\| \leq |\lambda - \gamma| \|u(\gamma)\|.$$

Hence, if $\lambda > \gamma$, then

$$\begin{aligned}\lambda \|u(\lambda)\| &\leq \|\lambda u(\lambda) - \gamma u(\gamma)\| + \gamma \|u(\gamma)\| \\ &\leq |\lambda - \gamma| \|u(\gamma)\| + \gamma \|u(\gamma)\| \\ &= (\lambda - \gamma + \gamma)\|u(\gamma)\| \\ &= \lambda \|u(\gamma)\|.\end{aligned}$$

Thus, for $\lambda > \gamma$, we have $\|u(\lambda)\| \leq \|u(\gamma)\|$, that is, the function $\lambda \to \|A_\lambda x\|$ is decreasing.

(c) Since X is smooth and uniformly convex, by Proposition 3.7.3, A^0 is a single-valued mapping from $D(A)$ into X. So, in order to show that $\lim_{\lambda \to 0^+} A_\lambda x = A^0 x$, we shall establish that there exists a sequence $(A_{\lambda_n} x)_{n \in \mathbb{N}}$ which converges to $A^0 x$ as $\lambda_n \to 0$.

On one hand, it is known that $\forall x \in D(A)$, $\|A_\lambda x\| \leq |Ax|$, then the sequence $(A_{\lambda_n} x)_{n \in \mathbb{N}}$ is bounded. On the other hand, the reflexivity of X implies that there exists a subsequence $(A_{\lambda_{n_k}} x)_{k \in \mathbb{N}}$ of $(A_{\lambda_n} x)_{n \in \mathbb{N}}$ such that $A_{\lambda_{n_k}} x \rightharpoonup u \in X$. Now the use of Proposition 3.4.1 guarantees

$$A_{\lambda_{n_k}} x \in AJ_{\lambda_{n_k}} x \quad \text{and} \quad J_{\lambda_{n_k}} x \to x.$$

Further, since X^* has Kadec–Klee property and is reflexive, it follows from Proposition 3.7.1(b) that A is demiclosed and then $(x, u) \in A$. According to the definition of $|Ax|$, we have

$$\|u\| \geq |Ax|. \tag{3.26}$$

Hence,

$$\|u\| \leq \liminf_{k \to +\infty} \|A_{\lambda_{n_k}} x\| \leq |Ax|. \tag{3.27}$$

Now using (3.26) and (3.27), we infer that $u = A^0 x$. The above argument allows us to ensure that

$$\|A_{\lambda_{n_k}} x\| \to \|u\| \quad \text{and} \quad A_{\lambda_{n_k}} x \rightharpoonup u. \tag{3.28}$$

Since X is uniformly convex, (3.28) implies that

$$A_{\lambda_{n_k}} x \to u = A^0 x.$$

Let us see that $\forall x \in D(A)$ we have $\lim_{\lambda \to 0^+} A^0 J_\lambda x = A^0 x$. Moreover, since $A_\lambda x \in AJ_\lambda x$, we have

$$\|A^0 J_\lambda x\| \leq \|A_\lambda x\| \leq |Ax|, \quad \forall \lambda > 0.$$

In order to get the result, it is enough to use the same argument as before. □

3.8 Domains of *m*-accretive operators

It is well known that in a Hilbert space, the closure of the domain of an m-accretive (equivalently, maximally monotone) operator is indeed convex [71]. If X is a reflexive Banach space and A is a maximally monotone operator in $X \times X^*$, then $\overline{D(A)}$ is also convex [45, 203]. The analogous statement for m-accretive operators is not true in all reflexive Banach spaces. As a matter of fact, S. Reich [200, p. 382] gave the following simple example.

Example 3.8.1. Consider the Banach space $(\mathbb{R}^2, \|\cdot\|_\infty)$ and the operator $A : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by

$$\begin{cases} D(A) := \{(x,y) \in \mathbb{R}^2 : |x| = y\}, \\ A(x,y) := \{(0,z) : z \in \mathbb{R}\}, \quad \text{for all } (x,y) \in D(A). \end{cases}$$

Then A is m-accretive, but $\overline{D(A)} = \{(x,y) : |x| = y\}$ is not a convex subset of $\mathbb{R}^2$.

It is, however, known (cf. [48]) that if A is m-accretive and X is uniformly convex, then $\overline{D(A)}$ *is* convex. Furthermore, it was observed in [200, p. 382] (without proof) that this is also true when the norm of X^* is merely Fréchet differentiable (and not *uniformly* Fréchet differentiable). In the present section, we first provide a detailed proof of this assertion (see [110]). It is natural to ask [200, p. 383] if this conclusion holds in *all* reflexive and strictly convex Banach spaces. Although this question remains open, we show that strict convexity by itself is not enough (see [110]).

Theorem 3.8.1. *Let X be a Banach space and let $A : X \to 2^X$ be an m-accretive operator. If X is reflexive, strictly convex, and has Kadec–Klee property, then $\overline{D(A)}$ is convex.*

Proof. Let $x, y \in D(A)$, $x \neq y$, $t \in (0,1)$, and consider $z := tx + (1-t)y$. We first prove that $\lim_{\lambda \to 0^+} J_\lambda(z) = z$, which yields that $z \in \overline{D(A)}$ as required. To see this, fix $u \in A(x)$, $v \in A(y)$, and set $u_\lambda = x + \lambda u$, $v_\lambda = y + \lambda v$. It is clear that

$$J_\lambda(u_\lambda) = x, \quad J_\lambda(v_\lambda) = y.$$

Thus,

$$\|J_\lambda(z) - x\| = \|J_\lambda(z) - J_\lambda(u_\lambda)\| \leq \|z - u_\lambda\| \leq \|z - x\| + \lambda\|u\|.$$

Analogously, one can also check that $\|J_\lambda(z) - y\| \leq \|z - y\| + \lambda\|v\|$.

Since X is reflexive, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n \to 0^+$ and $J_{\lambda_n} z \rightharpoonup w \in X$. By the weak lower semicontinuity of the norm, we have

$$\|w - x\| \le \liminf_{n\to+\infty} \big\|J_{\lambda_n}(z) - x\big\| \le \|z - x\| = (1-t)\|x - y\|$$

and

$$\|w - y\| \le \liminf_{n\to+\infty} \big\|J_{\lambda_n}(z) - y\big\| \le \|z - y\| = t\|x - y\|.$$

It is obvious that

$$\|w - x\| \ge \|x - y\| - \|w - y\| \ge \|x - y\| - t\|x - y\| = (1-t)\|x - y\|$$

and

$$\|w - y\| \ge \|x - y\| - \|w - x\| \ge \|x - y\| - (1-t)\|x - y\| = t\|x - y\|.$$

Accordingly,

$$\|w - x\| = \|z - x\| = (1-t)\|x - y\| \quad \text{and} \quad \|w - y\| = \|z - y\| = t\|x - y\|.$$

Suppose $w \neq z$. Since X is strictly convex, the latter equalities imply that

$$\big\|t(w - x) + (1-t)(z - x)\big\| < (1-t)\|x - y\|$$

and

$$\big\|t(w - y) + (1-t)(z - y)\big\| < t\|x - y\|.$$

Next, using the latter two estimates, we get

$$\begin{aligned}
\|x - y\| &= \big\|t(w - x) + (1-t)(z - x) - t(w - y) - (1-t)(z - y)\big\| \\
&\le \big\|t(w - x) + (1-t)(z - x)\big\| + \big\|t(w - y) + (1-t)(z - y)\big\| \\
&< (1-t)\|x - y\| + t\|x - y\| = \|x - y\|,
\end{aligned}$$

which is a contradiction; so, we infer that $w = z$. Thus $J_{\lambda_n}(z) \rightharpoonup z$, and therefore we may conclude that $J_\lambda(z) \rightharpoonup z$ as $\lambda \to 0^+$. Moreover,

$$\|z - x\| \le \liminf_{\lambda\to 0^+} \big\|J_\lambda(z) - x\big\| \le \limsup_{\lambda\to 0^+} \big\|J_\lambda(z) - x\big\| \le \|z - x\|,$$

and thus

$$\lim_{\lambda\to 0^+}\|J_\lambda(z)-x\| = \|z-x\| \quad \text{and} \quad J_\lambda(z)-x \rightharpoonup z-x.$$

Finally, since X has Kadec–Klee property, it follows that $J_\lambda(z)-x \to z-x$, that is, $J_\lambda(z) \to z$ as $\lambda \to 0^+$. This proves our claim. ☐

The following example shows that when we remove both the reflexivity and Kadec–Klee property assumptions, then again Theorem 3.8.1 no longer holds. In other words, strict convexity by itself is not enough.

Example 3.8.2. Consider the Banach space $X = (C([0,1]), \|\cdot\|_X)$, where the norm $\|\cdot\|_X$ is defined by

$$\|u\|_X := \|u\|_\infty + \left(\int_0^1 u^2(t)\,dt\right)^{1/2}.$$

It is well known that when the space X is equipped with this norm, it is strictly convex (see, for example, [78, Theorem 4.2.1, p. 100]). Now consider the mapping $\phi : C([0,1]) \to \mathbb{R}$ defined by $\phi(u) = \sin(\frac{1}{2}u(1))$ and define the operator $A : D(A) \to C([0,1])$ by

$$\begin{cases} A : D(A) \to C([0,1]), & Au = u', \\ D(A) := \{u \in C^1([0,1]) : u(0) = \phi(u)\}. \end{cases}$$

We claim that A is m-accretive on X, but $\overline{D(A)}$ is not convex. To see this, we first show that A is accretive. In other words, given $u, v \in D(A)$, $u \neq v$, and $\lambda > 0$, we have to prove that $\|u-v\|_X \le \|u-v+\lambda(u'-v')\|_X$. To this end, let $t_0 \in [0,1]$ be such that

$$\|u-v\|_\infty = \max_{0\le t\le 1}\bigl|u(t)-v(t)\bigr| = \bigl|u(t_0)-v(t_0)\bigr|.$$

Since $|u(0)-v(0)| = |\sin(\frac{1}{2}u(1)) - \sin(\frac{1}{2}v(1))| \le \frac{1}{2}|u(1)-v(1)|$, we infer that $t_0 \neq 0$. If $t_0 \in (0,1)$, then it is clear that $u'(t_0)-v'(t_0) = 0$ and therefore

$$\|u-v\|_\infty = \bigl|u(t_0)-v(t_0)+\lambda(u'(t_0)-v'(t_0))\bigr| \le \bigl\|u-v+\lambda(u'-v')\bigr\|_\infty.$$

Now suppose $t_0 = 1$. If $u(1)-v(1) \ge 0$, then $u'(1)-v'(1) \ge 0$ and

$$\|u-v\|_\infty = u(1)-v(1) \le u(1)-v(1)+\lambda\bigl(u'(1)-v'(1)\bigr) \le \bigl\|u-v+\lambda(u'-v')\bigr\|_\infty.$$

If $u(1)-v(1) \le 0$, then $u'(1)-v'(1) \le 0$ and therefore

$$\|u-v\|_\infty = v(1)-u(1) \le v(1)-u(1)+\lambda\bigl(v'(1)-u'(1)\bigr) \le \bigl\|u-v+\lambda(u'-v')\bigr\|_\infty.$$

Thus in all cases we find that $\|u-v\|_\infty \le \|u-v+\lambda(u'-v')\|_\infty$. Next, we compute

$$\begin{aligned}
&\int_0^1 [u(t)-v(t)+\lambda(u'(t)-v'(t))]^2\,dt\\
&\quad=\int_0^1 (u(t)-v(t))^2\,dt+\int_0^1 \lambda^2(u'(t)-v'(t))^2\,dt+2\lambda\int_0^1 (u(t)-v(t))(u'(t)-v'(t))\,dt\\
&\quad=\int_0^1 (u(t)-v(t))^2\,dt+\int_0^1 \lambda^2(u'(t)-v'(t))^2\,dt+\lambda\int_0^1 [(u(t)-v(t))^2]'\,dt\\
&\quad\geq\int_0^1 (u(t)-v(t))^2\,dt+\lambda[(u(1)-v(1))^2-(u(0)-v(0))^2]\\
&\quad\geq\int_0^1 (u(t)-v(t))^2\,dt+\lambda\Big[(u(1)-v(1))^2-\Big(\frac{1}{2}u(1)-\frac{1}{2}v(1)\Big)^2\Big]\\
&\quad\geq\int_0^1 (u(t)-v(t))^2\,dt.
\end{aligned}$$

Consequently, $\|u-v\|_X \leq \|u-v+\lambda(u'-v')\|_X$, which proves that A is accretive.

Now we prove that $\mathrm{R}(I+A)=C([0,1])$. Given $v\in C([0,1])$, we have to find a solution u belonging to $D(A)$ to the equation $u+u'=v$. The general solution of this equation is given by

$$u(t)=e^{-t}\Big(\int_0^t v(s)e^s\,ds+u(0)\Big).$$

In order to find a solution $u\in D(A)$, consider the integral operator $T: C([0,1])\to C([0,1])$ defined by

$$Tu(t)=e^{-t}\int_0^t e^s v(s)\,ds+e^{-t}\phi(u),$$

where $t\in[0,1]$. This operator is a contraction because we have, for each $u,w\in C([0,1])$,

$$\|Tu-Tw\|_\infty=\max_{0\leq t\leq 1}\{e^{-t}|\phi(u)-\phi(w)|\}\leq|\phi(u)-\phi(w)|\leq\frac{1}{2}\|u-v\|_\infty.$$

By Banach contraction principle (Theorem 6.2.1), there exists a unique function $u_0\in C([0,1])$ such that $Tu_0=u_0$, that is,

$$u_0(t)=e^{-t}\Big(\int_0^t v(s)e^s\,ds+\phi(u_0)\Big).$$

It is clear that $u_0 \in C^1([0,1])$ and $u_0(0) = \phi(u_0)$, so that $u_0 \in D(A)$.

Finally, we claim that $\overline{D(A)}$ is not convex. Indeed, it is not difficult to check that

$$\overline{D(A)}^{\|\cdot\|_X} = \{u \in C([0,1]) : u(0) = \phi(u)\}.$$

Now consider the mappings u and v defined by $u(t) = (1-t) + \pi t$ and $v(t) = 0$ for each $t \in [0,1]$. Both u and v belong to $\overline{D(A)}$, but $\frac{1}{2}(u(0) + v(0)) = \frac{1}{2}$ and $\sin[\frac{1}{4}(u(1) + v(1))] = \frac{\sqrt{2}}{2}$, that is, $\frac{1}{2}(u+v) \notin \overline{D(A)}$, as claimed.

Let X be a Banach space. If $A : D(A) \subseteq X \to X$ is a linear m-accretive operator, then, by Theorem 2.4.2, $-A$ is the generator of a C_0-semigroup of contractions; and then, by Proposition 2.3.1, $D(A)$ is dense in X. This result fails when $A : D(A) \to 2^X$ is a nonlinear m-accretive operator (see, for instance, Example 3.8.1). Next we shall discus under what conditions we can guarantee that the domain remains dense in X.

Definition 3.8.1. An operator $A : X \to 2^X$ is called bounded if it maps bounded subsets of $D(A)$ into bounded subsets of X.

Theorem 3.8.2. *If $A : D(A) \subseteq X \to 2^X$ is a bounded m-accretive operator, then $\overline{D(A)} = X$.*

Proof. Since A is m-accretive, it follows from Proposition 3.5.1 that, for all $\lambda > 0$, $\mathrm{R}(I + \lambda A) = X$. The proof will be performed in two steps.

Step 1. We are going to show that, if A is bounded, then the resolvent $J_1 : X \to D(A)$ maps unbounded subsets onto unbounded subsets. To get a contradiction, suppose that J_1 does not map unbounded subsets onto unbounded subsets. Let B be an unbounded subset of X such that $J_1(B)$ is bounded. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in B such that $\|u_n\| \to \infty$ as $n \to \infty$ and define $y_n = J_1(u_n)$. We have $u_n \in y_n + Ay_n$ and hence $u_n - y_n \in Ay_n$. Since $(y_n)_{n\in\mathbb{N}}$ is bounded and $(u_n)_{n\in\mathbb{N}}$ is unbounded, the sequence $(u_n - y_n)_{n\in\mathbb{N}}$ is unbounded and consequently, $A(\{y_n : n = 1, 2, \ldots\})$ is unbounded. Hence A is not bounded.

Step 2. Let $z \in X$. For each $\lambda > 0$, set

$$z_\lambda := \frac{1}{\lambda} z + \left(1 - \frac{1}{\lambda}\right) J_\lambda(z).$$

We claim that the set $\{z_\lambda : 0 < \lambda \leq 1\}$ is bounded. Indeed, suppose that it is not the case, then it would follow from Step 1 that the set $\{J_1(z_\lambda) : 0 < \lambda \leq 1\}$ is also unbounded. According to Proposition 3.4.1(e),

$$J_\lambda(z) = J_1\left(\frac{1}{\lambda} z + \left(1 - \frac{1}{\lambda}\right) J_\lambda(z)\right) = J_1(z_\lambda),$$

and consequently, $\{J_\lambda(z) : 0 < \lambda \leq 1\}$ is unbounded, too. Now, fix $x \in D(A)$, $u \in A(x)$, and define $u_\lambda = x + \lambda u$. It is clear that $J_\lambda(u_\lambda) = x$, and therefore,

$$\begin{aligned}\|J_\lambda(z)-x\| &= \|J_\lambda(z)-J_\lambda(u_\lambda)\| \le \|z-u_\lambda\| \\ &\le \|z-x\|+\lambda\|u\| \\ &\le \|z-x\|+\|u\|. \end{aligned} \tag{3.29}$$

This shows that the set $\{J_\lambda(z) : 0 \le \lambda \le 1\}$ is bounded. This is a contradiction; therefore the set $\{z_\lambda : 0 < \lambda \le 1\}$ is bounded as claimed. Since $A_\lambda(z) = z_\lambda - J_\lambda(z)$ for every $\lambda > 0$, we obtain that the set $\{A_\lambda(z) : 0 < \lambda \le 1\}$ is bounded as well. Therefore

$$\lim_{\lambda\to 0^+}\|z-J_\lambda(z)\| = \lim_{\lambda\to 0^+}\lambda\|A_\lambda(z)\| = 0,$$

that is, $J_\lambda(z) \to z$ as $\lambda \to 0^+$. This proves that $z \in \overline{D(A)}$ as required. ☐

The following example shows that if A is an m-accretive operator which is not bounded, then the above theorem no longer holds even if X is a finite-dimensional Hilbert space.

Example 3.8.3. Consider the space $X = (\mathbb{R}^2, \|\cdot\|_2)$ and the operator $A : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by

$$\begin{cases} D(A) := \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}, \\ A(x,0) = \{(0,z) : z \in \mathbb{R}\}, \quad \forall (x,0) \in D(A). \end{cases}$$

Then A is an m-accretive operator which is not bounded and $\overline{D(A)} \neq X$.

3.9 Perturbation of *m*-accretive operators

It is clear that if A and B are two accretive operators in a smooth Banach space X, then the operator $A+B$ with domain $D(A)\cap D(B)$ is also accretive in X. This is an immediate consequence of the fact the normalized duality mapping of a smooth Banach space is single-valued. Indeed, for $x, y \in D(A)\cap D(B)$, we know that

$$\langle u_1 - v_1, J(x-y)\rangle \ge 0, \quad \forall u_1 \in Ax,\ v_1 \in Ay,$$

and

$$\langle u_2 - v_2, J(x-y)\rangle \ge 0, \quad \forall u_2 \in Bx,\ v_2 \in By.$$

Therefore,

$$\langle u_1 + u_2 - v_1 - v_2, J(x-y)\rangle \ge 0, \quad \forall u_1+u_2 \in (A+B)x, \quad v_1+v_2 \in (A+B)y.$$

Nevertheless, if A and B are m-accretive operators, in general, the operator $A + B$ is not necessarily m-accretive. In this section we will study under what conditions $A + B$ is also m-accretive.

Lemma 3.9.1. *Let X be a uniformly smooth Banach space and let A, B be two m-accretive operators of $X \times X$ with $D(A) \cap D(B) \neq \emptyset$. Then, for each $y \in X$ and for each $\lambda > 0$, the equation*

$$x_\lambda + Ax_\lambda + B_\lambda x_\lambda \ni y \tag{3.30}$$

has a unique solution $x_\lambda \in D(A)$; moreover, the net $(\|x_\lambda\|)_{\lambda>0}$ is bounded. If $\|B_\lambda x_\lambda\|$ is bounded as $\lambda \to 0$, then the equation

$$x + Ax + Bx \ni y \tag{3.31}$$

has a unique solution $x \in D(A) \cap D(B)$ and $\lim_{\lambda\to 0} x_\lambda = x$.

Proof. The inclusion given in (3.30) is equivalent to $x_\lambda + Ax_\lambda \ni y - B_\lambda x_\lambda$. The definition of Yosida approximant of B yields

$$\lambda x_\lambda + \lambda Ax_\lambda \ni \lambda y - x_\lambda + J_\lambda^B x_\lambda,$$

and thus we can write

$$(\lambda + 1)x_\lambda + \lambda Ax_\lambda \ni \lambda y + J_\lambda^B x_\lambda.$$

Dividing by $1 + \lambda$, we obtain

$$x_\lambda + \frac{\lambda}{\lambda+1} Ax_\lambda \ni \frac{\lambda}{\lambda+1} y + \frac{1}{\lambda+1} J_\lambda^B x_\lambda.$$

Finally, it is clear from the definition of the resolvent of the operator A that (3.30) is equivalent to

$$x_\lambda = J^A_{\frac{\lambda}{\lambda+1}}\left(\frac{\lambda}{\lambda+1} y + \frac{1}{\lambda+1} J_\lambda^B x_\lambda\right). \tag{3.32}$$

Now we define the operator

$$T : X \to D(A), \quad T(x) := J^A_{\frac{\lambda}{\lambda+1}}\left(\frac{\lambda}{\lambda+1} y + \frac{1}{\lambda+1} J_\lambda^B x\right).$$

Let $x_1, x_2 \in X$. Since the resolvents of an accretive operator are nonexpansive, we have

$$\begin{aligned}\|T(x_1)-T(x_2\| &\leq \left\|J^A_{\frac{\lambda}{\lambda+1}}\left(\frac{\lambda}{\lambda+1}y+\frac{1}{\lambda+1}J^B_\lambda x_1\right)-J^A_{\frac{\lambda}{\lambda+1}}\left(\frac{\lambda}{\lambda+1}y+\frac{1}{\lambda+1}J^B_\lambda x_2\right)\right\|\\ &\leq \frac{1}{\lambda+1}\|J^B_\lambda x_1-J^B_\lambda x_2\|\\ &\leq \frac{1}{\lambda+1}\|x_1-x_2\|.\end{aligned}$$

Hence, the operator T is a $\frac{1}{1+\lambda}$-contractive mapping with $\frac{1}{1+\lambda}\in(0,1)$. By Banach contraction principle (Theorem 6.2.1), there exists a unique $x_\lambda\in D(A)$ such that $Tx_\lambda=x_\lambda$. This shows that the unique solution of equation (3.30) is x_λ.

Now, we will prove that $(\|x_\lambda\|)_{\lambda\geq 0}$ is bounded. Indeed, since B is m-accretive, by Proposition 3.4.1(c), we have $\|B_\lambda x\|\leq|Bx|$ whenever $x\in D(B)$. Given $x_1\in D(A)\cap D(B)$, consider $y_\lambda=x_1+z+B_\lambda x_1$ where z is a fixed element of Ax_1. Since A and B are accretive operators and recalling that, by Proposition 3.4.1(a), B_λ is accretive, the operator $A+B_\lambda$ is also accretive, and then

$$\langle y-x_\lambda-(y_\lambda-x_1),J(x_\lambda-x_1)\rangle\geq 0.$$

Hence

$$\|x_\lambda-x_1\|^2\leq\langle y-y_\lambda,J(x_\lambda-x_1)\rangle\leq\|y-y_\lambda\|\|x_\lambda-x_1\|,$$

and then

$$\|x_\lambda-x_1\|\leq\|y-y_\lambda\|.$$

This implies that

$$\|x_\lambda\|\leq\|x_1\|+\|y\|+\|y_\lambda\|\leq\|x_1\|+\|y\|+\|x_1+z\|+|Bx_1|,$$

which allows us to conclude that $(\|x_\lambda\|)_{\lambda>0}$ is bounded.

Next, assume that $\|B_\lambda x\|$ is bounded for small values of $\lambda>0$. We have to show that the limit $\lim_{\lambda\to 0}x_\lambda$ exists. To this end, it is enough to see that, given $\varepsilon>0$, there exists $\delta>0$ such that if $0<\lambda,\mu<\delta$, then $\|x_\lambda-x_\mu\|<\epsilon$. Since $y-x_\lambda\in Ax_\lambda+B_\lambda x_\lambda$, we can write this equation in the form

$$y-x_\lambda-B_\lambda x_\lambda\in Ax_\lambda.$$

Because A is accretive, we have

$$\langle y-x_\lambda-B_\lambda x_\lambda-(y-x_\mu-B_\mu x_\mu),J(x_\lambda-x_\mu)\rangle\geq 0.$$

Hence

$$0\leq\langle x_\mu-x_\lambda,J(x_\lambda-x_\mu)\rangle+\langle B_\mu x_\mu-B_\lambda x_\lambda,J(x_\lambda-x_\mu)\rangle$$

and therefore

$$\|x_\lambda - x_\mu\|^2 \leq \langle B_\mu x_\mu - B_\lambda x_\lambda, J(x_\lambda - x_\mu)\rangle.$$

By Proposition 3.4.1(a), we know that B_λ is an accretive operator and $B_\lambda x \in AJ_\lambda x$. It follows from the above inequality that

$$\|x_\lambda - x_\mu\|^2 \leq \langle B_\mu x_\mu - B_\lambda x_\lambda, J(x_\lambda - x_\mu)\rangle + \langle B_\lambda x_\lambda - B_\mu x_\mu, J(J_\lambda x_\lambda - J_\mu x_\mu)\rangle.$$

Therefore

$$\|x_\lambda - x_\mu\|^2 \leq \langle B_\mu x_\mu - B_\lambda x_\lambda, J(x_\lambda - x_\mu) - J(J_\lambda x_\lambda - J_\mu x_\mu)\rangle. \tag{3.33}$$

Bearing in mind that we are under the hypothesis that $(\|B_\lambda x\|)_{\lambda>0}$ is bounded for small values of $\lambda > 0$, it is clear that

$$\|J_\lambda x_\lambda - x_\lambda\| \leq C\lambda, \quad \text{for every } \lambda > 0 \text{ close enough to zero.}$$

Since X is uniformly smooth, it is well known that the normalized duality map is single-valued and uniformly continuous on bounded sets, then from equation (3.33) we derive that $\|x_\lambda - x_\mu\| < \epsilon$ whenever $\lambda, \mu > 0$ are small enough. This means that there exists $x \in X$ such that

$$\lim_{\lambda\to 0} x_\lambda = x.$$

Finally, we are going to see that x is the solution of equation (3.31). Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lambda_n \to 0$ as $n \to +\infty$, $B_{\lambda_n} x_{\lambda_n} \rightharpoonup w$, and $w_{\lambda_n} \rightharpoonup w$, where $w_{\lambda_n} = y - B_{\lambda_n} x_{\lambda_n} - x_{\lambda_n} \in Ax_{\lambda_n}$. Passing to limits with λ_n in equation (3.30) and using Corollary 3.7.1, we conclude that x is a solution of equation (3.31). The uniqueness of x follows readily from the m-accretivity of A and B. ☐

Theorem 3.9.1. *Let X be a uniformly smooth Banach space. Let A and B be two m-accretive operators in X. If the following two assumptions hold true:*

(a) $D(A) \cap D(B) \neq \emptyset$,

(b) $\langle y, J(B_\lambda x)\rangle \geq 0$, *for all* $(x,y) \in A$ *and for every* $\lambda > 0$,

then $A + B$ is m-accretive.

Proof. Since the sum of two accretive operators in a smooth Banach space is also an accretive operator, we just have to check that $\mathrm{R}(I + A + B) = X$. Let $z \in X$ and consider the following inclusion:

$$z \in x + Ax + Bx.$$

In order to solve the above inclusion, we shall use Lemma 3.9.1. Let $x_\lambda \in D(A)$ be the unique solution of equation (3.30) with $y = z$. To obtain the result, it suffices to show that $\|B_\lambda x_\lambda\|$ remains bounded as $\lambda \to 0$, but this is a trivial consequence of the following fact. Because $y - x_\lambda - B_\lambda x_\lambda \in Ax_\lambda$, it follows from hypothesis (b) that

$$\langle y - x_\lambda - B_\lambda x_\lambda, J(B_\lambda x_\lambda)\rangle \geq 0,$$

and then

$$\|B_\lambda x_\lambda\|^2 \leq \|y - x_\lambda\|\|B_\lambda x_\lambda\|.$$

Since $(\|x_\lambda\|)_{\lambda\geq 0}$ is bounded, we achieve the result. ☐

We shall finish this section by giving an example of an m-accretive operator. Let Ω be an open and bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let β be a maximal monotone operator in $\mathbb{R} \times \mathbb{R}$ such that $0 \in \beta(0)$. Let $\tilde{\beta} \subseteq L^p(\Omega) \times L^p(\Omega)$, $1 \leq p < \infty$, be the operator defined by

$$\begin{aligned} D(\tilde{\beta}) &:= \{u \in L^p(\Omega) : \exists v \in L^p(\Omega) \text{ such that } v(x) \in \beta(u(x)) \text{ a. e. on } \Omega\}, \\ \tilde{\beta}(u) &:= \{v \in L^p(\Omega) : \ v(x) \in \beta(u(x)) \text{ a. e. on } \Omega, \ u \in D(\tilde{\beta})\}. \end{aligned}$$

It is easy to show that $\tilde{\beta}$ is an m-accretive operator in $L^p(\Omega)$. Next we shall give the proof of this fact in $L^1(\Omega)$. It is not hard to see that

$$\begin{aligned} ((I + \lambda\tilde{\beta})^{-1}u)(x) &= (I + \lambda\beta)^{-1}u(x) \quad \text{a. e. } \Omega, \\ (\tilde{\beta}_\lambda u)(x) &= \beta_\lambda(u(x)) \quad \text{a. e. } \Omega, \end{aligned}$$

for each $u \in L^p(\Omega)$ and for every $\lambda > 0$.

Let $u \in L^1(\Omega)$. Since β is a maximal monotone operator with $0 \in \beta(0)$, for each $x \in \Omega$, we can define $v(x) := (I + \beta)^{-1}(u(x))$, and then

$$|v(x) - 0| \leq |(I + \beta)^{-1}(u(x)) - (I + \beta)^{-1}(0)| \leq |u(x)|.$$

Since u is a measurable function and $(I + \beta)^{-1}$ is a continuous function, we have that v is measurable and, since $|v| \leq |u|$, $v \in L^1(\Omega)$. Since $u \in v + \tilde{\beta}(v)$, the conclusion is obtained.

Proposition 3.9.1. *Let* $1 < p < \infty$ *and let* $B : L^p(\Omega) \to L^p(\Omega)$ *be the operator defined by*

$$\begin{aligned} D(B) &:= W_0^{1,p} \cap W^{2,p} \cap D(\tilde{\beta}), \\ Bu &= -\Delta u + \tilde{\beta}(u), \quad u \in D(B). \end{aligned}$$

Then B is m-accretive.

Proof. First, we notice that if $1 < p < +\infty$, by Example 3.3.5, the operator $-\Delta$ with domain $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is accretive in $L^p(\Omega)$. Moreover, it is a well-known result (although quite hard) that $R(I - \Delta) = L^p(\Omega)$ (see, for instance, [8]). To obtain the result, we are going to apply Theorem 3.9.1 where $X = L^p(\Omega)$, $A = -\Delta$, $D(A) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, and $B = \tilde{\beta}$. We have to show

$$\int_\Omega \Delta u(x)\beta_\lambda(u(x))|\beta_\lambda(u(x))|^{p-2}\,dx \le 0,$$

for every $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ and for each $\lambda > 0$. Since β_λ is an increasing and Lipschitz function, we have $\frac{d}{dr}\beta_\lambda(r) \ge 0$. This implies the result by standard methods. Indeed, by Green's theorem,

$$\begin{aligned}\int_\Omega \Delta u(x)\beta_\lambda(u(x))|\beta_\lambda(u(x))|^{p-2}\,dx &= \int_{\partial\Omega} \beta_\lambda(u(x))|\beta_\lambda(u(x))|^{p-2}\frac{\partial}{\partial n}\nabla(u)\,dA \\ &\quad - \int_\Omega \langle \nabla u(x), \nabla(\beta_\lambda(u(x))|\beta_\lambda(u(x))|^{p-2})\rangle\,dx.\end{aligned}$$

Since $u \in W_0^{1,p}(\Omega)$ and $\beta_\lambda(0) = 0$, the above integral is equal to

$$-\int_\Omega \|\nabla u(x)\|^2 (p-1)\beta_\lambda'(u(x))|\beta_\lambda(u(x))|^{p-2}\,dx \le 0. \qquad \square$$

3.10 Crandall–Liggett exponential formula

In this section we discuss the generation of semigroups of nonexpansive mappings by accretive operators. We first present Crandall–Liggett formula [70].

Lemma 3.10.1.

(a) *For every* $n, m \in \mathbb{N}$ *with* $m \le n$ *and* $\alpha, \beta > 0$, *let* $a_{k,j} \in \mathbb{R}$, $0 \le k \le n$ *such that*

$$a_{k,j} \le \alpha a_{k-1,j-1} + \beta a_{k,j-1}. \tag{3.34}$$

Then

$$a_{m,n} \le \sum_{j=0}^{m-1} \alpha^j \beta^{n-j} \binom{n}{j} a_{m-j,0} + \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} a_{0,n-j}. \tag{3.35}$$

(b) *For every* $0 < m \le n$ *and* $0 < \alpha \le 1$, *we have*

$$\sum_{j=0}^{m} \binom{n}{j} \alpha^j (1-\alpha)^{n-j}(m-j) \le [(n\alpha - m)^2 + n\alpha(1-\alpha)]^{\frac{1}{2}},$$

$$\sum_{j=m}^{n}\binom{j-1}{m-1}\alpha^{m}(1-\alpha)^{j-m}(n-j)\le\left[\frac{m(1-\alpha)}{\alpha^{2}}+\left(\frac{m(1-\alpha)}{\alpha}+m-n\right)^{2}\right]^{\frac{1}{2}}.$$

Proof. (a) Take $m=1$. We shall prove this assertion by induction on $n\in\mathbb{N}$. It is clear that

$$a_{1,n}\le\beta^{n}a_{1,0}+\sum_{i=1}^{n}\alpha\beta^{j-1}a_{0,n-j}. \tag{3.36}$$

For $n=1$, equation (3.35) is reduced to $a_{1,1}\le\beta a_{1,0}+\alpha a_{0,0}$, which holds true by hypothesis (3.34).

Suppose that inequality (3.36) holds for $n\in\mathbb{N}$. By (3.34), we know that $a_{1,n+1}\le\alpha a_{0,n}+\beta a_{1,n}$. Then

$$\begin{aligned}a_{1,n+1}&\le\alpha a_{0,n}+\beta^{n+1}a_{1,0}+\sum_{j=1}^{n}\alpha\beta^{j}a_{0,n-j}\\&=\beta^{n+1}a_{1,0}+\sum_{j=1}^{n+1}\alpha\beta^{j-1}a_{0,n+1-j}.\end{aligned}$$

We obtain that (3.36) also holds for $n+1$.

Assume that expression (3.35) holds for $m\le n$. Let us see that it also holds for $n+1$.

We know that

$$a_{m,n+1}\le\alpha a_{m-1,n}+\beta a_{m,n},$$

hence

$$\begin{aligned}a_{m,n+1}\le{}&\alpha\sum_{j=0}^{m-2}\alpha^{j}\beta^{n-j}\binom{n}{j}a_{m-j-1,0}+\alpha\sum_{j=m-1}^{n}\alpha^{m-1}\beta^{j+1-m}\binom{j-1}{m-2}a_{0,n-j}\\&+\beta\sum_{j=0}^{m-1}\alpha^{j}\beta^{n-j}\binom{n}{j}a_{m-j,0}+\beta\sum_{j=m}^{n}\alpha^{m}\beta^{j-m}\binom{j-1}{m-1}a_{0,n-j}\\={}&\sum_{j=0}^{m-2}\alpha^{j+1}\beta^{n-j}\binom{n}{j}a_{m-j-1,0}+\sum_{j=m-1}^{n}\alpha^{m}\beta^{j+1-m}\binom{j-1}{m-2}a_{0,n-j}\\&+\sum_{j=0}^{m-1}\alpha^{j}\beta^{n+1-j}\binom{n}{j}a_{m-j,0}+\sum_{j=m}^{n}\alpha^{m}\beta^{j+1-m}\binom{j-1}{m-1}a_{0,n-j}\\={}&\sum_{j=1}^{m-1}\alpha^{j}\beta^{n+1-j}\binom{n}{j-1}a_{m-j,0}+\sum_{j=m-1}^{n}\alpha^{m}\beta^{j+1-m}\binom{j-1}{m-2}a_{0,n-j}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{m-1} \alpha^j \beta^{n+1-j}\binom{n}{j} a_{m-j,0} + \sum_{j=m}^{n} \alpha^m \beta^{j+1-m}\binom{j-1}{m-1} a_{0,n-j} \\
&= \sum_{j=0}^{m-1} \alpha^j \beta^{n+1-j}\binom{n+1}{j} a_{m-j,0} + \sum_{j=m}^{n+1} \alpha^m \beta^{j-m}\binom{j-1}{m-1} a_{0,n+1-j}.
\end{aligned}
$$

(b) Now consider the function $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto f(t) = (\alpha t + \beta)^n$. Using Newton's formula, we obtain

$$f(t) = \sum_{j=0}^{n} \binom{n}{j} t^j \alpha^j \beta^{n-j}.$$

If we replace β by $1-\alpha$ and compute the value of $f(1)$, we have

$$f(1) = \sum_{j=0}^{n} \binom{n}{j} \alpha^j (1-\alpha)^{n-j} = 1.$$

Differentiating f with respect to t and replacing t by 1 and β by $1-\alpha$, we obtain

$$f'(1) = \sum_{j=0}^{n} j\binom{n}{j} \alpha^j (1-\alpha)^{n-j} = \alpha n,$$

computing the second derivative of f and substituting $t = 1$ and $\beta = 1-\alpha$, we have

$$f''(1) = \sum_{j=2}^{n} \binom{n}{j} j(j-1) \alpha^j (1-\alpha)^{n-j} = n(n-1)\alpha^2,$$

thus, we conclude that

$$\sum_{j=0}^{n} j^2 \binom{n}{j} \alpha^j (1-\alpha)^{n-j} = \alpha^2 n(n-1) + \alpha n.$$

Since

$$\sum_{j=0}^{n} \binom{n}{j} \alpha^j (1-\alpha)^{n-j}(m-j) = \sum_{j=0}^{n} \sqrt{\binom{n}{j} \alpha^j (1-\alpha)^{n-j}} \sqrt{\binom{n}{j} \alpha^j (1-\alpha)^{n-j}(m-j)^2},$$

applying Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
\sum_{j=0}^{n} \binom{n}{j} \alpha^j (1-\alpha)^{n-j}(m-j) &\le \sqrt{\sum_{j=0}^{n} \binom{n}{j} \alpha^j (1-\alpha)^{n-j}} \sqrt{\sum_{j=0}^{n} \binom{n}{j} \alpha^j (1-\alpha)^{n-j}(m-j)^2} \\
&= \sqrt{m^2 - 2mn\alpha + \alpha^2 n(n-1) + \alpha n} = \sqrt{(n\alpha - m)^2 + n\alpha(1-\alpha)}.
\end{aligned}
$$

For showing the last inequality of (b), remember the formula

$$\sum_{j=m}^{\infty} \binom{j-1}{m-1} t^{j-m} = (1-t)^{-m}, \quad \text{whenever } |t| < 1.$$

This formula can be written as

$$\sum_{j=m}^{\infty} \binom{j-1}{m-1} t^{j-m} (1-t)^m = 1. \tag{3.37}$$

Differentiating (3.37) with respect to t, we obtain

$$\sum_{j=m+1}^{\infty} (j-m) \binom{j-1}{m-1} t^{j-m-1} (1-t)^m - m(1-t)^{m-1} \sum_{j=m}^{\infty} \binom{j-1}{m-1} t^{j-m} = 0,$$

and hence, by (3.37),

$$\sum_{j=m+1}^{\infty} (j-m) \binom{j-1}{m-1} t^{j-m-1} (1-t)^m = \frac{m}{1-t}.$$

This means that

$$\frac{1}{t} \sum_{j=m}^{\infty} (j-m) \binom{j-1}{m-1} t^{j-m} (1-t)^m = \frac{m}{1-t}.$$

Then

$$\sum_{j=m}^{\infty} (j-m) \binom{j-1}{m-1} t^{j-m} (1-t)^m = \frac{mt}{1-t}.$$

Consequently,

$$\sum_{j=m}^{\infty} j \binom{j-1}{m-1} t^{j} (1-t)^m - m \sum_{j=m}^{\infty} \binom{j-1}{m-1} t^{j} (1-t)^m = \frac{mt}{1-t},$$

and, by using equality (3.37) again,

$$\sum_{j=m}^{\infty} j \binom{j-1}{m-1} t^{j} (1-t)^m - m = \frac{mt}{1-t}.$$

Thus we may conclude that

$$\sum_{j=m}^{\infty} j \binom{j-1}{m-1} t^{j} (1-t)^m = \frac{mt}{1-t} + m = \frac{mt}{1-t} + \frac{m(1-t)}{1-t} = \frac{m}{1-t}. \tag{3.38}$$

A similar argument when differentiating expression (3.38) with respect to t yields

$$\sum_{j=m}^{\infty} j^2 \binom{j-1}{m-1} t^{j-m}(1-t)^m = m^2 + m\frac{1-t}{t} + 2\frac{m^2t^2}{1-t} + \frac{m^2t^2}{2} + m\frac{t^2}{(1-t)^2}. \tag{3.39}$$

Finally, by using again Cauchy–Schwarz inequality, along with expressions (3.37) and (3.39), we derive

$$\begin{aligned}
&\sum_{j=m}^{n} \binom{j-1}{m-1} \alpha^m (1-\alpha)^{j-m}(n-j) \\
&\quad \le \sqrt{\sum_{j=m}^{\infty} \binom{j-1}{m-1} \alpha^m (1-\alpha)^{j-m}} \sqrt{\sum_{j=m}^{\infty} \binom{j-1}{m-1} \alpha^m (1-\alpha)^{j-m}(n-j)^2} \\
&\quad \le \sqrt{m\frac{1-\alpha}{\alpha^2} + \left(m\frac{1-\alpha}{\alpha} + m - n\right)^2}.
\end{aligned}$$

☐

Proposition 3.10.1. *Let X be a Banach space and let $A : D(A) \to 2^X$ be an accretive operator. Then:*

(a) $\|J_\lambda^n x - x\| \le n\lambda |Ax|$, *for all* $x \in D(J_\lambda^n) \cap D(A)$.

(b) *If* $x \in D_\lambda$, $\lambda > 0$, *then for every* $\gamma > 0$ *we have*

$$\frac{\gamma}{\lambda}x + \left(1 - \frac{\gamma}{\lambda}\right)J_\lambda x \in D_\lambda \quad \textit{and} \quad J_\lambda x = J_\gamma\left(\frac{\gamma}{\lambda}x + \left(1 - \frac{\gamma}{\lambda}\right)J_\lambda x\right).$$

(c) *For each* $x \in D(J_\lambda^n) \cap D(J_\lambda^m) \cap D(A)$, $0 < \gamma \le \lambda$, *and* $n, m \in \mathbb{N}$ *with* $m \le n$, *the following holds:*

$$\|J_\gamma^n x - J_\lambda^m x\| \le ([(m\lambda - n\gamma)^2 + n\gamma(\lambda - \gamma)]^{\frac{1}{2}} + [(m\lambda - n\gamma)^2 + m\lambda(\lambda - \gamma]^{\frac{1}{2}})|Ax|.$$

Proof. (a) Let $x \in D(J_\lambda^n) \cap D(A)$. Since $J_\lambda^0 x = x$, we have

$$J_\lambda^n x - x = \sum_{i=0}^{n} (J_\lambda^{n-i} x - J_\lambda^{n-i-1} x).$$

Hence, using the fact that, for all $\lambda > 0$, J_λ is nonexpansive, one can write

$$\begin{aligned}
\|J_\lambda^n x - x\| &\le \sum_{i=0}^{n} \|J_\lambda^{n-i} x - J_\lambda^{n-i-1} x\| \\
&= \sum_{i=0}^{n} \|J_\lambda^{n-i-1}(J_\lambda x) - J_\lambda^{n-i-1} x\| \le \sum_{i=0}^{n} \|J_\lambda x - x\| \\
&= n\|J_\lambda x - x\| = n\lambda \|A_\lambda x\|.
\end{aligned}$$

By Proposition 3.4.1, we know that $\|A_\lambda x\| \le |Ax|$, hence

$$\|J_\lambda^n x - x\| \le n\lambda |Ax|.$$

(b) This result has been established in Proposition 3.4.1(e).

(c) Let $m, n \in \mathbb{N}$, $m \le n$. For $0 \le k \le m$, $0 \le j \le n$, and $x \in D(J_\lambda^m) \cap D(J_\gamma^n) \cap D(A)$ with $0 < \gamma \le \lambda$, denote

$$a_{k,j} := \|J_\gamma^j x - J_\lambda^k x\|.$$

Since $J_\lambda^{j-1} x \in D_\lambda$, by (b) we have

$$\frac{\gamma}{\lambda} J_\lambda^{k-1} x + \left(1 - \frac{\gamma}{\lambda}\right) J_\lambda^k x \in D_\lambda$$

and, moreover,

$$J_\lambda^k x = J_\gamma\left(\frac{\gamma}{\lambda} J_\lambda^{k-1} x + \left(1 - \frac{\gamma}{\lambda}\right) J_\lambda^k x\right).$$

So, we have

$$\begin{aligned} a_{k,j} &= \left\| J_\gamma^j x - J_\gamma\left(\frac{\gamma}{\lambda} J_\lambda^{k-1} x + \left(1 - \frac{\gamma}{\lambda}\right) J_\lambda^k x\right)\right\| \\ &\le \left\| J_\gamma^{j-1} x - \left(\frac{\gamma}{\lambda} J_\lambda^{k-1} x + \left(1 - \frac{\gamma}{\lambda}\right) J_\lambda^k x\right)\right\| \\ &\le \frac{\gamma}{\lambda}\|J_\gamma^{j-1} x - J_\lambda^{k-1} x\| + \left(1 - \frac{\gamma}{\lambda}\right)\|J_\gamma^{j-1} x - J_\lambda^k x\| \\ &= \alpha a_{k-1,j-1} + (1-\alpha) a_{k,j-1}, \end{aligned}$$

where $\alpha := \frac{\gamma}{\lambda}$, $\beta = 1 - \alpha$. Finally, Lemma 3.10.1 and (a) yield

$$\begin{aligned} a_{m,n} &= \|J_\gamma^n x - J_\lambda^m x\| \\ &\le \sum_{j=0}^{m-1} \alpha^j \beta^{n-j} \binom{n}{j} a_{m-j,0} + \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} a_{0,n-j} \\ &\le \sum_{j=0}^{m} \alpha^j \beta^{n-j} \binom{n}{j} \lambda |Ax| (m-j) + \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1} (n-j)\gamma |Ax| \\ &\le \left\{\lambda \sum_{j=0}^{m} \alpha^j \beta^{n-j} \binom{n}{j} (m-j) + \gamma \sum_{j=m}^{n} \alpha^j \beta^{n-j} \binom{n}{j} (m-j)\right\} |Ax| \\ &\le \left\{\lambda [n\alpha - m)^2 + n\alpha(1-\alpha)]^{\frac{1}{2}} + \gamma\left[\frac{m(1-\alpha)}{\alpha^2} + \left(\frac{m(1-\alpha)}{\alpha} + m - n\right)^2\right]^{\frac{1}{2}}\right\} |Ax| \\ &= \{[(n\gamma - \lambda m)^2 + n\gamma(\lambda - \gamma)]^{\frac{1}{2}} + [m\lambda(\lambda - \gamma) + (m\lambda + (m\lambda - \gamma n)^2]^{\frac{1}{2}}\} |Ax|, \end{aligned}$$

which ends the proof. □

Definition 3.10.1. An accretive operator $A : D(A) \to 2^X$ is said to satisfy the *range condition* if $\overline{D(A)} \subseteq \bigcap_{\lambda>0} R(I + \lambda A)$.

The main result of this section is the following theorem due to M. G. Crandall and T. M. Liggett [70].

Theorem 3.10.1. *Let X be a Banach space and let $A : D(A) \to 2^X$ be an accretive operator satisfying the range condition. The following limit*

$$S(t)x := \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} x$$

exists for each $x \in \overline{D(A)}$ uniformly on compact sets of $[0, +\infty[$. *Moreover,*

$$S : [0, +\infty[\times \overline{D(A)} \to \overline{D(A)}, \quad S(t)(x) := \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} x$$

is a continuous semigroup of nonexpansive mappings on $\overline{D(A)}$.

Proof. Since A satisfies the range property, $D(A) \subseteq D(J_\lambda^n)$ $\forall n \in \mathbb{N}$, $\forall \lambda > 0$.

Let $x \in D(A), m, n \in \mathbb{N}$, $m \le n$. If, in Proposition 3.10.1(c), we take $\gamma = \frac{t}{n}$ and $\lambda = \frac{t}{m}$, then we obtain

$$\|J_{t/n}^n x - J_{t/m}^m x\| \le 2t\left(\frac{1}{m} - \frac{1}{n}\right)^{\frac{1}{2}} |Ax|. \tag{3.40}$$

Clearly, the sequence $(J_{t/n}^n x)$ is a Cauchy sequence and, by the boundedness obtained in (3.40), if we define

$$S(t)x := \lim_{n\to\infty} J_{t/n}^n x,$$

this limit is uniform on compact subsets of $[0, +\infty[$.

On the other hand, since $\|J_{t/n}^n x - J_{t/n}^n y\| \le \|x - y\|$ for all $x,y \in D(A)$ and $t > 0$, we obtain

$$\|S(t)x - S(t)y\| \le \|x - y\| \quad \forall x,y \in D(A),\ t > 0.$$

By continuity of $S(t)$ on $D(A)$, we can extend $S(t)$ to whole $\overline{D(A)}$. Since $J_{t/n}^n x \in D(A)$, $\forall x \in D(A)$, $n \in \mathbb{N}$, $t \ge 0$, we derive that $S(t)x \in \overline{D(A)}$, whenever $x \in \overline{D(A)}$.

Let us show that

$$\|S(t)x - S(s)x\| \le 2|Ax||t - s| \quad \forall x \in D(A),\ s,t \ge 0.$$

Let $0 \le s \le t$ and $n \in \mathbb{N}$. By Proposition 3.10.1(c), with $\gamma = \frac{s}{n}$ and $\lambda = \frac{t}{n}$, we deduce

$$\|J^n_{t/n}x - J^n_{s/n}x\| \leq |Ax|\left\{\left[(s-t)^2 + \frac{s(t-s)}{n}\right]^{\frac{1}{2}} + \left[(s-t)^2 + \frac{t(t-s)}{n}\right]^{\frac{1}{2}}\right\}$$

and, therefore,

$$\|S(t)x - S(s)x\| \leq 2|Ax||t-s|, \quad \text{as } n \to +\infty.$$

In particular, the function $t \mapsto S(t)x$ is continuous on $[0,+\infty[$ for all $x \in D(A)$, and consequently on $\overline{D(A)}$.

Finally, we need to check the semigroup conditions of S. It is not hard to see that for every $x \in D(A)$,

$$S(t)^2x = \lim_{n\to\infty} J^{2n}_{t/n}x$$

and, in the same way, we have

$$S(t)^m x = \lim_{n\to\infty} J^{nm}_{t/n}x.$$

Therefore, for each $m \in \mathbb{N}$, we have

$$S(mt)x = \lim_{n\to+\infty} J^n_{tm/n}x = \lim_{k\to+\infty} J^{mk}_{mt/mk}x = \lim_{k\to+\infty} J^{mk}_{t/k}x = S(t)^m x.$$

Let $t,s \in \mathbb{Q}$, that is, $t = \frac{p}{q}$, $s = \frac{r}{v}$ where p, q, v, r are nonnegative integers with $q, v \neq 0$. Then,

$$S(t+s)x = S\left(\frac{pv+rq}{qv}\right)x = S\left(\frac{1}{qv}\right)^{pv+rq}x = S\left(\frac{1}{qv}\right)^{pv}\left(S\left(\frac{1}{qv}\right)^{rq}x\right) = S(t)S(s)x.$$

By density of $\mathbb{Q}$ in $\mathbb{R}$, we deduce the result. ☐

Remark 3.10.1. It is seen in the proof of Theorem 3.10.1 that $\|S(t)x - S(s)x\| \leq |Ax||t-s|$. So, the function $t \mapsto S(t)x$ is absolutely continuous on the compact subsets of $[0,+\infty[$.

Corollary 3.10.1. *Under the conditions of Theorem* 3.10.1, *for each $x \in \overline{D(A)}$, the following holds:*

$$\lim_{\lambda\to0^+} J^{[t/\lambda]}_{\lambda}x = \lim_{n\to\infty} J^n_{t/\lambda}x = S(t)x.$$

Proof. By using Proposition 3.10.1, we can see that $\{J^{[t/\lambda]}_{\lambda}x\}_{\lambda\to0^+}$ is Cauchy and

$$\lim_{\lambda\to0^+} J^{[t/\lambda]}_{\lambda}x - J^n_{t/n}x = 0.$$

☐

3.11 Quasiaccretive operators

Definition 3.11.1. Let X be a Banach space. An operator $A : D(A) \to 2^X$ is called *quasiaccretive* (*respectively, m-quasiaccretive*) if there exists $\omega \geq 0$ such that the operator $A + \omega I : D(A) \to 2^X$ is accretive (respectively, m-accretive). In this case we also call the operator A ω-accretive (respectively, ω-m-accretive). □

Coming back to the definition of accretiveness, it is clear that an operator $A : D(A) \to 2^X$ is ω-accretive if and only if the following estimate holds true:

$$\langle u - v, x - y\rangle_+ \geq -\omega\|x - y\|^2, \quad \text{for all } (x, u), (y, v) \in \mathrm{Gr}(A).$$

According to Proposition 1.10.3, we have $\langle u - v, x - y\rangle_+ = \|x - y\|[x - y, u - v]_s$. Hence replacing $\langle u-v, x-y\rangle_+$ by $\|x-y\|[x-y, u-v]_s$ in Definition 3.11.1, we obtain an equivalent definition of quasiaccretiveness. An operator $A : D(A) \to 2^X$ is ω-accretive if and only if the following estimate holds true:

$$[x - y, u - v]_s \geq -\omega\|x - y\|, \quad \text{for all } (x, u), (y, v) \in A.$$

An immediate consequence of Lemma 1.10.1 is that A is ω-accretive if and only if

$$\big\|x_1 - x_2 + \lambda(y_1 - y_2)\big\| \geq (1 - \lambda\omega)\|x_1 - x_2\| \quad \text{for } 0 < \lambda < \frac{1}{\omega} \text{ and } y_i \in A(x_i),\ i = 1, 2.$$

In the following propositions we collect some properties of quasiaccretive operators which can be obtained by using similar arguments as in Section 3.4.

Proposition 3.11.1. *Let $A : \mathrm{D}(A) \to X$ an operator on a Banach space X. The following conditions are equivalent:*

(a) *A is an ω-accretive operator;*

(b) $\|J_\lambda x - J_\lambda y\| \leq \frac{1}{1-\lambda\omega}\|x - y\|$, *for all $\lambda \in (0, 1/\omega)$, and $x, y \in R(I + \lambda A)$.*

Proposition 3.11.2. *Let X be a Banach space and let A be an ω-accretive operator in $X \times X$. Then:*

(a) *A_λ is ω-accretive and Lipschitz continuous with Lipschitz constant not greater than $\frac{2}{1-\lambda\omega}$ in $R(I + \lambda A)$, $\lambda \in (0, 1/\omega)$.*

(b) *$A_\lambda x \in AJ_\lambda x$, for all $x \in \mathrm{R}(I + \lambda A)$, $\lambda \in (0, 1/\omega)$.*

(c) $(1 - \lambda\omega)\|A_\lambda x\| \leq |Ax|$.

(d) $\lim_{\lambda\to 0} J_\lambda x = x$, *for all $x \in \overline{D(A)} \bigcap_{0<\lambda<\frac{1}{\omega}} R(I + \lambda A)$.*

3.12 Bibliographical remarks

The basic references that we have used in the development of this chapter are [32, 48, 64, 178]. Moreover, if the reader is interested in monotone operators on Hilbert spaces, a good reference is [45]. The results of Section 3.8 appear in [110]. In Section 3.9 we have followed [31]. For an exhaustive study of quasiaccretive operators, we refer to [32].

4 Abstract Cauchy problem

4.1 Introduction

Let X be a real Banach space and let $A : D(A) \to 2^X$ be a quasiaccretive operator and $f \in L^1_{\mathrm{loc}}(0,\infty;X)$. In this chapter we shall present some results about the existence and uniqueness of the solution to Cauchy problems in the form

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \\ u(0) = x \in \overline{D(A)}. \end{cases} \tag{4.1}$$

When $-A$ is a generator of a C_0-semigroup and, in problem (4.1), $f \equiv 0$, Theorem 3.1.1 guarantees that this problem admits a unique classical solution. Let us see that, when A is a nonlinear operator, the evolution equation (4.1), even for easy examples with $f \equiv 0$, does not admit any classical solution.

Consider the quasilinear equation

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}(\phi(u(t,x))) = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{4.2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is strictly increasing, onto, continuous, and $\phi(0) = 0$. This type of equations are called *scalar conservation laws' equations*.

We will see that Eq. (4.2) may be governed by an accretive operator. Let A be the operator in $L^1(\mathbb{R})$ defined by

$$\begin{cases} Au = \phi(u)', \quad \text{for } u \in D(A), \\ D(A) = \{u \in L^1(\mathbb{R}) : \phi(u) \in AC_{\mathrm{loc}}(\mathbb{R}),\ \phi(u)' \in L^1(\mathbb{R})\}, \end{cases}$$

where $AC_{\mathrm{loc}}(\mathbb{R})$ denotes the set of absolutely continuous functions on bounded intervals of $\mathbb{R}$.

If we denote $v(t) := u(t,\cdot) \in L^1(\mathbb{R})$, one may rewrite Eq. (4.2) as $v'(t) + A(v(t)) = 0$.

Proposition 4.1.1. *The operator A is accretive in $L^1(\mathbb{R})$.*

Proof. To show that A is an accretive operator, we have to check that if $u, v \in D(A)$ there exists $\gamma \in J(u-v)$, i. e., $\gamma \in L^\infty(\mathbb{R})$ with $\gamma(x) \in \|u-v\|_1 \operatorname{sign}(u(x)-v(x))$ a. e., such that

$$\langle A(u) - A(v), \gamma \rangle = \int_{\mathbb{R}} \gamma(x)(\phi(u)' - \phi(v)')(x)\,dx \geq 0.$$

In this sense, we have

$$\int_{\mathbb{R}} \gamma(x)(\phi(u)' - \phi(v)')(x)\,dx = \int_{\mathbb{R}} \gamma(x)(\phi(u) - \phi(v))'(x)\,dx.$$

Since ϕ is strictly increasing, it follows that

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$$\operatorname{sign}(u(x)-v(x)) = \operatorname{sign}(\phi(u(x))-\phi(v(x))),$$

and then

$$\frac{d}{dx}\big|\phi(u(x))-\phi(v(x))\big| = \frac{\gamma(x)}{\|u-v\|_1}(\phi(u(x))-\phi(v(x)))' \quad \text{a. e. on } \mathbb{R}.$$

Therefore, for every $R > 0$, we have

$$\int_{-R}^{R} \gamma(x)(\phi(u(x))-\phi(v(x)))'\,dx = \|u-v\|_1\big(\big|\phi(u(R))-\phi(v(R))\big| - \big|\phi(u(-R))-\phi(v(-R))\big|\big).$$

Since $u, v \in D(A)$, we have $\phi(u), \phi(v) \in AC_{\text{loc}}(\mathbb{R})$. This means that $\phi(u(x))$ and $\phi(v(x))$ have limits as $x \to \pm\infty$ and the value of these limits must be the value of ϕ at zero because $u, v \in L^1(\mathbb{R})$. Thus, we can conclude that

$$\langle A(u)-A(v), \gamma\rangle = \lim_{R\to+\infty}\int_{-R}^{R}\gamma(x)(\phi(u(x))-\phi(v(x)))'\,dx = 0. \qquad \square$$

Now we will give an example which shows that scalar conservation laws' equations, in general, do not admit classical solutions.

Example 4.1.1 (Burgers's equation). Consider the following partial differential equation:

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0. \tag{4.3}$$

Notice that equation (4.3) is a scalar conservation laws' equation for the particular case where $\phi(x) = \frac{x^2}{2}$. It can also be written in the following form:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0. \tag{4.4}$$

Suppose that (4.3) has a classical solution, say w, satisfying the initial condition $w(0,x) = f(x)$. For all real numbers t_0, x_0 with $t_0 > 0$, let $x(t)$ be the solution of

$$\frac{d}{dt}x = w(t,x), \quad x(t_0) = x_0. \tag{4.5}$$

Such paths are called *characteristic paths*. Using the chain rule, we obtain that the solutions of (4.4) are constant on the characteristic paths. Indeed,

$$\frac{d}{dt}w(t,x(t)) = \frac{\partial}{\partial x}w\frac{d}{dt}x + \frac{\partial}{\partial t}w = w\frac{\partial}{\partial x}w + \frac{\partial}{\partial t}w = 0,$$

hence $w(t,x(t)) = \text{const}$. Moreover, these characteristics paths are straight lines (because by (4.5) they have constant slopes).

Let $x(t), y(t)$ be two characteristic lines, and suppose that $x(0) = z, y(0) = v$. This means that, for all $t \geq 0$, $w(t, x(t)) = f(z)$ and $w(t, y(t)) = f(v)$. Thus

$$x(t) = f(z)t + z \quad \text{and} \quad y(t) = f(v)t + v.$$

If $f(z) \neq f(v)$, then the characteristic lines intersect at one point, namely (t_1, x_1).

Consider the problem

$$\frac{d}{dt}x = w(t, x), \quad x(t_1) = x_1.$$

By the above argument, this problem has two solutions which are the characteristic lines $x(t), y(t)$, but this is a contradiction since this problem has a unique solution. Consequently, Burgers's equation does not have classical solutions.

4.2 Strong solutions for the Cauchy problem

Let X be a real Banach space, $A : D(A) \to 2^X$ a quasiaccretive operator, and $f \in L^1_{\text{loc}}(0, \infty; X)$. Consider the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \\ u(0) = x \in D(A). \end{cases} \tag{4.6}$$

As we have seen in the introduction, when A is a nonlinear operator, the concept of classical solution does not always make sense. Thus, we introduce the notion of a strong solution to (4.6).

Definition 4.2.1. A function $u : [0, +\infty[\to X$ is a strong solution of problem (4.6) if:

(a) u is absolutely continuous on every compact subset of $[0, +\infty[$,

(b) u is differentiable a. e. in $[0, +\infty[$,

(c) $u(0) = x$,

(d) $f(t) \in \frac{d}{dt}u(t) + Au(t)$ a. e. in $[0, +\infty[$.

Lemma 4.2.1 (Kato). *Let X be a Banach space and let $f : \mathbb{R} \to X$ be a function satisfying:*

(a) *$t \to \|f(t)\|$ is differentiable a. e. in $\mathbb{R}$,*

(b) *the weak derivative of f (say, f') exists a. e. in $\mathbb{R}$.*

Then

$$\|f(t)\| \frac{d}{dt}\|f(t)\| = \langle x^*, f'(t) \rangle \quad \forall x^* \in J(f(t)) \text{ a. e. in } \mathbb{R}.$$

Proof. For every $t, s \in \mathbb{R}$ and $x^* \in J(f(t))$, we have

$$\langle x^*, f(t) \rangle = \|f(t)\|^2, \quad \langle x^*, f(s) \rangle \leq \|f(t)\| \|f(s)\|,$$

and therefore,

$$\langle x^*, f(s) - f(t)\rangle \leq \|f(t)\|\|f(s)\| - \|f(t)\|^2 = \|f(t)\|(\|f(s)\| - \|f(t)\|).$$

Consider $t \in \mathbb{R}$ and suppose that f is weakly differentiable at t and $\|f(t)\|$ is differentiable. For each $x^* \in J(f(t))$, we know that

$$\begin{aligned}\langle x^*, f'(t)\rangle &= \lim_{h\to 0^+}\left\langle x^*, \frac{f(t+h) - f(t)}{h}\right\rangle \\ &\leq \lim_{h\to 0^+}\|f(t)\|\frac{\|f(t+h)\| - \|f(t)\|}{h} \\ &= \|f(t)\|\frac{d}{dt}\|f(t)\|.\end{aligned}$$

The same argument yields

$$\begin{aligned}\langle x^*, f'(t)\rangle &= \lim_{h\to 0^-}\left\langle x^*, \frac{f(t+h) - f(t)}{h}\right\rangle \\ &\geq \lim_{h\to 0^-}\|f(t)\|\frac{\|f(t+h)\| - \|f(t)\|}{h} = \|f(t)\|\frac{d}{dt}\|f(t)\|,\end{aligned}$$

and so the proof of the lemma is complete. ☐

Proposition 4.2.1. *Let $A \subseteq X \times X$ be an ω-accretive operator and $f, g \in L^1(0, T, X)$.*

(i) *If u and v are strong solutions of $u' + Au \ni f$ and $v' + Av \ni g$ on $[0, T]$, respectively, then*

$$\begin{aligned}\|u(t) - v(t)\|^2 - \|u(s) - v(s)\|^2 \leq{}& 2w\int_s^t \|u(\tau) - v(\tau)\|^2\, d\tau \\ &+ 2\int_s^t \langle f(\tau) - g(\tau), u(\tau) - v(\tau)\rangle_+\, d\tau, \qquad (4.7)\end{aligned}$$

for $0 \leq s < t \leq T$.

(ii) *The initial value problem $u' + Au \ni f$, $u(0) = x$, has at most one strong solution.*

Proof. By Lemma 4.2.1, for each $x^* \in J(u(t) - v(t))$, we have

$$\|u(t) - v(t)\|\frac{d}{dt}\|u(t) - v(t)\| = \left\langle x^*, \frac{d}{dt}(u(t) - v(t))\right\rangle \quad \text{a. e.} \qquad (4.8)$$

Since u and v are strong solutions of (4.6), we have

$$f(t) - \frac{d}{dt}u(t) \in Au(t) \quad \text{and} \quad g(t) - \frac{d}{dt}v(t) \in Av(t) \quad \text{a. e.},$$

and therefore

$$g(t) - \frac{d}{dt}v(t) - \left(f(t) - \frac{d}{dt}u(t)\right) \in Av(t) - Au(t) \quad \text{a. e.}$$

Since A is quasiaccretive, there exist $w \in \mathbb{R}$ and $j \in J(u(t) - v(t))$ such that

$$\left\langle j, f(t) - g(t) - \frac{d}{dt}(u(t) - v(t)) \right\rangle \geq -w\|u(t) - v(t)\|^2. \tag{4.9}$$

Now combining (4.8) and (4.9), we get

$$-w\|u(t) - v(t)\|^2 \leq \langle j, f(t) - g(t)\rangle - \|u(t) - v(t)\| \frac{d}{dt}\|u(t) - v(t)\|,$$

and consequently,

$$\|u(t) - v(t)\| \frac{d}{dt}\|u(t) - v(t)\| \leq w\|u(t) - v(t)\|^2 + \langle j, f(t) - g(t)\rangle. \tag{4.10}$$

Integration over $[s, t]$ yields

$$\frac{1}{2}(\|u(t) - v(t)\|^2 - \|u(s) - v(s)\|^2) \leq w \int_s^t \|u(\tau) - v(\tau)\|^2 \, d\tau + \int_s^t \langle f(\tau) - g(\tau), u(\tau) - v(\tau)\rangle_+ \, d\tau.$$

To obtain (ii), we note that (4.10) implies

$$\frac{d}{d\tau}(e^{-2w\tau}\|u(\tau) - v(\tau)\|^2) \leq 2e^{-2w\tau}\langle f(\tau) - g(\tau), u(\tau) - v(\tau)\rangle_+$$

and so by integration we get

$$\|u(t) - v(t)\|^2 \leq e^{2wt}\|u(0) - v(0)\|^2 + 2\int_0^t e^{2w(t-\tau)}\langle f(\tau) - g(\tau), u(\tau) - v(\tau)\rangle_+ \, d\tau. \tag{4.11}$$

Now, according to (4.11), it is clear that if $f = g$ and $u(0) = v(0)$, then strong solution (when it exists) is unique. ☐

4.3 The homogeneous problem for accretive operators

In this section we will study the existence of a strong solution to the problem

$$\begin{cases} u'(t) + A(u(t)) \ni 0, \\ u(0) = x \in D(A). \end{cases} \tag{4.12}$$

where X is a Banach space and $A : D(A) \to 2^X$ is an accretive operator in X.

Lemma 4.3.1. *Let $A : D(A) \to 2^X$ be an accretive operator satisfying the range condition (see Definition* 3.10.1*) and let $(S(t))_{t\geq 0}$ be the semigroup generated by Crandall–Liggett's exponential formula. Then, for each $x \in \overline{D(A)}$ and $(x_0, u_0) \in A$, the following estimate holds true:*

$$\limsup_{t\to 0^+} \left\langle \frac{S(t)x - x}{t}, x - x_0 \right\rangle_+ \leq \langle u_0, x_0 - x\rangle_+ .$$

Proof. Let $\lambda > 0$ and $n \in \mathbb{N}$. According to Proposition 3.4.1(a), we have $A_\lambda J_\lambda^{n-1} x \in AJ_\lambda^n x$. Since A is accretive and $u_0 \in Ax_0$ (by hypothesis), there exists $x^* \in J(x_0 - J_\lambda^n x)$ such that

$$\langle x^*, u_0 - A_\lambda J_\lambda^{n-1} x\rangle \geq 0. \tag{4.13}$$

By the definition of the Yosida approximant of A and using the inequality $ab \leq \frac{a^2+b^2}{2}$, we get

$$\begin{aligned}
\langle x^*, A_\lambda J_\lambda^{n-1} x\rangle &= \frac{1}{\lambda}\langle x^*, J_\lambda^{n-1} x - J_\lambda^n x\rangle \\
&= \frac{1}{\lambda}\langle x^*, x_0 - J_\lambda^n x\rangle + \frac{1}{\lambda}\langle x^*, J_\lambda^{n-1} x - x_0\rangle \\
&= \frac{1}{\lambda}\big(\langle x^*, x_0 - J_\lambda^n x\rangle - \langle x^*, x_0 - J_\lambda^{n-1} x\rangle\big) \\
&\geq \frac{1}{\lambda}\big(\|x_0 - J_\lambda^n x\|^2 - \|x_0 - J_\lambda^n x\|\|x_0 - J_\lambda^{n-1} x\|\big) \\
&\geq \frac{1}{2\lambda}\big(\|x_0 - J_\lambda^n x\|^2 - \|x_0 - J_\lambda^{n-1} x\|^2\big).
\end{aligned}$$

We know from (4.13) that $\langle x^*, u_0\rangle \geq \langle x^*, A_\lambda J_\lambda^{n-1} x\rangle$. So, the previous estimate yields

$$\|J_\lambda^n x - x_0\|^2 - \|x_0 - J_\lambda^{n-1} x\|^2 \leq 2\lambda\langle x^*, u_0\rangle. \tag{4.14}$$

Set

$$f(x, y) := \langle y, x\rangle_+ .$$

Inequality (4.14) can be written as

$$\|J_\lambda^n x - x_0\|^2 - \|x_0 - J_\lambda^{n-1} x\|^2 \leq 2 \int_{n\lambda}^{(n+1)\lambda} f(x_0 - J_\lambda^{[s/\lambda]} x, u_0)\, ds.$$

Hence, by induction, for each $n \in \mathbb{N}$ we get

$$\begin{aligned}\|J_\lambda^n x - x_0\|^2 - \|x_0 - x\|^2 &= \sum_{k=1}^{n}(\|J_\lambda^k x - x_0\|^2 - \|x_0 - J_\lambda^{k-1}x\|^2)\\ &\le 2\int_\lambda^{(n+1)\lambda} f(x_0 - J_\lambda^{[s/\lambda]}x, u_0)\, ds.\end{aligned}$$

Next, applying Corollary 3.10.1 to the above inequality, we obtain

$$\|S(t)x - x_0\|^2 - \|x - x_0\|^2 \le 2\int_0^t f(x_0 - S(s)x, u_0)\, ds. \tag{4.15}$$

Finally, we notice that (cf. Example 1.11.1), for every $x^* \in J(x - x_0) = \partial(\frac{1}{2}\|x - x_0\|^2)$, we have

$$\langle x^*, z - x\rangle \le \frac{1}{2}(\|z - x_0\|^2 - \|x - x_0\|^2),$$

and therefore,

$$\frac{1}{2}\|S(t)x - x_0\|^2 - \frac{1}{2}\|x - x_0\|^2 \ge \langle x^*, S(t)x - x\rangle.$$

Now, applying (4.15), we derive

$$\langle x^*, S(t)x - x\rangle \le \int_0^t f(x_0 - S(s)x, u_0)\, ds = \int_0^1 tf(x_0 - S(t\tau)x, u_0)\, d\tau, \quad \forall x^* \in J(x - x_0).$$

Next using the fact that f is upper semicontinuous (see Proposition 1.10.1(c)) and since, by Remark 3.10.1, the function $t \mapsto S(t)x$ is continuous, we conclude

$$\lim_{t\to 0^+}\left\langle x^*, \frac{S(t)x - x}{t}\right\rangle \le f(x_0 - x, u_0), \quad \forall x^* \in J(x - x_0). \qquad \square$$

Theorem 4.3.1. *Let X be a Banach space, let $A : D(A) \to 2^X$ be a closed accretive operator satisfying the range condition, and denote by $(S(t))_{t\ge 0}$ the semigroup generated by Crandall–Liggett's exponential formula. If $x \in D(A)$ and the function $t \mapsto S(t)x$ is differentiable a. e. in $[0, +\infty[$, then $u(t) := S(t)x$ is the strong solution of problem* (4.12).

Proof. Let $x \in D(A)$. We know that the function $t \mapsto S(t)x$ is absolutely continuous on the compact intervals of $[0, \infty)$ (see Remark 3.10.1). Thus, condition (a) of Definition 4.2.1 is fulfilled. Since $t \mapsto S(t)x$ is differentiable a. e., condition (b) in the definition of a strong solution is also satisfied. It is also clear that $S(0)x = x$, and so condition (c) of Definition 4.2.1 is satisfied. The only thing that we have to show is

$$0 \in S'(t)x + AS(t)x \quad \text{a. e.} \tag{4.16}$$

To establish (4.16), it is equivalent to check that

$$\big(S(t)x, -S'(t)x\big) \in A \quad \text{a. e.} \tag{4.17}$$

Let $t_0 \geq 0$ be such that the function $t \mapsto S(t)x$ is differentiable at t_0. Then there exists a function $r(\cdot, t_0) : \mathbb{R} \to X$ such that $\lim_{\lambda \to 0} r(\lambda, t_0) = 0$, and

$$S(t_0 - \lambda)x = S(t_0)x - \lambda S'(t_0)x + \lambda r(\lambda, t_0),$$

where $0 < \lambda < t_0$. Since $S(t_0 - \lambda)x \in \overline{D(A)}$ and A satisfies the range condition, we have

$$S(t_0 - \lambda)x \in \overline{D(A)} \subseteq \bigcap_{\lambda > 0} R(I + \lambda A).$$

This means that there exists $(x_\lambda, y_\lambda) \in A$ with

$$x_\lambda + \lambda y_\lambda = S(t_0 - \lambda)x = S(t_0)x - \lambda S'(t_0)x + \lambda r(\lambda, t_0). \tag{4.18}$$

Applying Lemma 4.3.1 to $S(t_0)x \in \overline{D(A)}$ and $(x_\lambda, y_\lambda) \in A$, we obtain

$$\big\langle S'(t_0)x, S(t_0)x - x_\lambda \big\rangle_+ \leq \big\langle y_\lambda, x_\lambda - S(t_0)x \big\rangle_+. \tag{4.19}$$

Since $J(x_\lambda - S(t_0)x)$ is weakly* compact in X^*, the supremum in (4.19) is achieved, that is, for all $x^* \in J(S(t_0)x - x_\lambda)$, there exists $x_\lambda^* \in J(x_\lambda - S(t_0)x)$ such that

$$\langle x_\lambda^*, y_\lambda \rangle \geq \langle x^*, S'(t_0)x \rangle$$

From the above $-x_\lambda^* \in J(S(t_0)x - x_\lambda)$, and then $\langle x_\lambda^*, y_\lambda \rangle \geq \langle -x_\lambda^*, S'(t_0)x \rangle$. In particular, we have

$$\langle -x_\lambda^*, y_\lambda + S'(t_0)x \rangle \leq 0.$$

Now, using (4.18), we conclude that

$$\left\langle -x_\lambda^*, \frac{S(t_0)x - x_\lambda}{\lambda} + r(\lambda, t_0) \right\rangle \leq 0$$

whenever $0 < \lambda < t_0$. Since $x_\lambda^* \in J(x_\lambda - S(t_0)x)$, we derive

$$\frac{1}{\lambda} \|S(t_0)x - x_\lambda\|^2 \leq \langle x_\lambda^*, r(\lambda, t_0) \rangle \leq \|S(t_0)x - x_\lambda\| \|r(\lambda, t_0)\|.$$

Hence $\frac{\|S(t_0)x - x_\lambda\|}{\lambda} \leq \|r(\lambda, t_0)\|$, and therefore

$$\lim_{\lambda \to 0^+} \frac{S(t_0)x - x_\lambda}{\lambda} = 0.$$

Accordingly, $\lim_{\lambda\to 0^+} x_\lambda = S(t_0)x$. Bearing in mind (4.18), we conclude that

$$\lim_{\lambda\to 0^+} y_\lambda = \lim_{\lambda\to 0^+}\left(\frac{S(t_0)x - x_\lambda}{\lambda} - S'(t_0)x + r(\lambda, t_0)\right) = -S'(t_0)x.$$

Next, since A is closed, we conclude that $(S(t_0)x, -S'(t_0)x) \in A$, which ends the proof. ☐

Corollary 4.3.1. *Let X be a reflexive Banach space and let $A : D(A) \to 2^X$ be an m-accretive operator. Then, for each $x \in D(A)$, problem* (4.1) *has a unique strong solution, which is given by*

$$S(t)x = \lim_{n\to+\infty}\left(I + \frac{t}{n}A\right)^{-n} x, \quad \textit{for all } t \geq 0.$$

Proof. Since A is m-accretive by Proposition 3.5.1, A satisfies the range condition. Moreover, Proposition 3.5.1 guarantees that A is maximal accretive, and the use of Proposition 3.7.1 shows that A is closed. On the other hand, Remark 3.10.1 says that the function $t \mapsto S(t)x$ is absolutely continuous on compact intervals of $[0, +\infty[$, and then, by Theorem 1.14.6, such a function is differentiable a. e. in $[0, +\infty[$. Now applying Theorem 4.3.1, we conclude the existence of strong solutions. The uniqueness follows from Proposition 4.2.1. ☐

Definition 4.3.1. A Banach space X has *Radon–Nikodym property* (RN, for short) if every absolutely continuous mapping $h : [0, T] \to X$ is differentiable a. e.

Remark 4.3.1. In Corollary 4.3.1 we have used the reflexivity of X to show that the function $t \to S(t)x$ is differentiable a. e. because it is absolutely continuous. This fact can be generalized to Banach spaces with Radon–Nikodym property. Thus, Corollary 4.3.1 holds if we replace "X is reflexive" by "X has Radon–Nikodym property." At this point, we refer to [79] where the reader will find a deep study of Banach spaces with Radon–Nikodym property. We mention, in particular, two very important classes of Banach spaces which enjoy this property: reflexive and separable dual Banach spaces.

Recall that $BV(0, T; X)$ is the subspace of $L^1(0, T; X)$ consisting of those functions f which satisfy

$$V(f, T) = \limsup_{h\downarrow 0} \int_0^{T-h} \frac{\|f(\tau + h) - f(\tau)\|}{h}\, d\tau < \infty,$$

and, if $f \in BV(0, T; X)$, then it is essentially bounded and it has an essential limit from the right denoted by $f(t+)$ at every $t \in [0, T)$. We will also use the notation

$$V(f, t+) := \limsup_{h\downarrow 0} \int_0^{t} \frac{\|f(\tau + h) - f(\tau)\|}{h}\, d\tau \quad \text{for } 0 \leq t < T.$$

4.4 Integral solutions

Let X be the Banach space $C([-1,1])$. In [178, Example 4.7] an m-accretive operator $A \subseteq X \times X$ is given such that its associated semigroup by Crandall–Liggett's exponential formula is not differentiable a. e. and then the corresponding Cauchy problem does not have any strong solution (evidently, X does not satisfy Radon–Nikodym property). In order to solve this problem, we should give a more general concept of a solution for the Cauchy problem (4.12).

Definition 4.4.1. Let X be a Banach space. A function $u : [0,+\infty) \to X$ is called an integral solution of the Cauchy problem (4.12) with initial data $x_0 \in \overline{D(A)}$ if u is continuous, $u(0) = x_0$, and

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \le 2 \int_s^t \langle -y, u(\tau) - x\rangle_+ \, d\tau,$$

for every $(x,y) \in A$, $0 \le s \le t < +\infty$.

Remark 4.4.1. Let A be an accretive operator satisfying the range condition. Let $x_0 \in \overline{D(A)}$. If the function $u(t)$ is a strong solution of the homogeneous problem, then it is also an integral solution of such a problem. To see this, assume that u is a strong solution to the Cauchy problem (4.12), thus we have

$$u'(t) + A(u(t)) \ni 0 \quad \text{a. e. and} \quad u(0) = x_0.$$

Then, $-u'(t) \in A(u(t))$ a. e. Therefore, for $(x,y) \in A$, we have $\langle -u'(t) - y, u(t) - x\rangle_+ \ge 0$ a. e. This means that there exists $j(t) \in J(u(t) - x)$ such that

$$\langle u'(t) - 0, j(t)\rangle \le \langle -y, u(t) - x\rangle_+.$$

Now by Kato's differentiation rule, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t) - x\|^2 \le \langle -y, u(t) - x\rangle_+,$$

and hence, by integration,

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \le 2 \int_s^t \langle -y, u(\tau) - x\rangle_+ \, d\tau.$$

This proves our claim.

Theorem 4.4.1. *Let X be a Banach space and let $A : D(A) \to 2^X$ be an accretive operator satisfying the range condition. Then, for the Cauchy problem*

$$\begin{cases} u'(t) + A(u(t)) \ni 0, \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{4.20}$$

the function $u(t) := S(t)(x_0) = \lim_{n\to+\infty}(I + \frac{t}{n}A)^{-n}(x_0)$ *is an integral solution.*

Proof. Let $\lambda > 0$, $n \in \mathbb{N}$, and $(x,y) \in A$. By Proposition 3.4.1(a), $A_\lambda J_\lambda^{n-1}x_0 \in AJ_\lambda^n x_0$. Since A is an accretive operator, there exists $x^* \in J(x - J_\lambda^n x_0)$ such that

$$\langle x^*, y - A_\lambda J_\lambda^{n-1}x_0\rangle \geq 0. \tag{4.21}$$

Using the definition of the Yosida approximant A_λ and the estimate $ab \leq \frac{a^2+b^2}{2}$, we get

$$\begin{aligned}
\langle x^*, A_\lambda J_\lambda^{n-1}x_0\rangle &= \frac{1}{\lambda}\langle x^*, J_\lambda^{n-1}x_0 - J_\lambda^n x_0\rangle = \frac{1}{\lambda}\langle x^*, x - J_\lambda^n x_0\rangle + \frac{1}{\lambda}\langle x^*, J_\lambda^{n-1}x_0 - x\rangle \\
&= \frac{1}{\lambda}(\langle x^*, x - J_\lambda^n x_0\rangle - \langle x^*, x - J_\lambda^{n-1}x_0\rangle) \\
&\geq \frac{1}{\lambda}(\|x - J_\lambda^n x_0\|^2 - \|x - J_\lambda^n x_0\|\|x - J_\lambda^{n-1}x_0\|) \\
&\geq \frac{1}{2\lambda}(\|x - J_\lambda^n x_0\|^2 - \|x - J_\lambda^{n-1}x_0\|^2).
\end{aligned}$$

From the above estimate and (4.21), we obtain

$$\|J_\lambda^n x_0 - x\|^2 - \|x - J_\lambda^{n-1}x_0\|^2 \leq 2\lambda\langle x^*, y\rangle. \tag{4.22}$$

Set $f(x,y) := \langle y, x\rangle_+$. Inequality (4.22) can be written as

$$\|J_\lambda^n x_0 - x\|^2 - \|x - J_\lambda^{n-1}x_0\|^2 \leq 2\int_{n\lambda}^{(n+1)\lambda} f(x - J_\lambda^{[s/\lambda]}x_0, y)\, ds.$$

Consequently,

$$\begin{aligned}
\|J_\lambda^n x_0 - x\|^2 - \|x - x_0\|^2 &= \sum_{k=1}^{n}(\|J_\lambda^k x_0 - x\|^2 - \|x - J_\lambda^{k-1}x_0\|^2) \\
&\leq 2\int_{\lambda}^{(n+1)\lambda} f(x - J_\lambda^{[s/\lambda]}x_0, y)\, ds.
\end{aligned}$$

Applying Corollary 3.10.1 to the latter inequality, we derive

$$\|S(t)x_0 - x\|^2 - \|x_0 - x\|^2 \leq 2\int_0^t f(x - S(s)x_0, y)\, ds. \tag{4.23}$$

Finally, if $0 \leq s < t < \infty$ and $u(t) = S(t)(x_0)$, it follows from (4.23) that

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \leq 2 \int_s^t f(x - u(\tau), y)\, d\tau = 2 \int_s^t \langle -y, u(\tau) - x \rangle_+\, d\tau. \qquad \square$$

To establish the uniqueness of integral solutions, we will use the following result.

Lemma 4.4.1. *Let K be a nonempty, convex, and closed subset of a Banach space X and let $T : K \to K$ be a nonexpansive mapping. Then for every $x \in K$, the equation*

$$\begin{cases} u'(t) + (I - T)(u(t)) = 0, \\ u(0) = x, \end{cases} \tag{4.24}$$

has a unique classical solution $u \in C^1(0, \infty; X)$ such that $u(t) \in K$ for all $t \geq 0$ and

$$\|u(n) - T^n(x)\| \leq \sqrt{n} \|x - Tx\|, \quad n \in \mathbb{N}.$$

Proof. Set $v(t) = e^t u(t)$, then $v'(t) = e^t u(t) + e^t u'(t) = e^t(u(t) + u'(t))$, and therefore, equation (4.24) can be written in the form

$$v'(t) = e^t T(u(t)) = e^t T(e^{-t} v(t)),$$

Thus, it is clear that problem (4.24) is equivalent to the integral equation

$$u(t) = e^{-t} x + e^{-t} \int_0^t e^s T(u(s))\, ds = e^{-t} x + \int_0^t e^{s-t} T(u(s))\, ds. \tag{4.25}$$

To establish the existence and uniqueness of the solution of problem (4.24), we are going to apply the Banach contraction principle. Let $T > 0$ and consider the Banach space $(C(0, T; X), \| \cdot \|_\infty)$ where $\|u\|_\infty := \sup\{\|u(t)\| : t \in [0, T]\}$.

Let $S := \{u \in C(0, T; X) : u(t) \in K\ \forall t \in [0, T]\}$. Clearly, S is closed and convex. Denote by Q the operator defined on S by

$$Qu(t) := e^{-t} x + \int_0^t e^{s-t} T(u(s))\, ds.$$

We first check that Q is well defined. To do this, we only have to see that the function $t \mapsto T(u(t))$ is Bochner integrable. Since this function is continuous, it is measurable. Moreover, the map $s \mapsto \|T(u(t))\|$ is Lebesgue integrable, and hence, by Theorem 1.14.3, we conclude that the map $t \mapsto T(u(t))$ is Bochner integrable.

Let us see now that $Q(S) \subseteq S$. Let $P = \{s_0, \ldots, s_n\}$ be a partition of $[0, t]$, then

$$\frac{\int_0^t e^{s-t} T(u(s))\, ds}{\int_0^t e^{s-t}\, ds} = \lim_{|P| \to 0} \frac{\sum_{i=1}^n e^{s_i - t} T(u(s_i))(s_i - s_{i-1})}{\sum_{i=1}^n e^{s_i - t}(s_i - s_{i-1})}.$$

Since, for each i, $i = 1, \dots, n$, $T(u(s_i)) \in K$, and K is convex, we have

$$\sum_{i=1}^{n} \frac{e^{s_i - t}(s_i - s_{i-1})}{\sum_{i=1}^{n} e^{s_i - t}(s_i - s_{i-1})} T(u(s_i)) \in K.$$

Using the closedness of K, we conclude that

$$\frac{\int_0^t e^{s-t} T(u(s))\, ds}{\int_0^t e^{s-t}\, ds} \in K,$$

and hence,

$$\int_0^t e^{s-t} T(u(s))\, ds \in (1 - e^{-t})K.$$

Again the convexity of K implies that

$$Qu(t) = e^{-t}x + \int_0^t e^{s-t} T(u(s))\, ds \in K.$$

Next, for any $u, v \in S$, we have

$$\begin{aligned} \|Qu - Qv\|_\infty &= \sup_{t\in[0,T]} \left\| \int_0^t e^{s-t}\big(Tu(s) - Tv(s)\big)\, ds \right\| \\ &\le \sup_{t\in[0,T]} \int_0^t e^{s-t} \|u(s) - v(s)\|\, ds \\ &\le (1 - e^{-T})\|u - v\|_\infty. \end{aligned}$$

This shows that Q is $(1 - e^{-T})$-contractive. Hence, by the Banach contraction principle, equation (4.25) has a unique solution $u \in S$. Since $T > 0$ is arbitrary, the first part of the lemma is proved.

To show the second part of the lemma, we note that, because T is a nonexpansive mapping, we have

$$\begin{aligned} \langle x - Tx - (y - Ty), x - y\rangle_+ &\ge \|x - y\|^2 - \langle Ty - Tx, x - y\rangle_+ \\ &\ge \|x - y\|^2 - \|Tx - Ty\|\|x - y\| \\ &\ge 0. \end{aligned}$$

This shows that the operator $I - T$ is accretive. If u is a solution of equation (4.24), by Kato's differentiation rule, we have

$$\|u(t)-x\|\frac{d}{dt}\|u(t)-x\| \geq \langle x^*,(T-I)u(t)\rangle \quad \text{for all } t \geq 0 \text{ and } x^* \in J(u(t)-x).$$

Since $I-T$ is accretive, there exists $x^* \in J(u(t)-x)$ such that

$$\|u(t)-x\|\frac{d}{dt}\|u(t)-x\| \leq \langle x^*,(T-I)x\rangle \quad \forall t \geq 0,$$

and therefore,

$$\frac{d}{dt}\|u(t)-x\| \leq \|Tx-x\| \quad \forall t \geq 0.$$

So, by integration we obtain

$$\|u(t)-x\| \leq t\|Tx-x\| \quad \forall t \geq 0. \tag{4.26}$$

Since u is a solution of equation (4.25), we have

$$u(t)-T^n x = e^{-t}(x-T^n x)+\int_0^t e^{s-t}(Tu(s)-T^n x)\,ds.$$

Next, because $\|x-T^n x\| \leq \|x-Tx\|+\|Tx-T^2x\|+\cdots+\|T^{n-1}x-T^n x\|$ and using the fact that T is nonexpansive, we get $\|x-T^n x\| \leq n\|x-Tx\|$. This, together with the latter equation, gives

$$\|u(t)-T^n x\| \leq ne^{-t}\|x-Tx\|+\int_0^t e^{s-t}\|u(s)-T^{n-1}x\|\,ds.$$

Now, if, for every $n \in \mathbb{N}$, we denote $\varphi_n(t) := \|u(t)-T^n x\|$, then

$$\varphi_n(t) \leq ne^{-t}\varphi_1(0)+\int_0^t e^{s-t}\varphi_{n-1}(s)\,ds.$$

Moreover, by (4.26), we know that $\varphi_0(t) \leq \varphi_1(0)t$.

We claim that, for all $n \in \mathbb{N}$ and $t \geq 0$, the inequality

$$\varphi_n(t) \leq \varphi_1(0)\sqrt{t+(n-t)^2} \tag{4.27}$$

holds.

We will prove inequality (4.27) by induction. For $n=0$, $\varphi_0(t) \leq t\varphi_1(0) \leq \varphi_1(0)\sqrt{t+t^2}$, $t \geq 0$, hence (4.27) is true. Suppose it is satisfied for $n-1$. In this case, we can write

$$\varphi_n(t) \leq ne^{-t}\varphi_1(0)+\int_0^t e^{s-t}\sqrt{s+(n-1-s)^2}\varphi_1(0)\,ds.$$

Next, set $\phi_n(t) := n + \int_0^t e^s\sqrt{s + (n-1-s)^2}\,ds$. Then we have

$$\varphi_n(t) \le \varphi_1(0)e^{-t}\phi_n(t).$$

To show (4.27), it suffices to observe that

$$\phi_n(t) \le e^t\sqrt{t + (n-t)^2}, \quad \forall t \ge 0.$$

Since $\frac{d}{dt}(e^t\sqrt{t+(n-t)^2}) = e^t\sqrt{t+(n-t)^2} + e^t(\frac{1}{2} - n + t)\frac{1}{\sqrt{t+(n-t)^2}}$, it is clear that

$$e^t\sqrt{t + (n-1-t)^2} \le \frac{d}{dt}(e^t\sqrt{t+(n-t)^2}),$$

and therefore

$$\phi_n'(t) = e^t\sqrt{t+(n-1-t)^2} \le \frac{d}{dt}(e^t\sqrt{t+(n-t)^2}).$$

Finally, by integration over $[0,t]$, we conclude that

$$\phi_n(t) \le e^t\sqrt{t+(n-t)^2}.$$ ☐

Corollary 4.4.1. *Assume that the conditions of Lemma* 4.4.1 *hold true and let u be the solution of problem* (4.24). *Then, for every* $\lambda > 0$, *the function* $u_\lambda(t) := u(\frac{t}{\lambda})$ *is the solution of the problem*

$$\begin{cases} u'(t) + \frac{(I-T)}{\lambda}(u(t)) = 0, \\ u(0) = x, \end{cases} \tag{4.28}$$

Moreover, for all $\lambda > 0$ *and* $n \in \mathbb{N}$, *we have*

$$\|u_\lambda(n\lambda) - T^n x\| \le \sqrt{n}\|x - Tx\|.$$

Lemma 4.4.2. *Let X be a Banach space and let* $A \subseteq X \times X$ *be an m-accretive operator. For every* $x \in D(A)$ *and* $\lambda > 0$, *the problem*

$$\begin{cases} u_\lambda' + A_\lambda u_\lambda = 0, \\ u_\lambda(0) = x, \end{cases}$$

has a unique classical solution and

$$\lim_{\lambda\to 0^+} u_\lambda(t) = S(t)(x)$$

uniformly on compacts subsets of $[0, +\infty)$.

Proof. Applying Corollary 4.4.1 to equation (4.28) and taking $T := J_\lambda$, we obtain the first part of the result, and for each $n \in \mathbb{N}$, we have

$$\|u_\lambda(n\lambda) - J_\lambda^n x\| \le \sqrt{n}\|(I - J_\lambda)x\| = \lambda\sqrt{n}\|A_\lambda x\|.$$

Moreover, applying Proposition 3.4.1(c), we get

$$\|u_\lambda(n\lambda) - J_\lambda^n x\| \le \lambda\sqrt{n}|Ax|.$$

According to Kato's differentiation rule, we have

$$\|u_\lambda(t) - x\|\frac{d}{dt}\|u_\lambda(t) - x\| = \langle x^*, -A_\lambda u(t)\rangle, \quad \text{for any } x^* \in J(u_\lambda(t) - x).$$

Since A_λ is an accretive operator (cf. Corollary 3.4.2), it follows that

$$\|u_\lambda(t) - x\|\frac{d}{dt}\|u_\lambda(t) - x\| \le \langle x^*, -A_\lambda x\rangle, \quad \text{for some } x^* \in J(u_\lambda(t) - x),$$

and consequently (use again Proposition 3.4.1(c)),

$$\frac{d}{dt}\|u_\lambda(t) - x\| \le \|A_\lambda x\| \le |Ax|.$$

So, by integration, we obtain the estimate

$$\|u_\lambda(t) - x\| \le t|Ax|.$$

Following the same argument, for any $x^* \in J(u_\lambda(t+h) - u_\lambda(t))$, we have

$$\|u_\lambda(t+h) - u_\lambda(t)\|\frac{d}{dt}\|u_\lambda(t+h) - u_\lambda(t)\| \le \langle x^*, A_\lambda u_\lambda(t) - A_\lambda u_\lambda(t+h)\rangle \le 0.$$

Since the map $t \mapsto \frac{1}{2}\frac{d}{dt}\|u_\lambda(t+h) - u_\lambda(t)\|^2$ is negative, the function $t \mapsto \frac{1}{2}\|u_\lambda(t+h) - u_\lambda(t)\|^2$ is decreasing, and therefore

$$\|u_\lambda(t+h) - u_\lambda(t)\| \le \|u_\lambda(h) - x\| \le h|Ax|. \tag{4.29}$$

For $t > 0$, let $n \in \mathbb{N}$ be such that $t = n\lambda + \gamma$, with $0 \le \gamma < \lambda$. Then, by (4.29), we have

$$\|u_\lambda(t) - u_\lambda(n\lambda)\| \le \gamma|Ax|.$$

Now, by Remark 3.10.1, we know that

$$\|S(t)(x) - S(n\lambda)x\| \le 2\gamma|Ax|,$$

and the use of Proposition 3.10.1 implies that

$$\|S(n\lambda)x - J_\lambda^n x\| \le \frac{2n\lambda}{\sqrt{n}}|Ax| = 2\lambda\sqrt{n}|Ax|.$$

Finally, making use of the above inequalities, we can write

$$\begin{aligned}\|u_\lambda(t) - S_t(x)\| &\le \|u_\lambda(t) - u_\lambda(n\lambda)\| + \|u_\lambda(n\lambda) - J_\lambda^n(x)\| \\ &\quad + \|J_\lambda^n(x) - S_{n\lambda}(x)\| + \|S_{n\lambda}(x) - S_t(x)\| \\ &\le \gamma|Ax| + \sqrt{n}\lambda|Ax| + 2\lambda\sqrt{n}|Ax| + 2\gamma|Ax| \\ &\le (\gamma + \sqrt{n}\lambda + 2\lambda\sqrt{n} + 2\gamma)|Ax|\end{aligned}$$

and, since $0 \le \gamma < \lambda$, we obtain the result. ☐

Remark 4.4.2. The above lemma also holds for accretive operators $A : D(A) \to 2^X$ satisfying the condition $\overline{\mathrm{co}}(D(A)) \subseteq \bigcap_{\lambda>0} R(I + \lambda A)$.

Lemma 4.4.3. *Let $\varphi : [a,b] \times [a,b] \to \mathbb{R}$ be a continuous function. If, for every s, t, α, β satisfying $a \le s < t \le b$ and $a \le \alpha < \beta \le b$, the estimate*

$$\int_\alpha^\beta (\varphi(\sigma,t) - \varphi(\sigma,s))\,d\sigma \le \int_s^t (\varphi(\alpha,\tau) - \varphi(\beta,\tau))\,d\tau$$

holds, then the function $t \mapsto \varphi(t,t)$ is monotone decreasing on $[a,b]$.

Proof. We define $F(t,s,\beta,\alpha) := \int_\alpha^\beta (\int_s^t \varphi(\tau,\xi)d\xi)\,d\tau$ for $t,s,\beta,\alpha \in [a,b]$. Since

$$\frac{\partial}{\partial t}F(t,s,\beta,\alpha) + \frac{\partial}{\partial s}F(t,s,\alpha,\beta) = \int_\alpha^\beta \varphi(\xi,t)\,d\xi - \int_\alpha^\beta \varphi(\xi,s)\,d\xi$$

and

$$\frac{\partial}{\partial \beta}F(t,s,\beta,\alpha) + \frac{\partial}{\partial \alpha}F(t,s,\beta,\alpha) = \int_s^t \varphi(\beta,\tau)\,d\tau - \int_s^t \varphi(\alpha,\tau)\,d\tau,$$

it follows from the assumptions, for s, t, α, β with $a < s < t < b$ and $a < \alpha < \beta < b$, that

$$\frac{\partial}{\partial t}F(t,s,\beta,\alpha) + \frac{\partial}{\partial s}F(t,s,\beta,\alpha) + \frac{\partial}{\partial \beta}F(t,s,\beta,\alpha) + \frac{\partial}{\partial \alpha}F(t,s,\alpha,\beta) \le 0.$$

Now, choose arbitrary s, t such that $a < s < t < b$, and set $G(h) := F(t+h, s+h, t+h, s+h)$ with $0 \le h \le b - t$. Then G is continuous on $[0, b-t]$ and differentiable on $(0, b-t)$, and we have

$$G'(h) = \frac{\partial}{\partial t}F(t+h, s+h, t+h, s+h) + \frac{\partial}{\partial s}F(t+h, s+h, t+h, s+h)$$

$$+\frac{\partial}{\partial\beta}F(t+h,s+h,t+h,s+h)+\frac{\partial}{\partial\alpha}F(t+h,s+h,t+h,s+h)$$
$$\le 0.$$

Hence G is monotone decreasing on $[0,b-t]$ and therefore, for every $h\in(0,b-t)$, we have $G(h)\le G(0)$, that is,

$$\int_s^t\left(\int_s^t\varphi(\xi+h,\tau+h)\,d\tau\right)d\xi\le\int_s^t\left(\int_s^t\varphi(\xi,\tau)\,d\tau\right)d\xi.$$

Dividing both sides by $(t-s)^2$, passing to the limit as $s\to t^-$, applying l'Hôpital's rule, and bearing in mind that φ is continuous, we obtain

$$\varphi(t+h,t+h)\le\varphi(t,t),\quad \text{for all } 0<h\le b-t.$$

This means that the function $t\mapsto\varphi(t,t)$ is monotone decreasing, which completes the proof. □

Theorem 4.4.2. *Let X be a Banach space and let $A:D(A)\to 2^X$ be an accretive operator satisfying $\overline{\mathrm{co}}(D(A))\subseteq\bigcap_{\lambda>0}R(I+\lambda A)$. Then the unique integral solution of the problem*

$$\begin{cases}u'(t)+A(u(t))\ni 0,\\ u(0)=x_0\in\overline{D(A)}.\end{cases}$$

is the function $u(t):=S(t)(x_0)=\lim_{n\to+\infty}(I+\frac{t}{n}A)^{-n}(x_0)$.

Proof. By Theorem 4.4.1, we know that $u(t):=S(t)(x_0)$ is an integral solution. Assume, toward a contradiction, that $v:[0,\infty)\to\overline{D(A)}$ is also an integral solution.

We claim that, for $0\le s\le t<\infty$, the inequality $\|u(t)-v(t)\|^2\le\|u(s)-v(s)\|^2$ holds. Indeed, since v is an integral solution, it satisfies

$$\|v(t)-x\|^2\le\|v(s)-x\|^2+2\int_s^t\langle -u,v(\tau)-x\rangle_+\,d\tau, \tag{4.30}$$

for $(x,u)\in A$ and $0\le s\le t<\infty$.

Let u_λ be the unique classic solution to the problem

$$\begin{cases}u_\lambda'+A_\lambda u_\lambda=0,\\ u_\lambda(0)=x_0,\end{cases}$$

(the existence of u_λ follows from Lemma 4.4.2 and Remark 4.4.2). Set $w_\lambda(t):=J_\lambda u_\lambda(t)$. Then

$$\lim_{\lambda\to 0^+} w_\lambda(t) = u(t),$$

uniformly on compact subsets of $[0,\infty)$. It is an easy consequence of the following fact:

$$\|w_\lambda(t) - u(t)\| \le \|J_\lambda u_\lambda(t) - J_\lambda u(t)\| + \|J_\lambda u(t) - u(t)\| \le \|u_\lambda(t) - u(t)\| + \|J_\lambda u(t) - u(t)\|.$$

Taking in (4.30) $x = w_\lambda(\sigma)$ and $u = -u'_\lambda(\sigma) = A_\lambda u_\lambda(\sigma) \in AJ_\lambda u_\lambda(\sigma) = Aw_\lambda(\sigma)$, we obtain

$$\|v(t) - w_\lambda(\sigma)\|^2 \le \|v(s) - w_\lambda(\sigma)\|^2 + 2\int_s^t \langle u'_\lambda(\sigma), v(\tau) - w_\lambda(\sigma)\rangle_+ \, d\tau.$$

On the other hand, by Kato's differentiation rule, we have

$$\frac{1}{2}\frac{d}{d\sigma}\|u_\lambda(\sigma) - v(\tau)\|^2 = \left\langle \frac{d}{d\sigma}u_\lambda(\sigma), u_\lambda(\sigma) - v(\tau)\right\rangle_+ .$$

Moreover, since $w_\lambda(t) - u_\lambda(t) = \lambda A_\lambda u_\lambda(t)$, it is clear that

$$\lim_{\lambda\to 0^+}\|u_\lambda(t) - w_\lambda(t)\| = 0 \quad \text{uniformly on compact subsets of } [0,\infty).$$

Thus, since the function $\langle\cdot,\cdot\rangle_+$ is upper semicontinuous, we get

$$\begin{aligned}
&\limsup_{\lambda\to 0^+}\int_\alpha^\beta\int_s^t \left\langle \frac{d}{d\sigma}u_\lambda(\sigma), v(\tau) - w_\lambda(\sigma)\right\rangle_+ d\tau\, d\sigma \\
&\quad\le \limsup_{\lambda\to 0^+}\int_\alpha^\beta\int_s^t -\frac{1}{2}\frac{d}{d\sigma}\|u_\lambda(\sigma) - v(\tau)\|^2\, d\tau\, d\sigma \\
&\quad= -\frac{1}{2}\int_s^t \left(\|u_\lambda(\beta) - v(\tau)\|^2 - \|u_\lambda(\alpha) - v(\tau)\|^2\right) d\tau,
\end{aligned}$$

for all $0 \le \alpha \le \beta$. Now, for $\sigma, \tau \ge 0$, define $\varphi(\sigma,\tau) := \frac{1}{2}\|u(\sigma) - v(\tau)\|^2$. As $\lambda \to 0^+$, the above inequalities yield

$$\int_\alpha^\beta (\varphi(\sigma,t) - \varphi(\sigma,s))d\sigma \le \int_s^t (\varphi(\alpha,\tau) - \varphi(\beta,\tau))\, d\tau.$$

Next, applying Lemma 4.4.3 to the latter inequality, we prove the claim.

Finally, because $u(0) = x_0 = v(0)$, we conclude that $u = v$. □

Remark 4.4.3. Suppose that $A : D(A) \to X$ is an accretive operator satisfying the range condition. Theorem 4.4.1 shows that $u(t) = S(t)(x_0)$ is an integral solution for the problem

$$\begin{cases} u'(t) + A(u(t)) \ni 0, \\ x_0 \in \overline{D(A)}. \end{cases}$$

On the other hand, in Theorem 4.4.2, we see that if $\overline{\text{co}}(D(A)) \subseteq \bigcap_{\lambda>0} R(I + \lambda A)$, then $u(t) = S(t)(x_0)$ is the unique integral solution for the above problem. Later, in Theorem 4.6.2, we will see that in order to obtain the existence and uniqueness of an integral solution, it is sufficient to assume that A is an accretive operator satisfying the range condition.

We close this subsection by showing that a single-valued accretive operator with the domain being the whole space is m-accretive whenever it is continuous.

Theorem 4.4.3. *Let X be a Banach space and let $A : D(A) = X \to X$ be a continuous accretive operator. Then A is m-accretive.*

Proof. To prove that A is an m-accretive operator, we must show that for any $y \in X$ there exists $x \in X$ such that $y = x + Ax$. Without loss of generality, we may assume that $y = 0$. Otherwise we could shift the range of A, i. e., we would use the operator $\tilde{A} := A - y$.

To get the result, let u_0 be an arbitrary, but fixed, element of X, and consider the differential equation

$$\begin{cases} u'(t) + Au(t) = 0, \\ u(0) = u_0. \end{cases} \tag{4.31}$$

This initial value problem has a unique classic solution in $C^1(0, \infty; X)$. Indeed, if we suppose that problem (4.31) has a unique classical solution whenever $A : D(A) = X \to X$ is continuous and accretive, then the problem

$$\begin{cases} u'(t) + (I + A)u(t) = 0, \\ u(0) = u_0 \end{cases} \tag{4.32}$$

has a unique classical solution, namely $u(\cdot) \in C^1(0, \infty; X)$. Hence,

$$u'(t + h) - u'(t) = -Au(t + h) + Au(t) - u(t + h) + u(t), \quad \forall t \geq 0,$$

and therefore, by Kato's differential rule and the fact that A is accretive,

$$\|u(t + h) - u(t)\| \frac{d}{dt} \|u(t + h) - u(t)\| \leq -\|u(t + h) - u(t)\|^2, \quad \forall t \geq 0.$$

By integration on $[0, t]$, we obtain

$$\|u(t + h) - u(t)\| \leq e^{-t} \|u(h) - u_0\|, \quad \forall t, h \geq 0.$$

Analogously, since $u'(t) = -Au(t) - u(t)$, we have

$$\|u(t)-u_0\|\frac{d}{dt}\|u(t)-u_0\| = \langle x^*, -Au(t)-u(t)\rangle, \quad \forall t \geq 0 \text{ and } x^* \in J(u(t)-u_0),$$

that is,

$$\|u(t)-u_0\|\frac{d}{dt}\|u(t)-u_0\| \leq -\|u(t)-u_0\|^2 + \|u(t)-u_0\|(\|Au_0\|+\|u_0\|).$$

Thus, by Gronwall's inequality,

$$\|u(t)-u_0\| \leq (1-e^{-t})(\|Au_0\|+\|u_0\|).$$

Hence

$$\|u'(t)\| \leq e^{-t}\lim_{h\to 0}\frac{(1-e^{-h})(\|Au_0\|+\|u_0\|)}{h} = e^{-t}(\|Au_0\|+\|u_0\|),$$

and therefore $\lim_{t\to+\infty} u'(t) = 0$. Since

$$\|u(t+h)-u(t)\| \leq e^{-t}\|u(h)-u_0\| \leq e^{-t}(1-e^{-h})(\|Au_0\|+\|u_0\|), \quad \forall t,h \geq 0,$$

we know that there exists $x \in X$ such that $x = \lim_{t\to+\infty} u(t)$. Finally, letting $t \to +\infty$ in $u'(t) + (I+A)u(t) = 0$, $u(0) = u_0$, we obtain $x + Ax = 0$, as needed.

Let us see that problem (4.31) has a unique classical solution. The uniqueness is a consequence of Proposition 4.2.1. For the existence, by the continuity of A, we can consider a ball $\mathbb{B}_r(u_0)$ such that $\|Ay\| \leq M$ for all $y \in \mathbb{B}_r(u_0)$ for some given $M > 0$. Let $T > 0$ be such that $MT < r$ and consider a sequence of positive numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \downarrow 0$. We will show that, for every $n \in \mathbb{N}$, we can build a partition $(t_i^n)_{i=0}^{k_n}$ of the interval $[0,T]$, and a function u_n defined on the partition $(t_i^n)_{i=0}^{k_n}$ by the recursive formula

$$u_n(t_i^n) = u_n(t_{i-1}^n) - (t_i^n - t_{i-1}^n)Au_n(t_{i-1}^n), \quad u_n(0) = u_0,\ i = 1,2,\ldots,k_n, \tag{4.33}$$

such that

$$\|y - u_n(t_{i-1}^n)\| \leq M(t_i^n - t_{i-1}^n) \Longrightarrow \|Ay - u_n(t_{i-1}^n)\| \leq \varepsilon_n. \tag{4.34}$$

Indeed, define $t_0^n = 0$ and $u_n(0) = u_0$; since A is continuous, there exists $\delta_0^n > 0$ such that

$$\|Ay - Au_n(t_0^n)\| \leq \varepsilon_n \quad \text{whenever } \|y - u_n(t_0^n)\| \leq \delta_0^n.$$

Define $t_1^n = t_0^n + \frac{\delta_0^n}{M}$, $u_n(t_1^n) = u_n(t_0^n) - \frac{\delta_0^n}{M}Au_n(t_0^n)$. Then there exists $\delta_1^n > 0$ such that

$$\|y - u_n(t_1^n)\| \leq \delta_1^n \Longrightarrow \|Ay - Au_n(t_1^n)\| \leq \varepsilon_n.$$

Define again $t_2^n = t_1^n + \frac{\delta_1^n}{M}$ and $u_n(t_2^n) = u_n(t_1^n) - \frac{\delta_1^n}{M} A u_n(t_1^n)$, and continue the process. We can suppose that at each step the number $\frac{\delta_i^n}{M}$ is maximal with respect to property (4.34). Let us see that after a finite number of steps the point T is achieved. Suppose to the contrary. Then $\lim_{i\to\infty} t_i^n = s < T$. Notice that $u_n(t_1^n) = u_0 - \frac{\delta_0^n}{M} A(u_0)$, then $u_n(t_1^n) \in \mathbb{B}_r(u_0)$. Moreover, since $u_n((t_2^n)) = u_0 - \frac{\delta_0^n}{M} A u_0 - \frac{\delta_1^n}{M} A u_n(t_1^n)$, one has

$$\|u_n(t_2^n) - u_0\| \le \left(\frac{\delta_0^n}{M} + \frac{\delta_1^n}{M}\right) M < r,$$

which proves that $u_n(t_2^n) \in \mathbb{B}_r(u_0)$.

Finally, if we assume that $u_n(t_{i-1}^n) \in \mathbb{B}_r(u_0)$, then the relation

$$u_n(t_i^n) = u_0 - \sum_{j=1}^{i} \frac{\delta_j^n}{M} A u_n(t_{i-1}^n)$$

yields

$$\|u_n(t_i^n) - u_0\| \le \left(\sum_{j=1}^{i} \frac{\delta_j^n}{M}\right) M \le TM < r.$$

Consequently, $u_n(t_i^n) \in \mathbb{B}_r(u_0)$. Now from (4.33) we obtain

$$\|u_n(t_i^n) - u_n(t_{i-1}^n)\| \le \delta_{i-1}^n, \quad i = 1, 2, \dots$$

Therefore the limit $\lim_{i\to+\infty} u_n(t_i^n)$ exists and is, say, v.

By the maximality of the numbers $\frac{\delta_i^n}{M}$, there is $y_i \in X$ such that $\delta_{i-1}^n \le \|y_i - u_n(t_{i-1}^n)\| \le \delta_{i-1}^n + \frac{1}{i}$ and

$$\|A y_i - A u_n(t_{i-1}^n)\| \ge \varepsilon_n. \tag{4.35}$$

However, $\lim_{i\to+\infty} y_i = \lim_{i\to+\infty} u_n(t_i^n) = v$. Since A is continuous, we conclude that

$$\lim_{i\to+\infty} A y_i = \lim_{i\to+\infty} A u_n(t_i^n) = Av.$$

This contradicts (4.35).

Define the function $u_n : [0, T) \to X$ by

$$u_n(t) = u_n(t_{i-1}^n) - (t - t_{i-1}^n) A u_n(t_{i-1}^n), \quad \text{for } t \in [t_{i-1}^n, t_i^n].$$

Then u_n is continuous on $[0, T)$ and $u_n'(t) + A u_n(t_{i-1}^n) = 0$, for $t \in (t_{i-1}^n, t_i^n)$. Using the fact that $\|u_n(t) - u_n(t_{i-1}^n)\| \le \delta_{i-1}^n$, it follows from (4.34) that

$$u_n'(t) + Au_n(t) = g_n(t), \quad t \in (t_{i-1}^n, t_i^n),$$

where g_n satisfies $\|g_n(t)\| \le \varepsilon_n$.

Analogously, for every $m, n \in \mathbb{N}$ and $t \in \triangle_{ij} := (t_{i-1}^n, t_i^n) \cap (t_{j-1}^m, t_j^m)$, it is clear that

$$u_n'(t) - u_m'(t) = Au_m(t_{j-1}^m) - Au_n(t_{i-1}^n).$$

Since $\|u_n(t) - u_n(t_{i-1}^n)\| \le \delta_{i-1}^n$ and $\|u_n(t) - u_n(t_{j-1}^n)\| \le \delta_{j-1}^m$, for all $t \in \triangle_{ij}$, from (4.34) we derive

$$u_n'(t) - u_m'(t) = Au_m(t) - Au_n(t) + g_{m,n}(t), \quad \forall t \in \triangle_{ij},$$

with $\|g_{m.n}(t)\| \le \varepsilon_m + \varepsilon_n$. Kato's differential rule yields

$$\frac{1}{2}\frac{d}{dt}\|u_n(t) - u_m(t)\|^2 = \langle x^*, Au_m(t) - Au_n(t)\rangle + \langle x^*, g_{m,n}(t)\rangle, \quad x^* \in J(u_n(t) - u_m(t)).$$

Since A is accretive, $\langle x^*, Au_m(t) - Au_n(t)\rangle \le 0$, and hence

$$\frac{1}{2}\frac{d}{dt}\|u_n(t) - u_m(t)\|^2 \le \|u_n(t) - u_m(t)\|\|g_{m,n}(t)\|, \quad t \in \triangle_{ij}.$$

Since $(0, T) = \bigcup_{i,j} \triangle_{ij}$, by integration we obtain

$$\|u_n(t) - u_m(t)\| \le (\varepsilon_m + \varepsilon_n)T, \quad \forall t \in [0, T].$$

This means that $u(t) := \lim_{n\to+\infty} u_n(t)$ exists uniformly in $[0, T]$.

Let us prove that u is a solution. Indeed, since $u_n'(t) + Au_n(t) = g_n(t)$, we have

$$u_n(t) - u_0 = -\int_0^t Au_n(s)\,ds + \int_0^t g_n(s)\,ds.$$

Since $u_n \to u$ uniformly on $[0, T]$ and $g_n \to 0$ uniformly on $[0, T]$ as $n \to \infty$, we infer that

$$u(t) = u_0 - \int_0^t Au(s)\,ds, \quad \forall t \in [0, T).$$

Hence u is a solution to problem (4.31) on $[0, T)$. Now, let $[0, T_0)$ be the largest interval of existence of u and suppose that $T_0 < +\infty$. Since u is a solution of the problem $u'(t) + Au(t) = 0$, $u(0) = u_0$, and A is accretive, then $\lim_{t\to T_0} u(t) = u(T_0)$ and, by the continuity of A, we obtain

$$\lim_{t\to T_0} u'(t) = -\lim_{t\to T_0} Au(t) = -Au(T_0).$$

Now, we can consider the initial value problem

$$\begin{cases} u'(t+T_0) + Au(t+T_0) = 0, \\ u(T_0) = u_0. \end{cases}$$

Proceeding as above, we may prove that there exists $T' > 0$ such that the above equation has a solution on $[0, T')$, and so u can be extended at the right of T_0, which is a contradiction. This proves that $T_0 = \infty$ and the proof of the theorem is complete. □

4.5 The inhomogeneous problem for accretive operators

Let X be a Banach space and let $A \subset X \times X$ be an m-accretive operator on X. In this section we will study the concept of a solution (called integral solution) of the following problem:

$$\begin{cases} u' + Au \ni f \quad \text{a.e. } t \in [0,T], \\ u(0) = x_0, \\ 0 < T < +\infty, \\ f \in L^1((0,T), X) \end{cases} \tag{4.36}$$

Integral solutions were introduced by Ph. Bénilan [36] and they allow solving problems of type (4.36) in very general contexts. Although, integral solutions cannot be interpreted as solutions of problem (4.36) in a pointwise sense, we will see that every strong solution is an integral solution and, moreover, under certain additional assumptions, it is possible to obtain a regularization of such solutions.

Definition 4.5.1. A function $u : [0,T] \to X$ is called an integral solution of (4.36) if u is continuous, $u(0) = x_0$, and

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \le 2 \int_s^t \langle f(\tau) - y, u(\tau) - x \rangle_+ \, d\tau,$$

for every $(x, y) \in A$ and $0 \le s \le t \le T$.

Remark 4.5.1. Let A be an accretive operator satisfying the range condition and let $x_0 \in \overline{D(A)}$. If $u(t)$ is a strong solution of the inhomogeneous problem (4.36), then it is also an integral solution of such a problem. Indeed, if u is a strong solution to problem (4.36), then u satisfies

$$u'(t) + A(u(t)) \ni f(t) \quad \text{a.e. and} \quad u(0) = x_0.$$

Hence, $f(t) - u'(t) \in A(u(t))$ a.e. Therefore, given $(x, y) \in A$, we have

$$\langle f(t) - u'(t) - y, u(t) - x\rangle_+ \geq 0 \quad \text{a. e.}$$

This means that there exists $j(t) \in J(u(t) - x)$ such that

$$\langle u'(t) - 0, j(t)\rangle \leq \langle f(t) - y, u(t) - x\rangle_+.$$

Now, it follows from Kato's differentiation rule that

$$\frac{1}{2}\frac{d}{dt}\|u(t) - x\|^2 \leq \langle f(t) - y, u(t) - x\rangle_+.$$

By integration, we get

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \leq 2\int_s^t \langle f(\tau) - y, u(\tau) - x\rangle_+ \, d\tau.$$

This proves that u is an integral solution of the inhomogeneous problem (4.36).

Remark 4.5.2. Let A be an m-accretive operator and $f \in L^1(0, T; X)$. Then, for $\lambda > 0$, the problem

$$\begin{cases} u_\lambda' + A_\lambda u_\lambda = f, \\ u_\lambda(0) = x \end{cases}$$

has a unique classical solution for every $x \in X$. This fact can be established using the Banach contraction principle as in the proof of Lemma 4.4.2.

Theorem 4.5.1. *Let $A \subset X \times X$ be an m-accretive operator on a Banach space X. Then, for each $f \in L^1((0, T), X)$ and, for every $x_0 \in \overline{D(A)}$, there exists a unique integral solution of problem* (4.36) *such that $u(t) \in \overline{D(A)}$, for every $t \in [0, T]$. Moreover, if u, v are two integral solutions of the problems*

$$\begin{cases} u' + Au \ni f, \quad v' + Av \ni g \text{ a. e. } t \in [0, T], \\ u(0) = v(0) = x_0, \\ f, g \in L^1((0, T), X), \\ 0 < T < +\infty, \end{cases}$$

then the inequality

$$\frac{1}{2}\|u(t) - v(t)\|^2 \leq \frac{1}{2}\|u(s) - v(s)\|^2 + \int_s^t \langle f(\tau) - g(\tau), u(\tau) - v(\tau)\rangle_+ \, d\tau \tag{4.37}$$

is satisfied for all reals s and t such that $0 \leq s \leq t \leq T$.

Proof. For the sake of clarify, the proof will be divided into three steps.

Step 1. We first show the existence of an integral solution u.

(i) Assume that $f(t) = y_0$ on $[0, T]$. It is easy to see that $A_0 = A - y_0$ is an m-accretive operator, and so it generates a semigroup

$$S_0 = \{S(t) : \overline{D(A)} \to \overline{D(A)}\}$$

defined by the Crandall–Liggett's exponential formula. Moreover, by Theorems 4.4.1 and 4.4.2, $u_0(t) = S(t)x_0$ will be the unique integral solution of the differential inclusion

$$\frac{d}{dt}u(t) + Au(t) \ni y_0, \quad u(0) = x_0.$$

The Yosida approximants of A_0 are $A_{0,\lambda}(x) = A_\lambda(x + \lambda y_0) - y_0$, $x \in X$. It follows from Lemma 4.4.2 that the problem

$$\frac{du_{0\lambda}}{dt} = -A_{0\lambda}u_{0\lambda} = -A_\lambda(u_{0\lambda} + \lambda y_0) + y_0, \quad u_{0\lambda}(0) = x_0,$$

has a unique classical solution $u_{0\lambda}$ and

$$\lim_{\lambda\to 0^+} u_{0\lambda}(t) = u_0(t)$$

exists uniformly in $[0, T]$. Now, consider the classical solution of the problem

$$\frac{du_\lambda}{dt} = -A_\lambda u_\lambda + y_0, \quad u_\lambda(0) = x_0,$$

and therefore

$$\frac{d}{dt}(u_\lambda(t) - u_{0\lambda}(t)) = -A_\lambda u_\lambda + A_\lambda(u_{0\lambda} + \lambda y_0).$$

Hence, using Kato's differentiation rule and the fact that the Yosida approximants are accretive operators, we infer that there exists $x^* \in J(u_\lambda(t) - v_\lambda(t) - \lambda y_0)$ such that

$$\|u_\lambda(t) - u_{0\lambda}(t) - \lambda y_0\| \frac{d}{dt}\|u_\lambda(t) - u_{0\lambda}(t) - \lambda y_0\| = \langle x^*, -A_\lambda u_\lambda(t) + A_\lambda(u_{0\lambda}(t) + \lambda y_0)\rangle \le 0,$$

and then

$$\frac{d}{dt}\|u_\lambda(t) - u_{0\lambda}(t) - \lambda y_0\| \le 0 \quad \text{a. e. on } [0, T].$$

By integration, we have

$$\|u_\lambda(t) - u_{0\lambda}(t) - \lambda y_0\| \le \lambda\|y_0\|, \quad t \in [0, T],$$

thus, the limit

$$\lim_{\lambda\to 0^+} u_\lambda(t) = \lim_{\lambda\to 0^+} u_{0\lambda}(t) = u_0(t)$$

exists uniformly on $[0, T]$.

(ii) Suppose that f is a simple function on $[0, T]$, namely, $f(t) = y_i$ for $t_{i-1} \le t < t_i$, with $i = 1, \dots, n$ where $t_0 = 0$ and $t_n = T$. Then the solutions u_λ of the equation

$$\frac{du}{dt} + A_\lambda u_\lambda = f, \quad u_\lambda(0) = x_0,\ t \in [0, T],$$

are given by

$$u_\lambda(t) = u_\lambda^i(t - t_{i-1}) \quad \text{for } t_{i-1} \le t < t_i,$$

where $u_\lambda^i, 1 \le i \le n$, are the solutions of the problems

$$\frac{du_\lambda}{dt} = -A_\lambda u_\lambda^i + y_i, \quad 0 \le t \le t_i - t_{i-1},\ u_\lambda^i(0) = u_\lambda(t_{i-1}) = u_\lambda^{i-1}(t_i - t_{i-1}).$$

Thus, applying (i), we conclude that $\lim_{\lambda\to 0^+} u_\lambda(t) = u(t)$ exists uniformly on $[0, T]$. Since $u(t)$ is an integral solution on every interval $[t_{i-1}, t_i]$, it is possible to verify that $u(t)$ is an integral solution of the initial value problem.

(iii) Finally, let $f \in L^1((0, T), X)$. In this case, there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ which converges to f. Let u_λ^n be the solutions of the Cauchy problem

$$\frac{du_\lambda^n}{dt} = -A_\lambda u_\lambda^n + f_n, \quad t \in [0, T],\ u_\lambda^n(0) = x_0,$$

and let u_λ be the solution of the equation

$$\frac{du}{dt} + A_\lambda u_\lambda = f, \quad u_\lambda(0) = x_0,\ t \in [0, T],$$

and consequently,

$$\frac{d}{dt}(u_\lambda - u_\lambda^n) = -A_\lambda u_\lambda + f + A_\lambda u_\lambda^n - f_n.$$

Hence, for some $x^* \in J(u_\lambda - u_\lambda^n)$,

$$\begin{aligned}\|u_\lambda - u_\lambda^n\| \frac{d}{dt}\|u_\lambda - u_\lambda^n\| &= \langle x^*, -A_\lambda u_\lambda + A_\lambda u_\lambda^n\rangle + \langle x^*, f - f_n\rangle \\ &\le \|u_\lambda - u_\lambda^n\| \|f - f_n\|.\end{aligned}$$

This implies that

$$\frac{d}{dt}\|u_\lambda - u_\lambda^n\| \le \|f(t) - f_n(t)\| \quad \text{a.e. on } [0, T],$$

and therefore,

$$\|u_\lambda(t) - u_\lambda^n(t)\| \le \int_0^T \|f(t) - f_n(t)\|\, dt.$$

Let $\lambda, \gamma > 0$. Then

$$\|u_\lambda(t) - u_\gamma(t)\| \le \|u_\lambda(t) - u_\lambda^n(t)\| + \|u_\lambda^n(t) - u_\gamma^n(t)\| + \|u_\gamma^n(t) - u_\gamma(t)\|.$$

Notice that the first and last terms on the right-hand side of the above expression converge to zero as $n \to \infty$. According to the previous point, for each fixed integer n, $u^n := \lim_{\lambda\to 0^+} u_\lambda^n$ exists uniformly on $[0, T]$. Hence, we may affirm that $u(t) := \lim_{\lambda\to 0^+} u_\lambda(t)$ exists.

Now we will check that $u(\cdot)$ is an integral solution of problem (4.36) which we are looking for. To this end, we first note that each $u^n(\cdot)$ is an integral solution of the corresponding problem and thus they have to verify

$$\|u^n(t) - x\|^2 - \|u^n(s) - x\|^2 \le 2 \int_s^t \langle f_n(\tau) - u, u^n(\tau) - u\rangle_+ \, d\tau,$$

for all $(x, u) \in A$, $0 \le s \le t \le T$. Since the limit $\lim_{n\to\infty} u^n(t) = u(t)$ exists uniformly on $[0, T]$, and $u_\lambda^n(t) \in \overline{D(A)}$, for all $t \in [0, T]$, we conclude that $u^n(t) \in \overline{D(A)}$. Consequently, $u(t) \in \overline{D(A)}$ for all $t \in [0, T]$. Next letting n go to $+\infty$, we infer that $u(\cdot)$ is an integral solution.

Step 2. Let us see that inequality (4.37) is satisfied. Suppose that u and v are integral solutions of problem (4.36). In this case

$$\|v(t) - x\|^2 \le \|v(s) - x\|^2 + 2 \int_s^t \langle g(\tau) - u, v(\tau) - x\rangle_+ \, d\tau, \tag{4.38}$$

for $(x, u) \in A$, $0 \le s \le t \le T$.

Consider the solutions u_λ of the problem

$$u_\lambda' + A_\lambda u_\lambda = f, \quad u_\lambda(0) = x_0,$$

and set $w_\lambda(t) = J_\lambda u_\lambda(t)$. Hence

$$\lim_{\lambda\to 0^+} w_\lambda(t) = u(t)$$

uniformly on $[0, T]$. In equation (4.38) we take $x = w_\lambda(\sigma)$ and

$$u = f(\sigma) - \frac{d}{d\sigma} u_\lambda(\sigma) = A_\lambda u_\lambda(\sigma) \in A J_\lambda u_\lambda(\sigma) = A w_\lambda(\sigma);$$

thus, we have

$$\|v(t)-w_\lambda(\sigma)\|^2 \le \|v(s)-w_\lambda(\sigma)\|^2 + 2\int_s^t \left\langle g(\tau)-f(\sigma)+\frac{d}{d\sigma}u_\lambda(\sigma), v(\tau)-w_\lambda(\sigma)\right\rangle_+ d\tau.$$

On the one hand, Kato's differentiation rule guarantees that

$$\frac{1}{2}\frac{d}{d\sigma}\|u_\lambda(\sigma)-v(\tau)\|^2 = \left\langle \frac{d}{d\sigma}u_\lambda(\sigma), u_\lambda(\sigma)-v(\tau)\right\rangle_+.$$

Since $w_\lambda(t)-u_\lambda(t)=\lambda A_\lambda u_\lambda(t)$, it is clear that

$$\lim_{\lambda\to 0}\|u_\lambda(t)-w_\lambda(t)\| = 0 \quad \text{uniformly on } [0,T].$$

On the other hand, the upper semicontinuity of the function $\langle\cdot,\cdot\rangle_+$ (cf. Proposition 1.10.1(c)) implies that

$$\begin{aligned}
&\limsup_{\lambda\to 0^+}\int_\alpha^\beta\int_s^t \left\langle g(\tau)-f(\sigma)+\frac{d}{d\sigma}u_\lambda(\sigma), v(\tau)-w_\lambda(\sigma)\right\rangle_+ d\tau\, d\sigma \\
&\quad\le \lim_{\lambda\to 0^+}\int_\alpha^\beta\int_s^t \langle g(\tau)-f(\sigma), v(\tau)-w_\lambda(\sigma)\rangle_+ d\tau\, d\sigma \\
&\quad\quad -\frac{1}{2}\int_s^t \left(\|u_\lambda(\beta)-v(\tau)\|^2 - \|u_\lambda(\alpha)-v(\tau)\|^2\right) d\tau,
\end{aligned}$$

and this holds for every $0\le\alpha\le\beta\le T$.

Now, for $\sigma,\tau\in[0,T]$, set

$$\varphi(\sigma,\tau) = \frac{1}{2}\|u(\sigma)-v(\tau)\|^2$$

and

$$\psi(\sigma,\tau) = \langle f(\sigma)-g(\tau), u(\sigma)-v(\tau)\rangle_+.$$

According the above inequalities, if λ tends to 0+, then we obtain

$$\int_\alpha^\beta (\varphi(\sigma,t)-\varphi(\sigma,s))\, d\sigma \le \int_\alpha^\beta\int_s^t \psi(\sigma,\tau)\, d\tau\, d\sigma + \int_s^t (\varphi(\alpha,\tau)-\varphi(\beta,\tau))\, d\tau. \tag{4.39}$$

Let φ_n and ψ_n be the regularizations of the functions φ and ψ, respectively, that is,

$$\varphi_n(t,s) = \int_0^T \int_0^T \rho_n(t-\sigma, s-\tau)\varphi(\sigma,\tau)\, d\tau\, d\sigma,$$
$$\psi_n(t,s) = \int_0^T \int_0^T \rho_n(t-\sigma, s-\tau)\psi(\sigma,\tau)\, d\tau\, d\sigma,$$

where

$$\rho_n(t,s) = n^2\rho(nt)\rho(ns), \quad \rho \in \mathcal{D}(\mathbb{R}),\ \rho \geq 0,$$

the support of ρ is contained in the interval $[-1,1]$, $\int_{-1}^{1}\rho(t)\,dt = 1$, and $\rho(t) = \rho(-t)$ for every $t \in \mathbb{R}$ (see, for example, [46, Chapter IV]).

Let $\frac{1}{n} \leq \alpha \leq \beta \leq T$ and $\frac{1}{n} \leq s \leq t \leq T$. It follows from (4.39) that

$$\int_\alpha^\beta (\varphi_n(\sigma,t) - \varphi_n(\sigma,s)\, d\sigma + \int_s^t (\varphi_n(\beta,\tau) - \varphi_n(\alpha,\tau)) \leq \int_\alpha^\beta d\sigma \int_s^t \psi_n(\sigma,\tau)\, d\tau,$$

and therefore

$$\frac{\partial}{\partial\tau}\varphi_n(\sigma,\tau) + \frac{\partial}{\partial\sigma}\varphi_n(\sigma,\tau) \leq \psi_n(\sigma,\tau),$$

whenever $\frac{1}{n} \leq \sigma, \tau \leq T$. By integration, this yields

$$\varphi_n(t,t) \leq \varphi_n(s,s) + \int_s^t \psi_n(\tau,\tau)\, d\tau$$

whenever $\frac{1}{n} \leq s \leq t \leq T$. Letting n to $+\infty$, we obtain

$$\varphi(t,t) \leq \varphi(s,s) + \int_s^t \psi(\tau,\tau)\, d\tau$$

for $0 \leq s \leq t \leq T$. This shows (4.37).

Step 3. The uniqueness of the integral solution is an easy consequence of (4.38) since (4.37) implies that the function $t \mapsto \|u(t) - v(t)\|$ is decreasing in $[0,T]$, and we know that $u(0) = v(0)$. ☐

Corollary 4.5.1. *Let $A \subset X \times X$ be an m-accretive operator on X. Then for, each $f \in L^1_{\text{loc}}((0,\infty), X)$ and for, any $x_0 \in \overline{D(A)}$, there exists a unique integral solution of the problem*

$$\begin{cases} u' + Au \ni f & a.\,e.\ t \in [0,+\infty[, \\ u(0) = x_0 \end{cases} \tag{4.40}$$

such that $u(t) \in \overline{D(A)}$, for every $t \in [0,T]$.

Proof. Given $n \in \mathbb{N}$, consider the unique integral solution u_n of the problem

$$\begin{cases} u' + Au \ni f & \text{a. e. } t \in [0, n], \\ u(0) = x_0 \end{cases}$$

and define the mapping $u : [0, \infty[\, \to \overline{D(A)}$ such that if $t \leq n$, then $u(t) = u_n(t)$. By Theorem 4.5.1, we know that u is the unique integral solution of problem (4.40). ☐

Corollary 4.5.2. *Let X be a real Banach space and let $A : D(A) \subseteq X \to 2^X$ be an m-accretive operator on X. If $B : X \to X$ is a k-Lipschitz mapping, then the problem*

$$\begin{cases} u'(t) + A(u(t)) \ni B(u(t)), & t \in (0, +\infty), \\ u(0) = x_0 \in \overline{D(A)} \end{cases} \tag{4.41}$$

has a unique integral solution.

The proof of this corollary is taken from [111].

Proof. Let $T > 0$ be a fixed real number. Consider the set $\mathcal{K} := \{u \in C(0, T; X) : u(0) = x_0\}$. Given $v \in \mathcal{K}$, the problem

$$\begin{cases} u'(t) + A(u(t)) \ni B(v(t)), & t \in (0, T), \\ u(0) = x_0, \end{cases} \tag{4.42}$$

has a unique integral solution, namely $S(v) \in \mathcal{K}$. Hence, we can introduce a mapping $S : \mathcal{K} \to \mathcal{K}$ which assigns to each $v \in \mathcal{K}$ the unique integral solution $S(v)$ of the above problem. From the definition of integral solution, it follows that

$$\|S(v)(t) - S(w)(t)\| \leq \int_0^t k\|v(\tau) - w(\tau)\| \, d\tau \leq kt \max\{\|v(s) - w(s)\| : s \in [0, t]\}.$$

An inductive process gives that, for each $n \in \mathbb{N}$,

$$\|S^n(v)(t) - S^n(w)(t)\| \leq \frac{(kt)^n}{n!} \max\{\|v(s) - w(s)\| : s \in [0, t]\},$$

and therefore,

$$\|S^n(v) - S^n(w)\|_\infty \leq \frac{(kT)^n}{n!} \|v - w\|_\infty.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that S^{n_0} is a contraction on $\mathcal{K}$. Since $\mathcal{K}$ is a closed set of the Banach space $C(0, T; X)$, the Banach contraction principle says that S has a unique fixed point in $\mathcal{K}$. This fixed point is the unique integral solution of problem (4.42).

Finally, we may define, given $t > 0$, $u(t) := u_T(t)$ where u_T is the unique integral solution of (4.42) with $T > t$. Consequently, u is the unique integral solution of problem (4.41). ☐

4.5.1 Linear case

Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be a single-valued linear operator. If $-A$ is the generator of a contraction C_0-semigroup, then Lumer–Phillips theorem (Theorem 2.4.2) shows that A is an m-accretive operator with dense domain and therefore Theorem 4.5.1 guarantees that problem (4.36) has a unique integral solution for every initial data.

The main goal of this subsection is to prove that, when $-A$ is the generator of a contraction C_0-semigroup, the integral solution of (4.36) has a special form.

Let $(S(t))$ be the C_0-semigroup generated by $-A$, if we assume that problem (4.36) has a classical solution, namely u. Then the function $g(s) := S(t-s)u(s)$ if differentiable for $0 < s < t$ (see Proposition 2.1.4) and

$$\begin{aligned} g'(s) &= AS(t-s)u(s) + S(t-s)u'(s) \\ &= AS(t-s)u(s) - S(t-s)\big(Au(s)\big) + S(t-s)f(s) \\ &= S(t-s)f(s), \end{aligned}$$

Since $f \in L^1(0,T;X)$, $S(t-s)f(s)$ is integrable, integrating the above equality $g'(s) = S(t-s)f(s)$ from 0 to t yields

$$u(t) = S(t)x_0 + \int_0^t S(t-s)f(s)\,ds. \tag{4.43}$$

Definition 4.5.2. Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be a linear operator such that $-A$ is the generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$. Let $x_0 \in X$ and $f \in L^1(0,T;X)$. The function $u \in C(0,T;X)$ given by

$$u(t) = S(t)x_0 + \int_0^t S(t-s)f(s)\,ds, \quad 0 \leq t \leq T$$

is called a mild solution of problem (4.36) on $[0,T]$.

The definition of a mild solution of problem (4.36) coincides, when $f \equiv 0$, with the corresponding integral solution for the homogeneous problem.

Theorem 4.5.2. *Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be a linear operator such that $-A$ is the generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$. If $f \in C^1(0,T;X)$, then problem* (4.36) *has a classic solution for every $x \in D(A)$.*

Proof. Define the function $v(t) := \int_0^t S(t-s)f(s)\,ds = \int_0^t S(s)f(t-s)\,ds$. It is clear that v is differentiable for $t > 0$ and

$$v'(t) = S(t)f(0) + \int_0^t S(s)f'(t-s)\,ds = S(t)f(0) + \int_0^t S(t-s)f'(s)\,ds.$$

Hence v is continuously differentiable on $(0, T)$.

Now, for $h > 0$, it is not hard to check the identity

$$\frac{S(h)-I}{h}v(t) = \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds. \tag{4.44}$$

From the continuity of f, we have that

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds = f(t).$$

Since v is continuously differentiable on $(0, T)$, it follows from (4.44) that $v(t) \in D(A)$ for $0 < t < T$ and $-Av(t) = v'(t) - f(t)$. Bearing in mind that $v(0) = 0$, we may conclude that $u(t) = T(t)x + v(t)$ is a classic solution of problem (4.36). ☐

For $f \in L^1(0, T; X)$, the initial value problem (4.36) has, by Definition 4.5.2, a unique mild solution. Now we are interested in showing that this mild solution coincides with the integral solution of (4.36).

We start with the following technical result, which is a variation of Gronwall's lemma.

Lemma 4.5.1. *Let $\varphi \in C([0, T])$ and $\chi \in L^1([0, T])$ be such that $\varphi, \chi \geq 0$. For every $s, t \in [0, T]$ with $s < t$, the following conditions are equivalent:*

(a) $\varphi(t) - \varphi(s) \leq \int_s^t \chi(\tau)\,d\tau$,

(b) $\varphi(t)^2 - \varphi^2(s) \leq 2\int_s^t \varphi(\tau)\chi(\tau)\,d\tau$.

Proof. (a) ⇒ (b) We define the functions $f : \mathbb{R} \to [0, +\infty)$ and $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(\tau) := \begin{cases} \varphi(\tau) & \text{if } \tau \in [0, T], \\ 0 & \text{if } \tau \notin [0, T], \end{cases} \qquad g(\tau) := \begin{cases} \chi(\tau) & \text{if } \tau \in [0, T], \\ 0 & \text{if } \tau \notin [0, T]. \end{cases}$$

Choose a nonnegative, infinitely differentiable function $\rho : \mathbb{R} \to [0, \infty)$ such that $\rho(\tau) = 0$ if $|\tau| > 1$ and $\int_{-\infty}^{+\infty} \rho(\tau)\,d\tau = 1$ and, for each $\varepsilon > 0$, define

$$\rho_\varepsilon(\tau) := \frac{1}{\varepsilon}\rho\Big(\frac{\tau}{\varepsilon}\Big), \quad f_\varepsilon(\tau) = \int_{-\infty}^{+\infty} \rho_\varepsilon(\sigma)f(\tau-\sigma)\,d\sigma, \quad \text{and} \quad g_\varepsilon(\tau) = \int_{-\infty}^{+\infty} \rho_\varepsilon(\sigma)g(\tau-\sigma)\,d\sigma,$$

where $\tau \in \mathbb{R}$. It is well known (see, for instance, [46, Chapter IV]) that $f_\varepsilon, g_\varepsilon \in C^\infty(\mathbb{R})$ and

$$\lim_{\varepsilon \to 0^+} f_\varepsilon(\tau) = f(\tau) = \varphi(\tau), \quad \tau \in (0, T).$$

Let $0 < 2\varepsilon < T$ and $\varepsilon \le s < t \le T - \varepsilon$. Then by (a) we have

$$\varphi(t - \xi) - \varphi(s - \xi) \le \int_s^t \chi(\tau - \xi)\, d\tau,$$

for any ξ with $|\xi| \le \varepsilon$. Multiplying both sides of the above inequality by ρ_ε and integrating with respect to ξ on $[-\varepsilon, \varepsilon]$, we obtain

$$f_\varepsilon(t) - f_\varepsilon(s) \le \int_s^t g_\varepsilon(\tau)\, d\tau.$$

Dividing both sides by $t - s$ and letting $s \to t^-$, we infer that $\frac{d}{dt} f_\varepsilon(t) \le g_\varepsilon(t)$. Hence, we obtain

$$\frac{d}{dt} f_\varepsilon(t)^2 \le 2 f_\varepsilon(t) g_\varepsilon(t), \quad t \in [\varepsilon, T - \varepsilon].$$

Integrating the latter equation over $[s, t]$, we get

$$f_\varepsilon^2(t) - f_\varepsilon^2(s) \le 2 \int_s^t f_\varepsilon(\tau) g_\varepsilon(\tau)\, d\tau, \quad \varepsilon \le s \le t \le T - \varepsilon.$$

Now, since $|f_\varepsilon(\tau)| \le M := \sup\{|\varphi(\tau)| : \tau \in [0, T]\}$ and

$$\begin{aligned} \left| \int_s^t (f_\varepsilon(\tau) - g_\varepsilon(\tau))\, d\tau - \int_s^t f_\varepsilon(\tau) g(\tau)\, d\tau \right| &\le \int_{-1}^1 \rho(\sigma) \left(\int_s^t |f_\varepsilon(\tau)(g(\tau - \varepsilon\sigma) - g(\tau)|)\, d\tau \right) d\sigma \\ &\le M \int_{-1}^1 \rho(\sigma) \left(\int_s^t \big|g(\tau - \varepsilon\sigma) - g(\tau)\big|\, d\tau \right) d\sigma. \end{aligned}$$

Letting $\varepsilon \to 0^+$, the above expression yields

$$\int_s^t f_\varepsilon(\tau) g(\tau)\, d\tau \to \int_s^t f(\tau) g(\tau)\, d\tau.$$

Accordingly,

$$f^2(t) - f^2(s) \le 2 \int_s^t f(\tau) g(\tau)\, d\tau, \quad 0 \le s \le t < T.$$

So, we conclude that

$$\varphi^2(t) - \varphi^2(s) \leq 2\int_s^t \varphi(\tau)\chi(\tau)\,d\tau, \quad 0 < s \leq t < T.$$

Here if we let $s \to 0^+$ and $t \to T^-$, the inequality holds for every $s, t \in [0, T]$ with $s \leq t$.

(b) $\Rightarrow$ (a) Consider the additional functional $\psi(t) := \frac{1}{2}\varphi^2(s) + \int_s^t \varphi(\tau)\chi(\tau)\,d\tau$. By (b), it is clear that $\frac{1}{2}\varphi^2(t) \leq \psi(t)$, and therefore $\varphi(t) \leq \sqrt{2\psi(t)}$. Bearing in mind the definition of ψ, we have

$$\psi'(t) = \chi(t)\varphi(t) \leq \chi(t)\sqrt{2\psi(t)},$$

and therefore

$$\frac{d}{dt}\sqrt{\psi(t)} = \frac{\psi'(t)}{2\sqrt{\psi(t)}} \leq \frac{\chi(t)}{\sqrt{2}}.$$

Integrating over $[s, t]$, we get

$$\sqrt{\psi(t)} - \sqrt{\psi(s)} \leq \frac{1}{\sqrt{2}}\int_s^t \chi(\tau)\,d\tau.$$

Since $\psi(s) = \frac{1}{2}\varphi^2(s)$, we conclude

$$\varphi(t) \leq \sqrt{2\psi(t)} \leq \varphi(s) + \int_s^t \chi(\tau)\,d\tau. \qquad \square$$

Theorem 4.5.3. *Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be a linear operator such that $-A$ is the generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$. The mild and integral solutions of problem* (4.36) *coincide.*

Proof. Since $D(A)$ is dense in X, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in $D(A)$ such that $x_n \to x_0$. Because $\overline{C^1(0, T; X)}^{L^1} = L^1(0, T; X)$, we take a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $C^1(0, T; X)$ such that $f_n \to f$ in $L^1(0, T; X)$.

Now, consider the problems

$$\begin{cases} u_n' + Au_n = f_n, & t \in [0, T], \\ u(0) = x_n, & 0 < T \leq +\infty. \end{cases} \tag{4.45}$$

By Theorem 4.5.2, we know that, for each $n \in \mathbb{N}$, problem (4.45) has a classical solution, namely u_n. Hence u_n is the mild and integral solution of that problem, we have

$$u_n(t) = S(t)x_n + \int_0^t S(t-s)f_n(s)\,ds \tag{4.46}$$

and

$$\|u_n(t)-x\|^2 - \|u_n(s)-x\|^2 \le 2\int_s^t \langle f_n(\tau) - A(x), u_n(\tau) - x\rangle_+\,d\tau, \quad \forall x \in D(A). \tag{4.47}$$

The relationship between solutions u_n and u_m is given in Theorem 4.5.1. This theorem, together with Lemma 4.5.1, yields

$$\|u_n(t) - u_m(t)\| \le \|x_n - x_m\| + \int_0^t \|f_n(s) - f_m(s)\|\,ds.$$

The conditions on $(x_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ imply that there exists $u \in C(0,T;X)$ such that $u_n \to u$. Finally, taking limits as $n \to \infty$ in (4.46) and (4.47), we easily see that u is the mild and integral solution of problem (4.36). ☐

4.5.2 Regularization of solutions

As we have seen in Theorem 4.5.1, the inhomogeneous Cauchy problem admits an integral solution in a very general framework. Next, we will show that this integral solution becomes a strong solution under special conditions.

Theorem 4.5.4. *Let X be a reflexive Banach space and let $A \subset X \times X$ be an m-accretive operator on X. If $u_0 \in D(A)$, then the problem*

$$\begin{cases} u' + Au \ni f \quad a.\,e.\ t \in [0,T], \\ u(0) = u_0, \\ f \in W^{1,1}((0,T),X), \\ 0 < T < +\infty \end{cases} \tag{4.48}$$

has a unique strong solution u such that $u(t) \in D(A)$ a. e. $t \in\,]0,T[$.

Proof. By Theorem 4.5.1, problem (4.48) has a unique integral solution, namely u. We will prove that u is a strong solution of this problem.

First, we will check that u is an absolutely continuous function on $[0,T]$, and $u' \in L^\infty(0,T,X)$, i. e., $u \in W^{1,\infty}(0,T,X)$.

Take $y \in Au_0$. According to the definition of an integral solution, u satisfies

$$\frac{1}{2}\|u(h) - u_0\|^2 \leq \int_0^h \|f(\tau) - y\|\|u(\tau) - u_0\|\, d\tau.$$

On the other hand, if we take $g(t) = f(t + h)$ in (4.37), then

$$\frac{1}{2}\|u(t+h) - u(t)\|^2 \leq \frac{1}{2}\|u(h) - u(0)\|^2 + \int_0^t \|f(\tau + h) - f(\tau)\|\|u(\tau + h) - u(\tau)\|\, d\tau,$$

and so, applying Lemma 4.5.1 to the above two expressions, we conclude that

$$\|u(h) - u_0\| \leq \int_0^h \|f(s) - y\|\, ds, \quad 0 \leq h \leq T.$$

Respectively,

$$\|u(t+h) - u(t)\| \leq \int_0^h \|f(s) - y\|\, ds + \int_0^t \|f(s+h) - f(s)\|\, ds. \tag{4.49}$$

Since $f \in W^{1,1}((0,T),X)$, it is absolutely continuous on $[0,T]$. This, together with the latter inequality, implies that u is also absolutely continuous on $[0,T]$. Now, since X is reflexive, the use Komura's theorem (Theorem 1.14.6) shows that u is differentiable a. e. on $[0,T]$. Hence, taking into account of the fact that (4.49) holds for every $y \in Au_0$, dividing by h, applying the mean value theorem for integrals, and passing to the limit as $h \to 0$, we get

$$\left\|\frac{d}{dt}u(t)\right\| \leq |Au_0| + \|f(0)\| + \int_0^t \left\|\frac{d}{ds}f(s)\right\| ds, \quad \text{a. e. }]0,T[.$$

This proves that $u \in W^{1,\infty}(0,T,X)$. To end the proof, it remains to show that u satisfies equation (4.48) a. e.

Let $t_0 \in \,]0,T[$ be such that $\frac{d}{dt}u(t_0)$ exists. By the definition of an integral solution and since

$$\langle u - x, x - v\rangle_+ \leq \frac{1}{2}(\|u - v\|^2 - \|x - v\|^2), \quad \forall u,x,v \in X,$$

we have

$$\left\langle \frac{u(t) - u(t_0)}{t - t_0}, u(t_0) - x\right\rangle_+ \leq \frac{1}{t - t_0}\int_{t_0}^t \langle f(\tau) - y, u(\tau) - x\rangle_+\, d\tau,$$

where the right-hand side of the above inequality is an upper semicontinuous function and hence integrable. Thus, letting $t \to t_0$, we obtain

$$\langle u'(t_0), u(t_0) - x\rangle_+ \leq \langle f(t_0) - y, u(t_0) - x\rangle_+. \tag{4.50}$$

To get (4.50), we have assumed that t_0 is a Lebesgue point of the function $t \mapsto \langle f(t) - y, u(t) - x\rangle_+$; however, this is without loss of generality because almost every point is a Lebesgue point (cf. Remark 1.14.1). From (4.50), we can guarantee the existence of some $f_0 \in J(u(t_0) - x)$ such that

$$\langle u'(t_0) + y - f(t_0), f_0\rangle \leq 0. \tag{4.51}$$

On the other hand, for each $0 \leq h \leq t_0$,

$$u(t_0 - h) = u(t_0) - hu'(t_0) + g(h),$$

where $\lim_{h\to 0} \frac{\|g(h)\|}{h} = 0$. Since A is an m-accretive operator, i. e., $X = \mathrm{R}(I + hA)$, there exists $(x_h, y_h) \in A$ such that $u(t_0 - h) + hf(t_0) = x_h + hy_h$. Taking $x = x_h$ and $y = y_h$ in (4.51), we obtain

$$\langle u(t_0) - x_h + g(h), f_0\rangle \leq 0,$$

or

$$\frac{\|u(t_0) - x_h\|}{h} \leq \frac{\|g(h)\|}{h} \to 0 \quad \text{as } h \to 0,$$

and consequently, $\lim_{h\to 0} y_h = -u'(t_0) + f(t_0)$. Moreover, since A is m-accretive, it follows from Proposition 3.7.1 that A is closed and therefore

$$-u'(t_0) + f(t_0) \in Au(t_0),$$

which concludes the proof. ☐

Remark 4.5.3. Note that, in Theorem 4.5.4, we used the reflexivity of X to show that the function $t \mapsto u(t)$ is differentiable a. e. because it is absolutely continuous. This fact can be generalized to Banach spaces with Radon–Nikodym property. Thus, Theorem 4.5.4 holds true also for Banach spaces X with Radon–Nikodym property.

Recall that $BV(0, T; X)$ is the subspace of $L^1(0, T; X)$ consisting of those functions f which satisfy

$$V(f, T) = \limsup_{h\downarrow 0} \int_0^{T-h} \frac{\|f(\tau + h) - f(\tau)\|}{h}\, d\tau < \infty,$$

and, if $f \in BV(0,T;X)$, then it is essentially bounded and has an essential limit from the right denoted by $f(t+)$ at every $t \in [0,T)$. We will also use the notation

$$V(f,t+) := \limsup_{h\downarrow 0} \int_0^t \frac{\|f(\tau+h) - f(\tau)\|}{h}\, d\tau \quad \text{for } 0 \le t < T.$$

Next, we shall prove that, if we replace the hypothesis $f \in W^{1,1}(0,T,X)$ by $f \in BV(0,T;X)$, Theorem 4.5.4 remains valid.

Lemma 4.5.2. *Let A be an m-accretive operator on a Banach space X, $f \in BV(0,T;X)$, $u_0 \in D(A)$, and u an integral solution of problem* (4.48). *Then u is Lipschitz continuous. More precisely, if*

$$\rho(t) = \limsup_{h\downarrow 0} \frac{\|u(t+h) - u(t)\|}{h}$$

and $y \in Au_0$, then

$$\rho(t) \le \|f(0+) - y\| + V(f,t+). \tag{4.52}$$

Proof. The Lipschitz continuity of u will be seen once (4.52) is established because a bound on $\rho(t)$ is a Lipschitz constant for u. First, we estimate $\rho(0)$. Bearing in mind that u is an integral solution of (4.48) and $v(t) = u_0$ is a constant integral solution to $v' + Av \ni y$, by Theorem 4.5.1 (inequality (4.37)) and Lemma 4.5.1, we have

$$\|u(h) - u_0\| \le \int_0^h \|f(\tau) - y\|\, d\tau.$$

Dividing in the previous inequality by $h > 0$ and taking the limit as $h \to 0$, we obtain

$$\rho(0) \le \|f(0+) - y\|.$$

To estimate $\rho(t)$, fix $h > 0$, put $s = 0$, and let $v(t) = u(t+h)$, $g(t) = f(t+h)$. Then

$$\|u(t+h) - u(t)\| \le \|u(h) - u_0\| + \int_0^t \|f(t+h) - f(\tau)\|\, d\tau.$$

Dividing the latter inequality by $h > 0$ and taking the limit as $h \to 0$, we conclude

$$\rho(t) \le \|f(0+) - u_0\| + V(f,t+). \qquad \square$$

Theorem 4.5.5. *Let X be a Banach space with Radon–Nikodym property. Let A be an m-accretive operator, $f \in BV(0,T;X)$, $u_0 \in D(A)$, and u an integral solution of problem* (4.48)*. Then u is a strong solution.*

Proof. Since $f \in BV(0,T;X)$, it belongs to $L^1(0,T;X)$. According to Theorem 4.5.1, problem (4.48) has a unique integral solution u. By Lemma 4.5.2, we know that u is Lipschitz continuous and so it is absolutely continuous, i. e., $u \in W^{1,1}(0,T,X)$, hence we can apply Remark 4.5.3 to conclude that u is a strong solution. ☐

4.6 The abstract Cauchy problem for quasiaccretive operators

In this section X denotes a Banach space, $A \subseteq X \times X$ is an m-quasiaccretive operator, that is, there exists $\omega \geq 0$ such that $A + \omega I$ is m-accretive, and we consider the Cauchy problem (4.6). Proposition 4.2.1 shows that, as far as the uniqueness and continuous dependence of solution on data are concerned, the class of quasiaccretive operators gives a suitable framework for the Cauchy problem. However, for the existence we have to extend the notion of the solution for the Cauchy problem (4.6) from differentiable to continuous functions.

A particular choice of v and g for which v is a strong solution of $v' + Av \ni g$ on $[0,T]$ is $g(t) = y$ and $v(t) = x$ for all $t \in [0,T]$, where $(x,y) \in A$. Bearing in mind Proposition 4.2.1 and using inequality (4.7), we find that a strong solution u of $u' + Au \ni f$ on $[0,T]$ must satisfy

$$\|u(t)-x\|^2 - \|u(s)-x\|^2 \leq 2\omega \int_s^t \|u(\tau)-x\|^2\, d\tau + 2\int_s^t \langle f(\tau)-y, u(\tau)-x\rangle_+\, d\tau, \tag{4.53}$$

Further, arguing as in the proof of Proposition 4.2.1, we get

$$e^{-2\omega t}\|u(t)-x\|^2 - e^{-2\omega s}\|u(s)-x\|^2 \leq 2\int_s^t e^{-2\omega\tau}\langle f(\tau)-y, u(\tau)-x\rangle_+\, d\tau,$$

for all $0 \leq s < t \leq T$ and all $(x,y) \in A$. This inequality, which holds for strong solutions of $u' + Au \ni f$, allows us to introduce the following concept of a solution.

Definition 4.6.1. A function $u : [0,+\infty) \to \overline{D(A)}$ is said to be an integral solution of (4.6) if it satisfies

(i) $u(0) = x_0$,
(ii) u is continuous,
(iii) for every $0 \leq r < t \leq T$ and for every $(x,y) \in A$,

$$e^{-2wt}\|u(t)-x_0\|^2 - e^{-2wr}\|u(r)-x_0\|^2) \leq 2\int_r^t e^{-2w\tau}\langle f(\tau)-y_0, u(\tau)-x_0\rangle_+\, d\tau.$$

Definition 4.6.2. Let $f \in L^1(0,T;X)$ and $\varepsilon > 0$ be given. An ε-discretization on $[0,T]$ of the equation $y' + Ay \ni f$ consists of a partition $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_n$ of the interval $[0,t_n]$ and a finite sequence $(f_i)_{i=1}^n$ such that

$$t_i - t_{i-1} < \varepsilon, \quad \text{for } i = 1,\dots,n,\ T - \varepsilon < t_n \le T, \tag{4.54}$$

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|f(s) - f_i\|\,ds < \varepsilon. \tag{4.55}$$

We denote by $D_A^\varepsilon(0 = t_0, t_1, \dots, t_n; f_1, \dots, f_n)$ this ε-discretization.

An ε-discretization $D_A^\varepsilon(0 = t_0, t_1, \dots, t_n; f_1, \dots, f_n)$ solution to (4.6) is a piecewise constant function $z : [0,t_n] \to X$ whose values z_i on $(t_{i-1}, t_i]$ satisfy the finite difference equation

$$\frac{z_i - z_{i-1}}{t_i - t_{i-1}} + Az_i \ni f_i, \quad i = 1,2,\dots,n. \tag{4.56}$$

Such a function $z = (z_i)_{i=1}^n$ is called an ε-approximate solution to problem (4.6) if it further satisfies

$$\|z(0) - x\| \le \varepsilon. \tag{4.57}$$

Definition 4.6.3. A mild solution of problem (4.6) is a continuous function $u \in C(0,T;X)$ with the property that, for each $\varepsilon > 0$, there exists an ε-approximate solution z of $y' + Ay \ni f$ on $[0,T]$ such that $\|u(t) - z(t)\| \le \varepsilon$ for all $t \in (0,T]$ and $u(0) = x$.

Remark 4.6.1. Note that every strong solution u of problem (4.6) is a mild solution. Indeed, let $0 = t_0 \le t_1 \le \cdots \le t_n$ be an ε-discretization of $[0,T]$ such that

$$\left\| u'(t) - \frac{u(t_i) - u(t_{i-1})}{t_i - t_{i-1}} \right\| \le \varepsilon, \quad t_i - t_{i-1} \le \delta,\ i = 1,2,\dots,n,$$

and taking t_i, $i = 1,2,\dots,n$, to be Lebesgue points of f,

$$\int_{t_{i-1}}^{t_i} \|f(t) - f(t_i)\|\,dt \le \varepsilon(t_i - t_{i-1}).$$

Then the simple function $z : [0,T] \to X$ defined by

$$t \mapsto z(t) = u(t_i) \quad \text{on } (t_{i-1}, t_i]$$

is a $D_A^\varepsilon(t_0, t_1, \dots, t_n; f_1, \dots, f_n)$ solution, and if we choose (t_i) so that $\|u(t) - u(s)\| \le \varepsilon$ for $t, s \in (t_{i-1}, t_i)$, we have that $\|u(t) - z(t)\| \le \varepsilon$ for all $t \in [0,T]$, as claimed.

In the sequel, $\mathcal{D}'(0,T)$ denotes the space of distributions on $(0,T)$ (see Section 1.16) and $\mathcal{D}(0,T)^+$ denotes the space of nonnegative, infinitely differentiable functions with compact support on $(0,T)$.

Theorem 4.6.1. *Let A be an ω-accretive operator on a Banach space X and $f,g \in L^1(0,T;X)$. If v is an integral solution of $v' + Av \ni g$ on $[0,T]$ and u is a mild solution of $u + Au \ni f$ on $[0,T]$, then*

$$\frac{1}{2}\frac{d}{dt}\|u(t)-v(t)\|^2 \le \omega\|u(t)-v(t)\|^2 + \langle f(t)-g(t), u(t)-v(t)\rangle_+ \tag{4.58}$$

in $\mathcal{D}'(0,T)$.

Writing expression (4.58)*, we mean that*

$$\|u(t)-v(t)\|^2 - \|u(s)-v(s)\|^2 \le 2\omega\int_s^t \|u(\tau)-v(\tau)\|^2\,d\tau + 2\int_s^t \langle f(\tau)-g(\tau), u(\tau)-v(\tau)\rangle_+\,d\tau.$$

Proof. Let u_k^n, $k = 1,2,\dots,N_n$, be a solution of the ε_n-discretization

$$D_A^{\varepsilon_n}(0 = t_0^n, t_1^n, \dots, t_{N_n}^n; f_1^n, \dots, f_{N_n}^n)$$

of $u' + Au \ni f$ on $[0,T]$. Set $t_k^n - t_{k-1}^n = h_k^n$ and let $0 \le a \le b \le T$. Since v is an integral solution of $v' + Av \ni g$, we have

$$\begin{aligned}
\|v(b)-u_k^n\|^2 - \|v(a)-u_k^n\|^2 \le{}& 2\omega\int_a^b \|v(\sigma)-u_k^n\|^2\,d\sigma \\
&+ 2\int_a^b \left\langle g(\sigma) - f_k^n + \frac{u_k^n - u_{k-1}^n}{h_k^n}, v(\sigma) - u_k^n\right\rangle_+ d\sigma \\
\le{}& 2\omega\int_a^b \|v(\sigma)-u_k^n\|^2\,d\sigma + 2\int_a^b \langle g(\sigma) - f_k^n, v(\sigma)-u_k^n\rangle_+\,d\sigma \\
&+ \frac{2}{h_k^n}\int_a^b (\|v(\sigma)-u_{k-1}^n\| - \|v(\sigma)-u_k^n\|)\|v(\sigma)-u_k^n\|\,d\sigma.
\end{aligned} \tag{4.59}$$

Multiplying (4.59) by h_k^n and summing over $k = j+1, j+2, \dots, i$, we obtain

$$\begin{aligned}
&\sum_{k=j+1}^{i} h_k^n(\|v(b)-u_k^n\|^2 - \|v(a)-u_k^n\|^2) \\
&\quad\le \sum_{k=j+1}^{i} h_k^n 2\int_a^b (\omega\|v(\sigma)-u_k^n\|^2 + \langle g(\sigma)-f_k^n, v(\sigma)-u_k^n\rangle_+)\,d\sigma
\end{aligned}$$

$$+\sum_{k=j+1}^{i} 2\int_a^b (\|v(\sigma)-u_{k-1}^n\| - \|v(\sigma)-u_k^n\|)\|v(\sigma)-u_k^n\|\,d\sigma. \tag{4.60}$$

Now we assume that $\varepsilon_n \downarrow 0$ and the ε_n-approximate solutions u_k^n of $u' + Au \ni f$ converge uniformly to the mild solution u on $[0,T]$. Set $\varphi_n(\eta,\tau) = \|v(\eta) - u_k^n\|^2$ for $0 \le \eta \le T$, $t_{k-1}^n < \tau \le t_k^n$. Then $\varphi_n(\eta,\tau) \to \|v(\eta) - u(\tau)\|^2$ uniformly on compact subsets of $[0,T) \times [0,T)$. If we choose i, j depending on n such that $t_j^n \to c$ and $t_i^n \to d$ as $n \to +\infty$, then

$$\sum_{k=j+1}^{i} h_k^n \|v(\eta) - u_k^n\|^2 \to \int_c^d \|v(\eta) - u(\tau)\|^2\,d\tau \quad \text{for } \eta \in [0,T] \tag{4.61}$$

and, moreover,

$$\int_a^b \|v(\sigma) - u_j^n\|^2 \to \int_a^b \|v(\sigma) - u(c)\|^2 d\sigma \quad \text{and} \quad \int_a^b \|v(\eta) - u_i^n\|^2 \to \int_a^b \|v(\sigma) - u(d)\|^2\,d\sigma. \tag{4.62}$$

These relations will help us take the limit in (4.60). To this end, we introduce the piecewise constant function

$$F_n(\tau) = \int_a^b \langle g(\sigma) - f_k^n, v(\sigma) - u_k^n\rangle_+\,d\sigma, \quad \text{for } t_k^n < \tau \le t_{k+1}^n.$$

Using the definition of ε-approximate solutions, we conclude that

$$\begin{aligned}
&\left|\sum_{k=j+1}^{i}\left(h_k^n F_n(\tau) - \int_{t_{k-1}^n}^{t_k^n}\int_a^b \langle g(\sigma) - f(\tau), v(\sigma) - u_k^n\rangle_+\,d\sigma\,d\tau\right)\right| \\
&\quad \le \sum_{k=j+1}^{i}\int_a^b\int_{t_{k-1}^n}^{t_k^n} \|f_k^n - f(\tau)\|\|v(\sigma) - u_k^n\|\,d\sigma\,d\tau \\
&\quad \le M\varepsilon_n(b-a),
\end{aligned}$$

and consequently,

$$\limsup_{n\to+\infty}\sum_{k=j+1}^{i} h_k^n F_n(\tau) = \limsup_{n\to+\infty}\sum_{k=j+1}^{i}\int_{t_{k-1}^n}^{t_k^n}\int_a^b \langle g(\sigma) - f(\tau), v(\sigma) - u_n^k\rangle_+\,d\sigma\,d\tau. \tag{4.63}$$

Since $u_k^n \to u(\tau)$ and $t_k^n \to \tau$ as $n \to \infty$, using the fact that $\langle\cdot,\cdot\rangle_+$ is upper semicontinuous (cf. Proposition 1.10.1(c)), it follows from (4.58) that

$$\limsup_{n\to+\infty}\sum_{k=j+1}^{i} h_k^n\int_a^b\langle g(\sigma)-f_k^n, v(\sigma)-u_k^n\rangle_+\,d\sigma=\limsup_{n\to+\infty}\int_{t_j^n}^{t_i^n}F_n(\tau)\,d\tau$$
$$\leq\int_c^d\int_a^b\big\langle g(\sigma)-f(\tau), v(\sigma)-u(\tau)\big\rangle_+\,d\sigma\,d\tau. \tag{4.64}$$

Thus, by (4.61), (4.62), and (4.64), if we pass to the limit in (4.60), we obtain

$$\begin{aligned}&\int_c^d\big(\|v(b)-u(\tau)\|^2-\|u(a)-u(\tau)\|^2\big)\,d\tau\\&\quad\leq 2w\int_c^d\int_a^b\|v(\sigma)-u(\tau)\|^2 d\sigma d\tau+2\int_c^d\int_a^b\big\langle g(\sigma)-f(\tau), v(\sigma)-u(\tau)\big\rangle_+\,d\sigma\,d\tau\\&\qquad+2\int_a^b\big(\|v(\sigma)-u(c)\|-\|v(\sigma)-u(d)\|\big)\|v(\sigma)-u(d)\|\,d\sigma.\end{aligned}$$

Set $\varphi(s,t)=\|v(s)-u(t)\|^2$ and $\chi(s,t)=\langle g(s)-f(t), v(s)-u(t)\rangle_+$. Let R be the rectangle $a\leq s\leq b, c\leq t\leq d$ and denote by ∂R its boundary. Let $\mathbf{n}$ be the outward pointing normal to ∂R, and $\phi=(\varphi,\varphi)$ be the vector field on R each of whose components is φ. We may rewrite the above inequality in the form

$$\int_{\partial R}\phi\cdot\mathbf{n}dl\leq 2\omega\iint_R\varphi\,d\sigma\,d\tau+2\iint_R\chi\,d\sigma\,d\tau, \tag{4.65}$$

where dl is the element of arc length on the boundary. To end the proof, we will show, using (4.65), that

$$\varphi(t,t)-\varphi(s,s)\leq 2\omega\int_s^t\varphi(\tau,\tau)\,d\tau+2\int_s^t\chi(\tau,\tau)\,d\tau, \tag{4.66}$$

for $0\leq s\leq t\leq T$, which is one of the equivalences of (4.58).

First, we assume that φ and χ are in $C^1(B)$ where $B:=[0,T]\times[0,T]$. If (4.65) holds for every subrectangle of B with sides parallel to the coordinate axes, then the use of Green's formula gives

$$\int_{\partial R}\phi\cdot\mathbf{n}dl=\iint_R\Big(\frac{\partial}{\partial\sigma}+\frac{\partial}{\partial\tau}\Big)\varphi(\sigma,\tau)\,d\sigma\,d\tau$$

for every subrectangle R of B. This, together with (4.65), yields

$$\frac{\partial}{\partial\sigma}\varphi(\sigma,\tau)+\frac{\partial}{\partial\tau}\varphi(\sigma,\tau)\le 2\omega\sigma(\sigma,\tau)+2\chi(\sigma,\tau).$$

This implies

$$\frac{d}{dt}\varphi(t,t)\le 2\omega\varphi(t,t)+2\chi(t,t),$$

and so, by integration over $[s,t]$, we obtain (4.66).

Now, we assume that φ and χ are not in $C^1(B)$, let $\rho\in C_0^\infty(\mathbb{R}\times\mathbb{R})$ be a nonnegative function, supported in $[-1,1]\times[-1,1]$, $\iint_{[-1,1]\times[-1,1]}\rho(s,t)\,dsdt=1$. For $\varepsilon>0$, set $\rho_\varepsilon(\sigma,\tau)=\varepsilon^{-2}\rho(\sigma/\varepsilon,\tau/\varepsilon)$ and denote by φ_ε, χ_ε the convolutions of φ and χ with ρ_ε, respectively, which are smooth. A simple computation shows that (4.65) holds replacing φ_ε, χ_ε by φ and χ if R is a subrectangle of the region $\varepsilon\le s\le t\le T-\varepsilon$. Hence, the result in the case of smooth functions discussed above gives

$$\varphi_\varepsilon(t,t)-\varphi_\varepsilon(s,s)\le 2\omega\int_s^t\varphi_\varepsilon(\tau,\tau)\,d\tau+2\int_s^t\chi_\varepsilon(\tau,\tau)\,d\tau, \tag{4.67}$$

for $\varepsilon\le s\le t\le T-\varepsilon$. Since φ is continuous,

$$\lim_{\varepsilon\downarrow 0}\varphi_\varepsilon(t,t)=\varphi(t,t)$$

holds uniformly on compacts subsets of $(0,T)\times(0,T)$. For $\varepsilon<\alpha\le\beta<T-\varepsilon$, we get

$$\int_\alpha^\beta\chi_\varepsilon(t,t)\,dt=\iint_R\rho(\sigma,\tau)\int_\alpha^\beta\chi(t-\varepsilon\sigma,t-\varepsilon\tau)\,dt\,d\sigma\,d\tau. \tag{4.68}$$

By the definitions of χ and $\langle\cdot,\cdot\rangle_+$ and by (4.68), we have

$$\limsup_{\varepsilon\downarrow 0}\int_\alpha^\beta\chi_\varepsilon(t,t)\le\int_\alpha^\beta\chi(t,t)\,dt. \tag{4.69}$$

Finally, inequality (4.66) follows from inequalities (4.67), (4.68), and (4.69). ☐

Theorem 4.6.2. *Let A be a ω-accretive operator on a Banach space X and $f\in L^1(0,T;X)$. Then:*

(i) *Every mild solution of the problem $u'+Au\ni f$ on $[0,T]$ is also an integral solution on $[0,T]$.*

(ii) *The initial value problem $u'+Au\ni f, u(0)=x$, has at most one mild solution on $[0,T]$.*

(iii) *If the initial value problem $u'+Au\ni f, u(0)=x$ on $[0,T]$ has a mild solution u, then u is also the unique integral solution of this problem.*

Proof. Let u be a mild solution of $u' + Au \ni f$ on $[0, T]$. If $(x, y) \in A$, $v(t) = x$, and $g(t) = y$ for all $t \in [0, T]$, then v is a strong solution and hence an integral solution of $v' + Av \ni g$. Substituting $v = x$ and $g = y$ in (4.58), we infer that u is an integral solution.

To show that (ii) holds, we may use (4.58). This inequality is valid if v is an integral solution and u is a mild solution. By (i), that inequality holds when u and v are mild solutions. But putting $s = 0, f = g$, and $u(0) = v(0) = x$, we obtain the uniqueness. The same arguments prove (iii), since v could have been an arbitrary integral solution. ☐

Theorem 4.6.3. *Let A be an m-ω-accretive operator on a Banach space X and $f \in L^1(0, T; X)$. Then:*

(i) *For every $x \in \overline{D(A)}$, the initial value problem $u' + Au \ni f, u(0) = x$ has a unique mild solution on* $[0, T]$.

(ii) *For every $x \in \overline{D(A)}$, the initial value problem $u' + Au \ni f, u(0) = x$ has a unique integral solution on* $[0, T]$. *Furthermore, this integral solution is the mild solution.*

(iii) *An integral solution of $u' + Au \ni f$ on $[0, T]$ is a mild solution of the same problem.*

Proof. The uniqueness asserted in (i) comes from Theorem 4.6.2(ii), and the existence follows from Theorem 4.5.1 if we take $\omega = 0$. The general case can be found, for instance, in [47, 32, 178].

(ii) This assertion follows from (i) and Theorem 4.6.2(iii).

(iii) We will do the proof for $\omega > 0$. It suffices to show that if u is an integral solution of $u' + Au \ni f$, then $u(0) \in \overline{D(A)}$.

Choose $0 < \lambda < \frac{1}{\omega}$ and put $x_\lambda = J_\lambda u(0) \in D(A)$. We will show that $x_\lambda \to u(0)$ as $\lambda \to 0^+$. By the definition of integral solutions, we know that

$$\begin{aligned} e^{-2\omega t}\|u(t) - x_\lambda\|^2 - \|u(0) - x_\lambda\|^2 &\le 2\int_0^t e^{-2\omega\tau}\left\langle f(\tau) - \frac{u(0) - x_\lambda}{\lambda}, u(\tau) - x_\lambda\right\rangle_+ d\tau \\ &\le 2\int_0^t e^{-2\omega\tau}\left\|f(\tau) - \frac{u(0) - x_\lambda}{\lambda}\right\|\|u(\tau) - x_\lambda\|\, d\tau. \end{aligned}$$

Now, applying Lemma 4.5.1, the above inequality is equivalent to

$$\begin{aligned} e^{-\omega t}\|u(t) - x_\lambda\| - \|u(0) - x_\lambda\| &\le \int_0^t e^{-\omega t}\left\|f(\tau) - \frac{u(0) - x_\lambda}{\lambda}\right\| d\tau \\ &\le \int_0^t \left\|f(\tau) - \frac{u(0) - x_\lambda}{\lambda}\right\| d\tau \\ &\le \int_0^t \|f(\tau)\|\, d\tau + \frac{1}{\lambda}\int_0^t (\|u(\tau) - u(0)\| - \|u(\tau) - x_\lambda\|)\, d\tau. \end{aligned}$$

Rearranging, it implies

$$\frac{1}{t}\int_0^t \|u(\tau)-x_\lambda\|\,d\tau \le \frac{\lambda}{t}\left(\|u(0)-x_\lambda\| + \int_0^t \|f(\tau)\|\,d\tau\right) + \frac{1}{t}\int_0^t \|u(\tau)-u(0)\|\,d\tau.$$

Since

$$\|u(0)-x_\lambda\| = \frac{1}{t}\int_0^t \|u(0)-x_\lambda\|\,d\tau \le \frac{1}{t}\int_0^t (\|u(\tau)-x_\lambda\| + \|u(\tau)-u(0)\|)\,d\tau,$$

that is,

$$\|u(0)-x_\lambda\| - \frac{1}{t}\int_0^t \|u(\tau)-u(0)\|\,d\tau \le \frac{1}{t}\int_0^t \|u(\tau)-x_\lambda\|\,d\tau,$$

we may conclude that, for every $0 < \lambda < t$,

$$\left(1-\frac{\lambda}{t}\right)\|u(0)-x_\lambda\| \le \frac{\lambda}{t}\int_0^t \|f(\tau)\|\,d\tau + \frac{2}{t}\int_0^t \|u(\tau)-u(0)\|\,d\tau.$$

It follows immediately from above inequality that

$$\limsup_{\lambda\to 0^+}\|u(0)-x_\lambda\| \le \frac{2}{t}\int_0^t \|u(\tau)-u(0)\|\,d\tau.$$

Because u is assumed to be continuous, the right-hand side of the above inequality tends to zero as $t \to 0$, which completes the proof. □

4.6.1 Regularization

We know, invoking Theorem 4.6.3, that (4.6) has an integral solution whenever A is an m-quasiaccretive operator and $x \in \overline{D(A)}$. In general, integral solutions cannot be interpreted in a pointwise sense since the integral solutions are continuous but are not necessarily differentiable. In this section we are going to investigate a question of great interest, namely that of circumstances under which mild or integral solutions are strong solutions.

Definition 4.6.4. Let X be a Banach space and let $u \in C(0,T;X)$. For $0 \le t < T$, we define $D^+u(t)$ as follows:

$$D^+u(t) := \left\{z \in X : \exists \delta_n \downarrow 0 \text{ such that } w\text{-}\lim_{n\to+\infty} \frac{u(t+\delta_n)-u(t)}{\delta_n} = z\right\}.$$

That is, $D^+u(t)$ is the set of weak limits of right difference quotients of u at t. In particular, if u is either weakly or strongly differentiable from the right at t with derivative $u_r'(t)$, then $D^+u(t) = \{u_r'(t)\}$.

In Remark 1.14.1 we introduced the notion of Lebesgue's points. Now we will need the concept of a right Lebesgue point s of f, that is, a point s at which

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_s^t \|f(\tau) - f(s)\|\, d\tau = 0.$$

Proposition 4.6.1. *Let A be an ω-accretive operator on a Banach space X, $f \in L^1(0,T;X)$, and u a mild solution of $u' + Au \ni f$ on $[0,T]$. Let $s \in [0,T)$ and $C := \overline{\mathrm{co}}(D^+u(s))$. If s is a right Lebesgue point of f, then*

$$\mathrm{Gr}(A) \cup \{(u(s), f(s) - z) : z \in C\}$$

is also an ω-accretive operator. Moreover, if $z \in C$, $v \in X$ and $\mathrm{Gr}(A) \cup \{(u(s), f(s) - z) : z \in C\}$ is ω-accretive, then $\|z\| \le \|v\|$.

Proof. Since u is an integral solution, for every $(x,y) \in A$ and $0 \le s < t \le T$ we have

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \le 2\omega \int_s^t \|u(\tau) - x\|^2 + 2\int_s^t \langle f(\tau) - y, u(\tau) - x\rangle_+ \, d\tau. \tag{4.70}$$

Moreover, for all $x^* \in J(u(s) - x)$, we have

$$\langle x^*, u(t) - u(s)\rangle \le \|u(t) - x\|\|u(s) - x\| - \|u(s) - x\|^2 \le \frac{\|u(t) - x\|^2 - \|u(s) - x\|^2}{2}. \tag{4.71}$$

The definition of an integral solution, along with inequality (4.71), yields

$$\left\langle x^*, \frac{u(t) - u(s)}{t-s} \right\rangle \le \frac{\omega}{t-s} \int_s^t \|u(\tau) - x\|^2\, d\tau + \frac{1}{s-t} \int_s^t \langle f(\tau) - y, u(\tau) - x\rangle_+ \, d\tau. \tag{4.72}$$

If $z \in D^+u(s)$ and letting $\delta_n > 0$ be such that

$$z = w - \lim_{n\to\infty} \frac{u(s+\delta_n) - u(s)}{\delta_n},$$

we can take $t = s + \delta_n$ in (4.72). Taking limits, since $\langle \cdot, \cdot \rangle_+$ is upper semicontinuous and s is a right Lebesgue point of f, we have

$$\langle x^*, z\rangle \le \omega \|x - u(s)\|^2 + \langle f(s) - y, u(s) - x\rangle_+. \tag{4.73}$$

The set of points z for which (4.73) holds is closed and convex, thus the same is also true for $z \in C$. Since (4.73) is satisfied for every $x^* \in J(u(s) - x)$, it is clear that

$$-\omega\|u(s) - x\|^2 \leq \langle f(s) - y, u(s) - x\rangle_+ - \langle x^*, z\rangle \quad \text{for } (x, y) \in A,$$

or again

$$-\omega\|u(s) - x\|^2 \leq \langle f(s) - z - y, u(s) - x\rangle_+ \quad \text{for } (x, y) \in A.$$

This means that $\mathrm{Gr}(A) \cup \{(u(s), f(s) - z) : z \in C\}$ is w-accretive and the first part of the proposition is established.

Finally, let $\hat{A} := \mathrm{Gr}(A) \cup \{(u(s), f(s) - v)\}$. Then u is clearly an integral solution of $u' + \hat{A}u \ni f$. By assumption, $\hat{A}$ is ω-accretive, so (4.70) is satisfied with $x = u(s)$ and $y = f(s) - v$. Substituting these values in (4.70) and estimating the last integral, we get

$$\|u(t) - u(s)\|^2 \leq 2\omega \int_s^t \|u(\tau) - u(s)\|^2 + 2\int_s^t \|f(\tau) - f(s) + v\|\|u(\tau) - u(s)\|\, d\tau.$$

By Lemma 4.5.1, we have

$$\|u(t) - u(s)\| \leq \int_s^t (\omega\|u(\tau) - u(s)\| + \|f(\tau) - f(s) + v\|)\, d\tau,$$

and consequently,

$$\left\|\frac{u(t) - u(s)}{t - s}\right\| \leq \frac{\omega}{t - s}\int_s^t \|u(\tau) - u(s)\|\, d\tau + \frac{1}{t - s}\int_s^t \|f(\tau) - f(s) + v\|\, d\tau. \tag{4.74}$$

Putting $t = s + \delta_n$ in (4.74), since the norm is weakly lower semicontinuous, taking limits as $n \to \infty$, we conclude that $\|z\| \leq \|v\|$ for $z \in D^+u(s)$ and then for $z \in C$. ☐

Proposition 4.6.1 shows that if a mild or integral solution u of $u' + Au \ni f$ has a weak right derivative u'_{wr} at some point s which is a right Lebesgue point of f, then $(u(s), f(s) - u'_{wr}(s)) \in X \times X$ extends A as an ω-accretive operator. Therefore, if A is an m-quasiaccretive operator, then A does not admit any extension, and therefore $(u(s), f(s) - u'_{wr}(s)) \in \mathrm{Gr}(A)$.

Theorem 4.6.4. *Let X be a Banach space and let A be an m-ω-accretive operator in X and $f \in L^1(0, T; X)$. Then the following conditions are equivalent:*

(a) *u is a strong solution of $u' + Au \ni f$ on $[0, T]$,*

(b) *u is a mild solution on $[0, T]$ of $u' + Au \ni f$ and u is absolutely continuous and differentiable a. e.*

Proof. Since almost every point of $[0, T]$ is a Lebesgue point (see Remark 1.14.1) of $f \in L^1(0, T; X)$, if u is a mild solution of the initial value problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), & t \in [0, T], \\ u(0) = x \in D(A), \end{cases}$$

which is absolutely continuous and differentiable almost everywhere on $[0, T]$, then, according to Proposition 4.6.1, if s is a right Lebesgue point of f such that u is differentiable at s, then $u(s) \in D(A)$, $u'_{wr}(s) + Au(s) \ni f(s)$, which was the only new thing to establish. ☐

A refinement of Lemma 4.5.2 yields:

Lemma 4.6.1. *Let A be an m-ω-accretive operator on a Banach space X, $f \in BV(0, T; X)$, $u_0 \in D(A)$, and u an integral solution of problem* (4.48)*. Then u is Lipschitz continuous. More precisely, if*

$$\rho(t) = \limsup_{h \downarrow 0} \frac{\|u(t + h) - u(t)\|}{h}$$

and $y \in Au_0$, then

$$\rho(t) \le e^{\omega t} \|f(0+) - y\| + V(f, t+) + \omega \int_0^t e^{\omega(t-\tau)} V(f, \tau+).$$

As a consequence of this lemma, we have.

Theorem 4.6.5. *Let X be a Banach space with Radon–Nikodym property. Let A be an m-ω-accretive operator, $f \in BV(0, T; X)$, $x \in D(A)$, and u an integral solution of problem* (4.48)*. Then u is a strong solution*

Proof. By Lemma 4.6.1, we know that u is Lipschitz continuous, and so it is absolutely continuous. Hence $f \in W^{1,1}(0, T, X)$ and therefore we can apply Theorem 4.6.4 to derive that u is a strong solution. ☐

An important concept of solution to the following problem:

$$\begin{cases} u'(t) + Au(t) \ni f(t), & 0 \le t \le T, \\ u(0) = x_0 \end{cases} \tag{4.75}$$

can be found in [31, p. 134].

Definition 4.6.5. Let X be a Banach space. A function $u \in \mathcal{C}([0, T], X)$ is said to be a weak solution of problem (4.75) if there exist sequences $(u_n)_{n \in \mathbb{N}} \subset W^{1,\infty}(0, T, X)$ and $(f_n)_{n \in \mathbb{N}} \subset L^1(0, T, X)$ satisfying

(a) $\frac{d}{dt}u_n(t) + Au_n(t) \ni f_n(t)$ a. e. $t \in]0, T[$, $n = 1, 2, \dots$,
(b) $\lim_{n\to+\infty} u_n(t) = u(t)$ uniformly on $[0, T]$,
(c) $u(0) = x_0$ and $\lim_{n\to+\infty} f_n = f$ in $L^1(0, T, X)$.

It is not hard to see that if u is a weak solution, then it is an integral solution.

Corollary 4.6.1. *Let X be a Banach space with Radon–Nikodym property and let $A \subset X\times X$ be an m-ω-accretive operator. Then problem* (4.36) *has a unique weak solution for any $x_0 \in \overline{D(A)}$.*

Proof. Consider a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ converging in $L^1(0, T, X)$ to the function f. For each $n \in \mathbb{N}$, consider the Cauchy problem

$$u_n'(t) + Au_n(t) \ni f_n(t), \quad u_n(0) = x_n,\ 0 \le t \le T,$$

where $(x_n)_{n\in\mathbb{N}}$ is a sequence in $D(A)$ which converges to x_0. Since every simple function is of bounded variation, Theorem 4.6.5 allows us to say that there exists a unique strong solution $u_n \in W^{1,\infty}(0, T, X)$. Furthermore, by (4.11), we have that

$$e^{-2\omega t}\|u_n(t) - u_m(t)\|^2 \le e^0\|u_n(0) - u_m(0)\|^2 + 2\int_0^t e^{-2\omega\tau}\langle u_n(\tau) - u_m(\tau), f_n(\tau) - f_m(\tau)\rangle_+ \, d\tau.$$

Next, applying Lemma 4.5.1, we get

$$\|u_n(t) - u_m(t)\| \le e^{\omega T}\|x_n - x_m\| + e^{\omega T}\int_0^T \|f_n(\tau) - f_m(\tau)\| \, d\tau.$$

From the latter inequality, we may infer that the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly convergent to a function, namely u, so that u must be the solution that we were looking for. ☐

4.6.2 Lipschitz perturbations of *m*-quasiaccretive operators

Lemma 4.6.2. *Let X be a Banach space and let $B : X \to X$ be a κ-Lipschitz mapping. Then, for all $x, y \in X$ and $j \in J(x - y)$, we have*

$$\langle Bx - By, j\rangle \ge -\kappa\|x - y\|^2.$$

Proof. Let $j \in J(x - y)$. By definition of $\langle\cdot,\cdot\rangle$ and bearing in mind that B is κ-Lipschitz, we obtain that

$$\langle Bx - By, j\rangle \ge -\|Bx - By\|\|x - y\| \ge -\kappa\|x - y\|^2. \qquad ☐$$

Note that Lemma 4.6.2 implies that if B is a κ-Lipschitz mapping, then B is a quasi-accretive operator. Furthermore, B has the following additional property:

$$\inf\{\langle Bx - By, j\rangle : j \in J(x-y)\} \geq -\kappa\|x-y\|^2.$$

Theorem 4.6.6. *Let X be a Banach space and let $A : D(A) \subseteq X \to 2^X$ be an m-quasiaccretive operator. Let $B : X \to X$ be a κ-Lipschitz mapping. Then $A + B$ is an m-quasiaccretive operator on X.*

Proof. Since A is m-quasiaccretive, there exists $\omega \geq 0$ such that $A + \omega I$ is m-accretive. Applying Lemma 4.6.2, we infer that $A + B$ is $(\omega + \kappa)$-accretive, that is, $A + B$ is a quasi-accretive operator. On the one hand, by Lemma 4.6.2, we know that $B + \kappa I : X \to X$ is a continuous accretive operator, and then, the use of Theorem 4.4.3, together with a simple calculation, shows that $B + \kappa I$ is a 2κ-Lipschitz m-accretive operator.

Now, if we denote $\tilde{A} := A + \omega I$ and $\tilde{B} := B + \kappa I$, by the above comments, $\tilde{A}$ and $\tilde{B}$ are two m-accretive operators. Moreover, from Lemma 4.6.2, it is easy to see that $\tilde{A} + t\tilde{B}$ is an accretive operator for $0 \leq t \leq 1$.

Finally, we will show that there exists a $\delta > 0$ such that if $\tilde{A} + t\tilde{B}$ is m-accretive and $0 \leq t < 1$, then $\tilde{A} + s\tilde{B}$ is also m-accretive provided that $s \in [0,1]$ and $|t-s| < \delta$. Once this is obtained, we can reach the value $t = 1$ by a finite number of steps of length $\frac{\delta}{2}$, starting at $t = 0$, and thus $\tilde{A} + \tilde{B}$ is m-accretive.

Let $t \in [0,1)$ and suppose that $\tilde{A} + t\tilde{B}$ is m-accretive. Given $z \in X$, consider the equation

$$z = x + y + (t+\tau)\tilde{B}x, \quad y \in \tilde{A}x.$$

If we denote $w = x + y + t\tilde{B}x$ and $J := (I + \tilde{A} + t\tilde{B})^{-1}$, the above equation becomes

$$w + \tau\tilde{B}Jw = z.$$

If we define $P_\tau : X \to X$ by $P_\tau(v) = z - \tau\tilde{B}J(v)$, since J is nonexpansive and $\tilde{B}$ is 2κ-Lipschitz, the operator P_τ is a contraction whenever $0 \leq \tau < \frac{1}{2\kappa}$; hence, by the Banach contraction principle (Theorem 6.2.1), there exists a unique $w_\tau \in X$ such that $w_\tau = P_\tau(w_\tau)$, i. e., $\tilde{A} + (t+\tau)\tilde{B}$ is m-accretive whenever $|\tau| \leq \delta$. This completes the proof of the theorem. □

4.6.3 Local Lipschitz perturbations

Consider the initial value problem

$$\begin{cases} y'(t) + Ay(t) + Fy(t) \ni f(t), & t \in [0,T], \\ y(0) = y_0, \end{cases} \tag{4.76}$$

where A is an m-quasiaccretive operator in X and $F : X \to X$ is a locally Lipschitz mapping, that is,

$$\|Fu - Fv\| \leq L_R\|u - v\|, \quad \forall u, v \in \mathbb{B}_R,\ \forall R > 0.$$

Theorem 4.6.7. *Let X be a Banach space with Radon–Nikodym property and let A be an m-quasiaccretive operator in X. Let $f \in BV(0, T; X)$ and let $F : X \to X$ be a locally Lipschitz operator on X. Then for each $y_0 \in D(A)$ there is $T(y_0) \in (0, T)$ such that problem* (4.76) *has a strong solution y in $[0, T(y_0)]$. If, moreover,*

$$\langle F(y(t)) - f(t), v\rangle \geq -\gamma_1\|y(t)\|^2 + \gamma_2, \quad \forall v \in J(y(t)),$$

then the strong solution y is global in $[0, T]$.

Proof. We truncate F as follows: consider the operator $F_R : X \to X$ defined by

$$F_R(y) = \begin{cases} F(y) & \text{if } \|y\| \leq R, \\ F(\frac{Ry}{\|y\|}) & \text{if } \|y\| \geq R. \end{cases}$$

It is not hard to see that F_R is $2L_R$-Lipschitz on X. Therefore, applying Theorem 4.6.6, we conclude that $A + F_R$ is an m-quasiaccretive operator on X. Hence, by Theorem 4.6.5, for each $R > 0$ there is a unique strong solution y_R to

$$\begin{cases} y_R'(t) + Ay_R(t) + F_R y_R(t) \ni f(t), & t \in [0, T], \\ y(0) = y_0. \end{cases} \tag{4.77}$$

Since y_R is a strong solution to (4.77), there is $s(t) \in Ay_R(t)$ such that $y_R'(t) = f(t) - s(t) - F_R y_R(t)$ a. e. Now, if we apply Lemma 4.2.1, given $v(t) \in J(y_R(t))$, then we have

$$\|y_R(t)\| \frac{d}{dt}\|y_R(t)\| = \langle f(t) - s(t) - F_R(y_R(t)), v(t)\rangle \quad \text{a. e. in } [0, T].$$

Next, using the fact that A is ω-accretive, we get (without loss of generality, we may assume that $0 \in A(0)$)

$$\|y_R(t)\| \frac{d}{dt}\|y_R(t)\| \leq \|f(t)\|\|y_R(t)\| + \langle 0 - s(t), v\rangle - \langle F_R(y_R(t)), v\rangle \quad \text{a. e. in } [0, T],$$

and therefore

$$\frac{d}{dt}\|y_R(t)\| \leq \|f(t)\| + \|F_R(0)\| + (\omega + 2L_R)\|y_R(t)\|.$$

Since $f \in BV(0, T; X)$, it is bounded, and so, by Gronwall's inequality (differential form), for all $t \in (0, T)$, we have

$$\begin{aligned}\|y_R(t)\| &\le e^{(2L_R+\omega)t}\|y_0\| + \int_0^t e^{(2L_R+\omega)(t-s)}(\|f(s)\| + \|F_R(0)\|)\,ds\\ &\le e^{(2L_R+\omega)t}\|y_0\| + \frac{M}{2L_R+\omega}(e^{(2L_R+\omega)t} - 1).\end{aligned}$$

This implies that $\|y_R(t)\| \le R$ for $0 \le t \le T_R$ and $R > 0$ large enough if $T_R > 0$ is suitably chosen. Hence, on $[0, T_R]$, $\|y_R(t)\| \le R$, and so equation (4.77) reduces on this interval to (4.77). Therefore (4.77) has a unique strong solution on $[0, T_R]$.

On the other hand, if $\langle F(y(t)) - f(t), v\rangle \ge -\gamma_1\|y(t)\|^2 + \gamma_2$, $\forall v \in J(y(t))$, then we have

$$\frac{1}{2}\frac{d}{dt}\|y_R(t)\|^2 \le \gamma_1\|y_R(t)\|^2 + \gamma_2, \quad \text{a. e. in } (0, T).$$

Consequently, using Gronwall's inequality (differential form), we get

$$\|y_R(t)\|^2 \le e^{2\gamma_1 t}\|y_0\|^2 + \frac{\gamma_2}{\gamma_1}(e^{2\gamma_1 T} - 1) \le R \quad \text{for } t \in [0, T],$$

if R is large enough. Hence, for such R, y_R is the strong solution to (4.76) on $[0, T]$. □

4.7 Bibliographical remarks

The main references that we have followed in Sections 4.2 and 4.3 are [31, 64]. Section 4.5.1 comes from [195]. Sections 4.6 and 4.6.2 have been obtained from [32, 37]. The readers interested in the existence of mild solutions to abstract Cauchy problems can check the book [234], where the author gives a detailed exposition of the compactness methods developed and used to deal with the existence of mild solutions of abstract evolution equations governed by perturbations of m-accretive operators.

5 Solvability of nonlinear evolution equations

This chapter aims to present some examples of nonlinear evolution equations to illustrate the field of applications of the results presented in Chapters 3 and 4. We shall discuss the existence and uniqueness of solutions to some evolution equations arising in the dynamic of populations and neutron transport theory. Examples constituting this chapter are recent. For equations derived from models arising in population dynamics, it is natural to suppose that the boundary conditions are modeled by nonlinear reproduction laws. Hence, in Sections 5.1–5.2, the boundary conditions are modeled by nonlinear operators. Moreover, we discuss both local and nonlocal boundary conditions.

5.1 Lebowitz–Rubinow model

In a proliferating population of cells, mother cells undergo fission to form two new daughter cells. The time between cell birth and division is called the cell cycle length. In 1974, J. L. Lebowitz and S. I. Rubinow [161] presented a model for growing cell population in which individual cells are distinguished by age a and cell cycle length l. In their viewpoint, the cell cycle length of individual cells is an inherent characteristic determined at birth. Let $f(t,a,l)$ be the density of the population with respect to age a and cell cycle length l at time t. The age of cells a is defined so that a cell is born at $a = 0$ (daughter cells) and divided at $a = l$ (mother cells). The equations of the model are

$$\begin{cases} \frac{\partial f}{\partial t}(t,a,l) = -\frac{\partial f}{\partial a}(t,a,l) - \sigma(a,l)f(t,a,l), \\ f(0,a,l) = f_0(a,l), \end{cases} \tag{5.1}$$

for $t \geq 0$, $0 < a < \ell$, and $0 < \ell_1 < l < \ell_2 < +\infty$, where f_0 stands for the initial data. The constant $\ell_1 > 0$ (resp. ℓ_2) denotes the minimum cycle length (resp. the maximum cycle length) of cells. The function $\sigma(\cdot,\cdot)$ is the rate of cell mortality or loss due to causes other than division. Problem (5.1) is complemented with a biological reproduction rule given by

$$f(t,0,l) = [Rf(t,\cdot,\cdot)](l), \tag{5.2}$$

where R stands for an operator defined on suitable trace spaces, called the transition operator.

Problem (5.1) was considered later by G. F. Webb in [239, 240] in the case where (5.2) has the form

$$f(t,0,l) = 2r(l)\int_0^1 f(t,l',l')\,dl' + 2cf(t,l,l), \quad 0 \leq l \leq 1,\ t \geq 0. \tag{5.3}$$

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The latter equation has the form $f(t,0,l) = 2\int_0^1 \kappa(l,l')f(t,l',l')\,dl'$ with the transition probability kernel $\kappa(l,l') = r(l) + c\delta(l-l')$ (here $\delta(\cdot)$ stands for Dirac's measure). The fraction of cells born from mother cells with cycle length l', which will have cycle length between l and $l+dl$, is $k(l,l')\,dl$. Since every daughter cell of a mother cell with cycle length l' must have some cycle length l, the kernel κ satisfies $\int_0^1 k(l,l')\,dl = 1$. If $c = 0$, then there is no correlation between cycle lengths of mother and daughter cells. If $r(l) = 0$ and $c = 1$, then there is perfect inheritance of cycle lengths. In [154], K. Latrach and M. Mokthar-Karroubi provided a detailed analysis of the solution to problem (5.1)–(5.2) in the case where R is an abstract bounded linear operator which covers all linear biological laws of the transition for distribution of mother cells with cycle length l to daughter cells with cycle length l considered previously in the literature. We note that several mathematical aspects concerning the linear Cauchy problem (5.1)–(5.2) (that is, the operator R is linear): well-posedness (generation results), spectral analysis, time asymptotic behavior of solution (when it exists) were discussed in many papers (we refer, for example, to the works [13, 39, 40, 154, 166, 217, 239, 240] and the references therein).

It was observed by R. Rotenberg [205] that the linear model seems not adequate. Indeed, the cells under consideration are in contact with a nutrient environment which is not part of the mathematical formulation. Fluctuations in nutrient concentration and other density-dependent effects such as contact inhibition of growth make the transition rates functions of the population density, thus creating a nonlinear problem. On the other hand, the biological boundary of ℓ_1 and ℓ_2 are fixed and tightly coupled throughout mitosis. The conditions present at the boundaries are felt throughout the system and cannot be remote. This phenomenon suggests that at mitosis the daughter and parent cells are related by a nonlinear reproduction rule. Hence, at mitosis, daughter and mother cells are related by a nonlinear rule which describes the boundary conditions. It is written in the shape $f(t,0,l) = [Rf(t,\cdot,\cdot)](l)$ where R denotes a nonlinear operator on suitable trace spaces and is intended to model the transition from mother cells with cycle length l to daughter cells with cycle length l (cf. [12, 100]).

5.1.1 Lebowitz–Rubinow model for finite maximum cell cycle length

In this section we shall discuss the existence and uniqueness of solutions to an initial boundary value problem which models a nonlinear age structured cell population dynamics of the form

$$\begin{cases} \frac{\partial f}{\partial t}(t,a,l) = -\frac{\partial f}{\partial a}(t,a,l) + \sigma(a,l,f(t,a,l)), \\ f(0,a,l) = f_0(a,l), \\ f(t,0,l) = [Rf(t,\cdot,\cdot)](l), \end{cases} \tag{5.4}$$

where $t > 0$, $0 < a < l$, and $0 < l_1 < l < l_2$.

Next, we introduce the functional framework which will be used in order to discuss the existence and uniqueness of solutions to problem (5.4).

Let $1 \le p < +\infty$ and set

$$\mathsf{X}_p = L_p(\Omega, da\, dl)$$

where $\Omega := \{(a,l) : 0 < a < l, \ell_1 < l < \ell_2\}$ with $0 < \ell_1 < l < \ell_2 < \infty$. The space X_p endowed with its natural norm is a Banach space and, for any $u \in \mathsf{X}_p$, by $\|u\|_p$ we denote the norm of u. We denote by X_p^1 and X_p^2 the boundary spaces

$$\mathsf{X}_p^1 = L_p(\Gamma_1, dl), \quad \mathsf{X}_p^2 = L_p(\Gamma_2, dl),$$

where

$$\Gamma_1 := \{(0,l) : l \in (\ell_1, \ell_2)\} \quad \text{and} \quad \Gamma_2 := \{(l,l) : l \in (\ell_1, \ell_2)\}.$$

Let W_p be the partial Sobolev space defined by

$$\mathsf{W}_p = \left\{ f \in \mathsf{X}_p : \frac{\partial f}{\partial a} \in \mathsf{X}_p \right\},$$

which equipped with the norm

$$\|f\|_{\mathsf{W}_p} = \left[\|f\|_p^p + \left\| \frac{\partial f}{\partial a} \right\|_p^p \right]^{1/p}$$

is a Banach space.

Now let us recall that if $f \in \mathsf{W}_p$, since Ω is bounded, we can define the traces of f as follows:

$$f_{|\Gamma_1}(l) := f(a,l) - \int_0^a \frac{\partial}{\partial \xi} f(\xi, l)\, d\xi \quad \text{and} \quad f_{|\Gamma_2}(l) := f(a,l) + \int_a^l \frac{\partial}{\partial \xi} f(\xi, l)\, d\xi.$$

This allows us to identify $f(0,l) = f_{|\Gamma_1}(l)$ and $f(l,l) = f_{|\Gamma_2}(l)$.

Lemma 5.1.1. *Let $f \in \mathsf{W}_p$. If $f_{|\Gamma_1} \in \mathsf{X}_p^1$, then $f_{|\Gamma_2} \in \mathsf{X}_p^2$, and vice versa.*

Proof. Let $f \in \mathsf{W}_p$ such that $f_{|\Gamma_1} \in \mathsf{X}_p^1$. We have

$$f(l,l) = f(0,l) + \int_0^l \frac{\partial f}{\partial a}(a,l)\, da. \tag{5.5}$$

Applying Hölder inequality, we get

$$|f(l,l)| \leq |f(0,l)| + \ell_2^{1/q}\left(\int_0^l \left|\frac{\partial f}{\partial a}(a,l)\right|^p da\right)^{1/p},$$

where q is the conjugate exponent of p. And consequently,

$$|f(l,l)|^p \leq 2^p\left[|f(0,l)|^p + \ell_2^{p/q}\int_0^l \left|\frac{\partial f}{\partial a}(a,l)\right|^p da\right].$$

This yields

$$\begin{aligned}\int_{\ell_1}^{\ell_2} |f(l,l)|^p\, dl &\leq 2^p\left[\int_{\ell_1}^{\ell_2} |f(0,l)|^p\, dl + \ell_2^{p/q}\int_{\ell_1}^{\ell_2}\int_0^l \left|\frac{\partial f}{\partial a}(a,l)\right|^p da\, dl\right]\\ &\leq 2^p\left[\|f_{|\Gamma_1}\|_{\mathsf{X}_p^1}^p + \ell_2^{p/q}\left\|\frac{\partial f}{\partial a}\right\|_p^p\right] < +\infty.\end{aligned}$$

Writing (5.5) in the form $f(0,l) = f(l,l) - \int_0^l \frac{\partial \varphi}{\partial a}(a,l)\, da$ and arguing as above, we derive the converse. ☐

Let $\widetilde{\mathsf{W}}_p$ denote the space

$$\widetilde{\mathsf{W}}_p = \{f \in \mathsf{W}_p \text{ such that } f_{|\Gamma_1} \in \mathsf{X}_p^1\}.$$

By Lemma 5.1.1, we also have

$$\widetilde{\mathsf{W}}_p = \{f \in \mathsf{W}_p \text{ such that } f_{|\Gamma_2} \in \mathsf{X}_p^2\}.$$

Hence, any function $f \in \widetilde{\mathsf{W}}_p$ has traces $f_{|\Gamma_1}$ and $f_{|\Gamma_2}$ belonging to the boundary spaces X_p^1 and X_p^2, respectively.

Remark 5.1.1. For simplicity, we identify X_p^1 and X_p^2 with the space $\mathsf{Y}_p := L_p([\ell_1,\ell_2], dl)$.

As indicated in the introduction, the boundary conditions are modeled by an operator R defined from Y_p into Y_p. It is a nonlinear reproduction rule relating mother to daughter cells. We suppose that R satisfies the following condition:

(H1) there exists $\kappa > 0$ such that, for all $u, v \in \mathsf{Y}_p$, we have

$$\|R(u) - R(v)\|_{Y_p} \leq \kappa\|u - v\|_{\mathsf{Y}_p}.$$

Let $\mathcal{S}_R$ be the operator defined by

$$\begin{cases}\mathcal{S}_R : D(\mathcal{S}_R) \subset \mathsf{X}_p \longrightarrow \mathsf{X}_p,\\ f \mapsto (\mathcal{S}_R f)(a,l) = \frac{\partial f}{\partial a}(a,l),\\ D(\mathcal{S}_R) = \{f \in \widetilde{\mathsf{W}}_p : f_{|\Gamma_1} = R(f_{|\Gamma_2})\}.\end{cases}$$

We shall now discuss the m-accretivity of the operator $\mathcal{S}_R$ on the spaces X_p, $1 \le p < +\infty$.

Lemma 5.1.2. *Assume that the transition operator R is κ-contractive ($\kappa \in [0,1)$), and $\ell_1 \ge 0$. Then $\mathcal{S}_R$ is m-accretive on X_p.*

Proof. Bearing in mind that $\mathcal{S}_R$ is an accretive operator if and only if, for every $\varphi_1, \varphi_2 \in D(T_R)$, we have

$$[\varphi_1 - \varphi_2, \mathcal{S}_R(\varphi_1) - \mathcal{S}_R(\varphi_2)]_s \ge 0.$$

We consider first the case $p = 1$. For $\varphi_1, \varphi_2 \in D(\mathcal{S}_R)$, we have

$$\begin{aligned}
&[\varphi_1 - \varphi_2, \mathcal{S}_R(\varphi_1) - \mathcal{S}_R(\varphi_2)]_s && (5.6)\\
&\quad \ge \int_{\ell_1}^{\ell_2} \left(\int_0^l \frac{\partial}{\partial a}(\varphi_1(a,l) - \varphi_2(a,l)) \operatorname{sgn}_0(\varphi_1(a,l) - \varphi_2(a,l))\, da \right) dl \\
&\quad = \int_{\ell_1}^{\ell_2} \left(\int_0^l \frac{\partial}{\partial a}(\left|\varphi_1(a,l) - \varphi_2(a,l)\right|)\, da \right) dl. && (5.7)
\end{aligned}$$

Because trace mappings are linear and continuous, the last term of inequality (5.7) is greater than or equal to

$$\int_{\ell_1}^{\ell_2} (|\varphi_{2|\Gamma_2} - \varphi_{1|\Gamma_2}| - |\varphi_{2|\Gamma_1} - \varphi_{1|\Gamma_1}|)\, dl. \tag{5.8}$$

Since $\varphi_1, \varphi_2 \in D(\mathcal{S}_R)$, we obtain that expression (5.8) is greater than or equal to

$$\int_{\ell_1}^{\ell_2} (|\varphi_{2|\Gamma_2} - \varphi_{1|\Gamma_2}| - \left|R(\varphi_{2|\Gamma_2}) - R(\varphi_{1|\Gamma_2})\right|)\, dl. \tag{5.9}$$

Finally, the κ-contractivity of R yields that

$$\begin{aligned}
[\varphi_1 - \varphi_2, \mathcal{S}_R(\varphi_1) - \mathcal{S}_R(\varphi_2)]_s &\ge \|\varphi_{2|\Gamma_2} - \varphi_{1|\Gamma_2}\| - \|R(\varphi_{2|\Gamma_2}) - R(\varphi_{1|\Gamma_2})\| \\
&\ge (1-\kappa)\|\varphi_{2|\Gamma_2} - \varphi_{1|\Gamma_2}\| \ge 0. && (5.10)
\end{aligned}$$

The latter inequality implies that $\mathcal{S}_R$ is an accretive operator on X_1.

We now prove that $\mathcal{S}_R$ is an accretive operator on X_p with $p \in (1, +\infty)$. Let $\varphi_1, \varphi_2 \in D(\mathcal{S}_R)$ and put $\psi = \varphi_1 - \varphi_2$. Then we have

$$\begin{aligned}[\psi, \mathcal{S}_R(\varphi_1) - \mathcal{S}_R(\varphi_2)]_s &\geq \|\psi\|_p^{1-p} \int_{\ell_1}^{\ell_2} \int_0^l |\psi|^{p-1} \frac{\partial}{\partial a}(\psi(a,l))\, \operatorname{sgn}_0(\psi)\, da\, dl \\ &= \|\psi\|_p^{1-p} \int_{\ell_1}^{\ell_2} \left(\int_0^l |\psi|^{p-1} \frac{\partial}{\partial a}(|\psi(a,l)|)\, da \right) dl \\ &= \|\psi\|_p^{1-p} \frac{1}{p} \int_{\ell_1}^{\ell_2} \left(\int_0^l \frac{\partial}{\partial a}(|\psi(a,l)|^p)\, da \right) dl. \end{aligned} \tag{5.11}$$

Since $\varphi_1, \varphi_2 \in D(\mathcal{S}_R)$, the right-hand side of (5.11) is greater than or equal to

$$\frac{\|\psi\|_p^{1-p}}{p} \int_0^{\ell_2} \left(|(\varphi_{1|\Gamma_2} - \varphi_{2|\Gamma_2})(l)|^p - |R(\varphi_{1|\Gamma_2})(l) - R(\varphi_{2|\Gamma_2})(l)|^p \right) dl.$$

Using the fact that R is κ-contractive, we conclude that

$$[\psi, \mathcal{S}_R(\varphi_1) - \mathcal{S}_R(\varphi_2)]_s \geq \frac{(1-\kappa^p)\|\psi\|_p^{1-p}}{p} \|\varphi_{1|\Gamma_2} - \varphi_{2|\Gamma_2}\|_{\mathsf{Y}_p}^p \geq 0.$$

This proves that $\mathcal{S}_R$ is accretive on X_p.

To complete the proof, it suffices to establish that $\mathsf{R}(I + \mathcal{S}_R) = \mathsf{X}_p$ where $\mathsf{R}(I + \mathcal{S}_R)$ denotes the range of the operator $(I + \mathcal{S}_R)$ and $p \in [1, +\infty)$. To this end, for a given function $g \in \mathsf{X}_p$, we seek for a function $\varphi \in D(\mathcal{S}_R)$ such that $\varphi + \mathcal{S}_R\varphi = g$. Thus, we look for the solution to the equation

$$\varphi(a,l) + \tfrac{\partial \varphi}{\partial a}(a,l) = g(a,l). \tag{5.12}$$

Since Eq. (5.12) is linear with respect to the variable a, we obtain the following solution:

$$\varphi(a,l) = e^{-a}\chi(l) + \int_0^a g(s,l) e^{s-a}\, ds. \tag{5.13}$$

We claim that, if $\chi \in \mathsf{Y}_p$, then $\varphi \in \widetilde{\mathsf{W}}^p$.

Indeed, the use of the estimate $(|a+b|)^p \leq 2^p(|a|^p + |b|^p)$, together with Hölder inequality, yields

$$\begin{aligned}\|\varphi\|_p^p &\leq 2^p \int_{\ell_1}^{\ell_2} \int_0^l \left[|\chi(l)|^p + \left(\int_0^a e^{s-a} |g(s,l)|\, ds \right)^p \right] da\, dl \\ &\leq 2^p \int_{\ell_1}^{\ell_2} \int_0^l |\chi(l)|^p\, da\, dl + 2^p \int_{\ell_1}^{\ell_2} \int_0^l \left[\int_0^a e^{s-a} |g(s,l)|\, ds \right]^p da\, dl\end{aligned}$$

$$\le 2^p \ell_2 \|\chi\|^p_{\mathsf{Y}_p} + 2^p \int_{\ell_1}^{\ell_2} \int_0^l \left[l^{\frac{1}{q}} \left(\int_0^l |g(s,l)|^p \, ds \right)^{\frac{1}{p}} \right]^p da \, dl$$
$$\le 2^p \ell_2 \|\chi\|^p_{Y_p} + 2^p \ell_2^p \|g\|^p_p.$$

Hence

$$\|\varphi\|_p \le 2\ell_2^{\frac{1}{p}} \|\chi\|_{\mathsf{Y}_p} + 2\ell_2 \|g\|_p.$$

Moreover, the use of (5.12) gives

$$\left\| \frac{\partial \varphi}{\partial a} \right\|_p = \|g - \varphi\|_p \le \|g\|_p + \|\varphi\|_p.$$

The last two inequalities imply that $\varphi \in \widetilde{\mathsf{W}}_p$, which proves our claim.

Hence, to end the proof, it suffices to check that functions belonging to $D(\mathcal{S}_R)$ are in the form (5.13) where $\varphi_{|\Gamma_1} = \chi \in \mathsf{Y}_p$. To do so, set

$$\psi(a,l) = \int_0^a g(s,a) e^{s-a} \, ds.$$

The following condition should be satisfied:

$$\varphi_{|\Gamma_2} = (e^{-a}\chi)_{|\Gamma_2} + \psi_{|\Gamma_2}. \tag{5.14}$$

Since $\varphi \in D(\mathcal{S}_R)$, we have $\varphi_{|\Gamma_1} = R(\varphi_{|\Gamma_2})$ and, consequently,

$$\varphi_{|\Gamma_2}(l) = e^{-l} R(\varphi_{|\Gamma_2})(l) + (\psi_{|\Gamma_2})(l). \tag{5.15}$$

Now define the operator $P : \mathsf{Y}_p \longrightarrow \mathsf{Y}_p$ by

$$P(u)(l) = e^{-l} R(u)(l).$$

It is clear that

$$\left\| P(u_1) - P(u_2) \right\|_{\mathsf{Y}_p} \le e^{-\ell_1} \kappa \|u_1 - u_2\|_{\mathsf{Y}_p} \tag{5.16}$$

for any $u_1, u_2 \in \mathsf{Y}_p$. Since $e^{-\ell_1}\kappa < 1$, for $u, v \in \mathsf{Y}_p$, we have

$$\left\| (I - P)u - (I - P)v \right\|_{\mathsf{Y}_p} \ge (1 - e^{-\ell_1}\kappa) \|u - v\|_{\mathsf{Y}_p}.$$

Hence the operator $I - P$ is injective. Next, putting $y_1 = (I - P)u$ and $y_2 = (I - P)v$, we get

$$\|(I-P)^{-1}y_1-(I-P)^{-1}y_2\|_{\mathsf{Y}_p}\leq\frac{1}{1-e^{-\ell_1}\kappa}\|y_1-y_2\|_{\mathsf{Y}_p},$$

which proves the continuity of the mapping $(I-P)^{-1}$.

Now, we shall check that the operator $(I-P)$ is surjective. Indeed, letting g be a function in Y_p, we have to prove that there exists $u\in\mathsf{Y}_p$ such that $(I-P)u=g$. To this end, define the operator $\Pi:\mathsf{Y}_p\to\mathsf{Y}_p$ by

$$\Pi(u)=e^{-l}R(u)(l)+g.$$

It is easy to see that

$$\|\Pi(u_1)-\Pi(u_2)\|_{\mathsf{Y}_p}\leq e^{-\ell_1}\kappa\|u_1-u_2\|_{\mathsf{Y}_p}$$

for any $u_1,u_2\in\mathsf{Y}_p$. Hence Π is $e^{-\ell_1}\kappa$-contractive. Applying the Banach contraction principle (Theorem 6.2.1), we infer that the equation $(I-P)u=g$ has a unique solution, which proves the surjectivity of $(I-P)$.

Summarizing the previous steps, we can write

$$\varphi(a,l):=(I+\mathcal{S}_R)^{-1}(g)=e^{-a}R(I-P)^{-1}(\psi_{|\Gamma_2})(l)+\psi(a,l)\in\widetilde{W}_p.$$

This yields that $\mathsf{R}(I+\mathcal{S}_R)=\mathsf{X}_p$ and completes the proof. ☐

Remark 5.1.2.

(a) It follows from (5.16) that the result of Lemma 5.1.2 is valid even if $l_1=0$. The condition $l_1=0$ means that there are cells which are born as mothers and daughters simultaneously. This presents a biological pathology.

(b) If $\ell_1>0$, then Lemma 5.1.2 holds even if $\kappa=1$.

(c) Following the proof of Lemma 5.1.2, it is clear that if $\kappa>1$, then we cannot guarantee the accretivity of $\mathcal{S}_R$ on X_p.

In the structured population dynamics framework, we have in general a proliferation of the population, thus it is more appropriate to consider multiplying boundary conditions ($\|R(u)\|\geq\|u\|$ for all $u\in\mathsf{Y}_p$) than dissipative ones ($\|R(u)\leq\|u\|$ for all $u\in\mathsf{Y}_p$). That is the reason to study the boundary value problem (5.4) when the operator $R:\mathsf{Y}_p\to\mathsf{Y}_p$ is κ-Lipschiz with $\kappa>1$. According to Remark 5.1.2(c), if $\kappa>1$, then we cannot guarantee the accretivity of $\mathcal{S}_R$, so we introduce the following spaces.

Let ω be an arbitrary real number satisfying

$$\omega>\max\left(0,\frac{1}{\ell_1}\ln(\kappa)\right),$$

where κ is the constant appearing in the hypothesis (H1) and define the weighted space X_p^{ω} by

$$\mathsf{X}_p^\omega = L_p(\Omega, h^\omega\, da\, dl),$$

where the weight function $h^\omega(\cdot,\cdot)$ is given by

$$h^\omega(a,l) = e^{-\omega(l-a)}.$$

The space X_p^ω is endowed with the norm

$$\|f\|_{p,\omega} = \Big(\int_\Omega |f|^p h^\omega(a,l)\, da\, dl\Big)^{1/p} = \Big(\int_{\ell_1}^{\ell_2}\Big(\int_0^l |f|^p h^\omega(a,l)\, da\Big) dl\Big)^{1/p}$$

is a Banach space.

Lemma 5.1.3. *Let* $1 \le p < +\infty$. *If* $f \in \mathsf{X}_p$, *then* $f \in \mathsf{X}_p^\omega$ *and conversely. In particular, we have*

(i) $\|f\|_{p,\omega} \le \|f\|_p \le e^{\frac{\omega\ell_2}{p}} \|f\|_{p,\omega}$,

(ii) $\|f\|_1 \le e^{\omega\ell_2}(\frac{\ell_2^2-\ell_1^2}{2})^{\frac{1}{q}} \|f\|_{p,\omega}$,

where q *denotes the conjugate exponent of* p.

Proof. Let $f \in \mathsf{X}_p$. Since $l \ge a$, we have

$$\|f\|_{p,\omega}^p = \int_{\ell_1}^{\ell_2}\int_0^l e^{-\omega(l-a)}|f(a,l)|^p\, da\, dl \le \int_{\ell_1}^{\ell_2}\int_0^l |f(a,l)|^p\, da\, dl = \|f\|_p^p,$$

hence $f \in \mathsf{X}_p^\omega$.

Conversely, let $f \in \mathsf{X}_p^\omega$. We can write

$$\begin{aligned}\|f\|_p^p = \int_{\ell_1}^{\ell_2}\int_0^l e^{\omega(l-a)}e^{-\omega(l-a)}|f(a,l)|^p\, da\, dl &\le e^{\omega\ell_2}\int_{\ell_1}^{\ell_2}\int_0^l e^{-\omega(l-a)}|f(a,l)|^p\, da\, dl\\ &= e^{\omega\ell_2}\|f\|_{p,\omega}^p.\end{aligned}$$

This proves (i).

(ii) It is clear that

$$\|f\|_1 = \int_{\ell_1}^{\ell_2}\int_0^l |f(a,l)|\, da\, dl \le e^{\omega\ell_2}\Big(\int_{\ell_1}^{\ell_2}\int_0^l e^{-\omega(l-a)}|f(a,l)|\, da\, dl\Big).$$

Applying Hölder inequality, we obtain

$$\|f\|_1 \le e^{\omega\ell_2}\left(\frac{\ell_2^2-\ell_1^2}{2}\right)^{\frac{1}{q}}\left(\int_{\ell_1}^{\ell_2}\int_0^l e^{-\omega(l-a)}|f(a,l)|^p\,da\,dl\right)^{1/p} \le e^{\omega\ell_2}\left(\frac{\ell_2^2-\ell_1^2}{2}\right)^{\frac{1}{q}}\|f\|_{p,\omega},$$

which ends the proof. □

The above lemma, among other things, proves that the Banach spaces X_p and X_p^ω are equivalent.

Let W_p^ω be the space defined by

$$\mathsf{W}_p^\omega = \left\{f \in \mathsf{X}_p^\omega : \frac{\partial f}{\partial a} \in X_p^\omega\right\}$$

equipped with the norm

$$\|f\|_{\mathsf{W}_p^\omega} = \left[\|f\|_{p,\omega}^p + \left\|\frac{\partial f}{\partial a}\right\|_{p,\omega}^p\right]^{1/p}.$$

It is clear that W_p^ω is a Banach space.

We shall now prove a result similar to that of Lemma 5.1.1 for the space X_p^ω.

Lemma 5.1.4. *Let $f \in \mathsf{W}_p^\omega$. If $f_{|\Gamma_1} \in \mathsf{Y}_p$, then $f_{|\Gamma_2} \in \mathsf{Y}_p$, and vice versa.*

Proof. Let $f \in \mathsf{W}_p^\omega$ such that $f_{|\Gamma_1} \in \mathsf{Y}_p$. We have

$$f(l,l) = f(0,l) + \int_0^l \frac{\partial f}{\partial a}(a,l)\,da. \tag{5.17}$$

Now observe that

$$1 \le e^{\omega(l-a)}e^{-\frac{1}{q}\omega(l-a)}e^{-\frac{1}{p}\omega(l-a)} \le e^{\ell_2\omega}e^{-\frac{1}{p}\omega(l-a)},$$

where q is the exponent conjugate of p. Then, applying Hölder inequality, we get

$$\left|f(l,l)\right| \le \left|f(0,l)\right| + e^{l_2\omega}\ell_2^{1/q}\left(\int_0^l e^{-\omega(l-a)}\left|\frac{\partial f}{\partial a}(a,l)\right|^p da\right)^{1/p}$$

and consequently,

$$\left|f(l,l)\right|^p \le 2^p\left(\left|f(0,l)\right|^p + e^{p\ell_2\omega}\ell_2^{p/q}\int_0^l e^{-\omega(l-a)}\left|\frac{\partial f}{\partial a}(a,l)\right|^p da\right).$$

This yields

$$\int_{\ell_1}^{\ell_2} |f(l,l)|^p \, dl \leq 2^p \left(\int_{l_1}^{l_2} |f(0,l)|^p \, dl + e^{p\ell_2\omega} l_2^{p/q} \int_{\ell_1}^{\ell_2} \int_0^l e^{-\omega(l-a)} \left| \frac{\partial f}{\partial a}(a,l) \right|^p da \, dl \right)$$

$$\leq 2^p \left(\|f_{|\Gamma_1}\|_{\mathsf{Y}_p}^p + e^{p\ell_2\omega} \ell_2^{p/q} \left\| \frac{\partial f}{\partial a} \right\|_{p,\omega}^p \right) < +\infty.$$

Writing (5.17) in the form $f(0,l) = f(l,l) - \int_0^l \frac{\partial \varphi}{\partial a}(a,l)\, da$ and arguing as above, we derive the converse assertion. ☐

Next, define the space

$$\widetilde{\mathsf{W}}_p^\omega = \{f \in \mathsf{W}_p^\omega \text{ such that } f_{|\Gamma_1} \in \mathsf{Y}_p\}. \tag{5.18}$$

Clearly, by Lemma 5.1.4, we have

$$\widetilde{\mathsf{W}}_p^\omega = \{f \in \mathsf{W}_p^\omega \text{ such that } f_{|\Gamma_2} \in \mathsf{Y}_p\}.$$

Accordingly, any function $f \in \widetilde{\mathsf{W}}_p^\omega$ has traces $f_{|\Gamma_1}$ and $f_{|\Gamma_2}$ belonging to the boundary spaces.

Let $\mathcal{S}_R^\omega$ be the operator defined by

$$\begin{cases} \mathcal{S}_R^\omega : D(\mathcal{S}_R^\omega) \subset \mathsf{X}_p^\omega \longrightarrow \mathsf{X}_p^\omega, \\ f \mapsto \mathcal{S}_R^\omega(f)(a,l) = \frac{\partial f}{\partial a}(a,l), \\ D(\mathcal{S}_R^\omega) = \{f \in \widetilde{\mathsf{W}}_p^\omega : f_{|\Gamma_1} = R(f_{|\Gamma_2})\}. \end{cases}$$

We shall now discuss the ω-m-accretivity of $\mathcal{S}_R^\omega$ on the space X_p^ω where $1 \leq p < +\infty$.

Lemma 5.1.5. Assume that the hypothesis (H1) is satisfied. Then $\mathcal{S}_R^\omega$ is an ω-m-accretive operator on X_p^ω.

Proof. We first prove that, for every $\mu > 0$, $\mathsf{R}(\mu(\mathcal{S}_R^\omega + \omega I) + I) = \mathsf{X}_p^\omega$ with $p \in [1, +\infty)$. This fact is equivalent to proving that, given $\lambda \in (0, 1/\omega)$, for each $g \in \mathsf{X}_p^\omega$ the equation $\varphi + \lambda \mathcal{S}_R^\omega(\varphi) = g$ has a solution $\varphi \in D(\mathcal{S}_R^\omega)$. Arguing as in the proof of Lemma 5.1.2, we look for a function φ satisfying

$$\varphi(a,l) = e^{-\frac{a}{\lambda}} \chi(l) + \frac{1}{\lambda} \int_0^a g(s,l) e^{\frac{s-a}{\lambda}} \, ds. \tag{5.19}$$

We break the argument into two steps.

Step 1. Assume that $\chi \in \mathsf{Y}_p$. We shall show that necessarily $\varphi \in \widetilde{\mathsf{W}}_p^\omega$. As in the previous lemma, the use of the estimate $(|a+b|)^p \leq 2^p(|a|^p + |b|^p)$ and Hölder inequality leads to

$$\begin{aligned}
\|\varphi\|_{p,\omega}^{p} &= \int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}\left| e^{-\frac{a}{\lambda}}\chi(l) + \left(\frac{1}{\lambda}\int_{0}^{a} e^{\frac{s-a}{\lambda}} g(s,l)\,ds\right)\right|^{p} da\,dl \\
&\leq 2^{p}\int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}\left|e^{-\frac{a}{\lambda}}\chi(l)\right|^{p} da\,dl \\
&\quad + 2^{p}\int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}\left(\frac{1}{\lambda}\int_{0}^{a} e^{\frac{s-a}{\lambda}}\left|g(s,l)\right|ds\right)^{p} da\,dl \\
&= 2^{p}I + 2^{p}J,
\end{aligned}$$

where

$$I = \int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}\left|e^{-\frac{a}{\lambda}}\chi(l)\right|^{p} da\,dl$$

and

$$J = \int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}\left(\frac{1}{\lambda}\int_{0}^{a} e^{\frac{s-a}{\lambda}}\left|g(s,l)\right|ds\right)^{p} da\,dl.$$

It is clear that

$$I = \int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}e^{-p\frac{a}{\lambda}}\left|\chi(l)\right|^{p} da\,dl \leq \int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}\left|\chi(l)\right|^{p} da\,dl \leq \frac{1}{\omega}\|\chi\|_{Y_p}^{p}.$$

Moreover, simple calculations using the fact that $\omega(l-s) \geq \omega(a-s)$ and Hölder inequality show that

$$\begin{aligned}
J &\leq \int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}\left(\frac{1}{\lambda}\int_{0}^{l} e^{-\omega(l-s)}\left|g(s,l)\right|ds\right)^{p} da\,dl \\
&\leq \int_{\ell_1}^{\ell_2}\int_{0}^{l} e^{-\omega(l-a)}\left(\frac{1}{\lambda^{p}}\left(\int_{0}^{l} e^{-\omega(l-s)}\,ds\right)^{\frac{1}{q}}\left(\int_{0}^{l} e^{-\omega(l-s)}\left|g(s,l)\right|^{p} ds\right)^{\frac{1}{p}}\right)^{p} da\,dl \\
&\leq \int_{\ell_1}^{\ell_2} \frac{\ell_2^{\frac{p}{q}}}{\lambda^{p}}\ell_2\left(\int_{0}^{l} e^{-\omega(l-s)}\left|g(s,l)\right|^{p} ds\right) dl \\
&= \left(\frac{\ell_2}{\lambda}\right)^{p}\|g\|_{p,\omega}^{p},
\end{aligned}$$

where q denotes the conjugate exponent of p. Hence,

$$\|\varphi\|_{p,\omega} \leq \ 2\max\left(\frac{1}{\omega^{\frac{1}{p}}}, \frac{\ell_2}{\lambda}\right)\left[\|\chi\|_{\mathsf{Y}_p}^p + \|g\|_{p,\omega}^p\right]^{\frac{1}{p}}. \tag{5.20}$$

It follows from the equality $\lambda \frac{\partial\varphi}{\partial a} = g - \varphi$ that

$$\left\|\frac{\partial\varphi}{\partial a}\right\|_{p,\omega} = \frac{1}{\lambda}\|g - \varphi\|_{p,\omega} \leq \frac{1}{\lambda}\left(\|g\|_{p,\omega} + \|\varphi\|_{p,\omega}\right). \tag{5.21}$$

Accordingly, φ belongs to $\widetilde{W}_p^\omega$ whenever χ belongs to Y_p.

Step 2. Let φ be a function in $D(\mathcal{S}_R^\omega)$. Now we will prove that, if φ satisfies (5.19), then $\varphi_{|\Gamma_1} = \chi$. To this end, put

$$\psi_\lambda(a, l) = \int_0^a g(s, a)e^{\frac{s-a}{\lambda}}\, ds.$$

The following condition should be satisfied:

$$\varphi_{|\Gamma_2} = \left(e^{-\frac{a}{\lambda}}\chi\right)_{|\Gamma_2} + \frac{1}{\lambda}(\psi_\lambda)_{|\Gamma_2}. \tag{5.22}$$

Since $\varphi \in D(\mathcal{S}_R)$, we have $\varphi_{|\Gamma_1} = K(\varphi_{|\Gamma_2})$ and, consequently,

$$\varphi_{|\Gamma_2}(l) = e^{-l/\lambda}R(\varphi_{|\Gamma_2})(l) + \left(\frac{1}{\lambda}(\psi_\lambda)_{|\Gamma_2}\right)(l). \tag{5.23}$$

Let P_λ be the operator defined by

$$P_\lambda : \mathsf{Y}_p \longrightarrow \mathsf{Y}_p, \quad u \mapsto P_\lambda(u)(l) = e^{-\frac{a}{\lambda}}R(u)(l).$$

It is clear that, for any $u_1, u_2 \in \mathsf{Y}_p$, we have

$$\left\|P_\lambda(u_1) - P_\lambda(u_2)\right\|_{\mathsf{Y}_p} \leq e^{-\frac{l_1}{\lambda}}\kappa\|u_1 - u_2\|_{\mathsf{Y}_p}. \tag{5.24}$$

Since $e^{-\frac{\ell_1}{\lambda}}\kappa < 1$, we have

$$\left\|(I - P_\lambda)u - (I - P_\lambda)v\right\|_{\mathsf{Y}_p} \geq \left(1 - e^{-\frac{\ell_1}{\lambda}}\kappa\right)\|u - v\|_{\mathsf{Y}_p} \quad \forall u, v \in \mathsf{Y}_p,$$

and therefore the operator $I - P_\lambda$ is injective. Next, for $u, v \in \mathsf{Y}_p$, putting $y_1 = (I - P_\lambda)u$ and $y_2 = (I - P_\lambda)v$, we derive

$$\left\|(I - P_\lambda)^{-1}y_1 - (I - P_\lambda)^{-1}y_2\right\|_{\mathsf{Y}_p} \leq \frac{1}{1 - e^{-\frac{\ell_1}{\lambda}}\kappa}\|y_1 - y_2\|_{\mathsf{Y}_p},$$

which proves the continuity of $(I - P_\lambda)^{-1}$.

Let us now check that $(I-P_\lambda)$ is surjective. To do so, let $g \in \mathsf{Y}_p$. We seek for a function $u \in Y_p$ such that $(I-P_\lambda)u = \frac{1}{\lambda}g$. Let $\Pi_\lambda : Y_p \to \mathsf{Y}_p$ be defined by $\Pi_\lambda(u) = e^{\frac{-l}{\lambda}}R(u)(l) + \frac{1}{\lambda}g$. Let $u_1, u_2 \in Y_p$. The use of (H1) shows that Π_λ is a contraction. By the Banach contraction principle (Theorem 6.2.1), we infer that Π_λ has a unique fixed point in Y_p which is the solution of the equation $(I-P_\lambda)u = g$. Hence the operator $(I-P_\lambda)$ is surjective.

Summarizing the steps above, we get

$$\varphi(a,l) := \left(I + \lambda\mathcal{S}_R^\omega\right)^{-1}(g) = e^{-\frac{a}{\lambda}}R(I-P_\lambda)^{-1}(\psi_\lambda)_{|\Gamma_2}(l) + \frac{1}{\lambda}\psi_\lambda(a,l) \in \widetilde{\mathsf{W}}_p^\omega.$$

Accordingly, $\mathsf{R}(I+\lambda\mathcal{S}_R^\omega) = \mathsf{X}_p^\omega$ for all $\lambda \in (0, 1/\omega)$.

To complete the proof, we have to prove that the operator $\mathcal{S}_R^\omega$ is ω-accretive (via Proposition 3.11.1). We first consider the spaces X_p^ω with $p \in (1, +\infty)$. Let $\lambda \in (0, 1/\omega)$ and consider $g_1, g_2 \in \mathsf{X}_p^\omega$. Using the result of the first part of the proof, there exist two function φ_1 and φ_2 belonging to $D(\mathcal{S}_R^\omega)$ such that $\varphi_1 = (I+\lambda\mathcal{S}_R^\omega)^{-1}(g_1)$ and $\varphi_2 = (I+\lambda\mathcal{S}_R^\omega)^{-1}(g_2)$.

Putting $\psi = \varphi_1 - \varphi_2$, we can write

$$\begin{aligned}
\|\psi\|_{p,\omega} &= \left\|\left(I+\lambda\mathcal{S}_R^\omega\right)^{-1}(g_1) - \left(I+\lambda\mathcal{S}_R^\omega\right)^{-1}(g_2)\right\|_{p,\omega} \\
&\leq -\lambda\|\psi\|_{p,\omega}^{1-p}\int\limits_\Omega h^\omega(a,l)|\psi|^{p-1}\frac{\partial(\psi)}{\partial a}\,\mathrm{sgn}_0(\psi)\,da\,dl \\
&\quad + \|\psi\|_{p,\omega}^{1-p}\int\limits_\Omega h^\omega(a,l)|\psi|^{p-1}(g_1-g_2)\,\mathrm{sgn}_0(\psi)\,da\,dl \\
&\leq -\lambda\frac{\|\psi\|_{p,\omega}^{1-p}}{p}\left(\int\limits_{\ell_1}^{\ell_2}\left(\int\limits_0^l\frac{\partial(h^\omega(a,l)|\psi|^p)}{\partial a}\,da\right)dl\right) \\
&\quad + \lambda\frac{\|\psi\|_{p,\omega}^{1-p}}{p}\left(\int\limits_{\ell_1}^{\ell_2}\left(\int\limits_0^l|\psi|^p\left(\frac{\partial h^\omega(a,l)}{\partial a}\right)da\right)dl\right) \\
&\quad + \|\psi\|_{p,\omega}^{1-p}\left(\int\limits_{\ell_1}^{\ell_2}\left(\int\limits_0^l h^\omega(a,l)|\psi|^{p-1}|g_1-g_2|\,da\right)dl\right) \\
&\leq -\lambda\frac{\|\psi\|_{p,\omega}^{1-p}}{p}\int\limits_{\ell_1}^{\ell_2}\left(\left|\varphi_{1|\Gamma_2}(l) - \varphi_{2|\Gamma_2}(l)\right|^p\right. \\
&\quad \left. - e^{-\omega lp}\left|R(\varphi_{1|\Gamma_2})(l) - R(\varphi_{2|\Gamma_2})(l)\right|^p\right)dl \\
&\quad + \lambda\omega\|\psi\|_{p,\omega}^{1-p}\int\limits_{\ell_1}^{\ell_2}\left(\int\limits_0^l h^\omega(a,l)|\psi|^p\,da\right)dl \\
&\quad + \|\psi\|_{p,\omega}^{1-p}\int\limits_{\ell_1}^{\ell_2}\left(\int\limits_0^l h^\omega(a,l)|\psi|^{p-1}|g_1-g_2|\,da\right)dl.
\end{aligned}$$

$$\begin{aligned}&\leq \lambda \frac{\|\psi\|_{p,\omega}^{1-p}}{p}\left(e^{-\ell_1 p\omega}\kappa^p - 1\right)\|\varphi_{1|\Gamma_2} - \varphi_{2|\Gamma_2}\|_{\mathsf{Y}_p}^p + \lambda\omega\|\psi\|_{p,\omega}^{1-p}\|\psi\|_{p,\omega}^p \\ &\quad + \|\psi\|_{p,\omega}^{1-p}\int_{\ell_1}^{l_2}\left(\int_0^l h^\omega(a,l)|\psi|^{p-1}|g_1 - g_2|\,da\right)dl.\end{aligned}$$

Because $\omega > \max(0, \frac{1}{\ell_1}\ln(\kappa))$, we have $e^{-\ell_1\omega}\kappa - 1 < 0$, and therefore

$$\|\psi\|_{p,\omega} \leq \lambda\omega\|\psi\|_{p,\omega} + \|\psi\|_{p,\omega}^{1-p}\int_{\ell_2}^{\ell_2}\left(\int_0^l h^\omega(a,l)|\psi|^{p-1}|g_1 - g_2|\,da\right)dl.$$

Using Hölder inequality gives $\|\psi\|_{p,\omega} \leq \lambda\omega\|\psi\|_{p,\omega} + \|g_1 - g_2\|_{p,\omega}$ and then

$$\left\|\left(I + \lambda\mathcal{S}_R^\omega\right)^{-1}(g_1) - \left(I + \lambda\mathcal{S}_R^\omega\right)^{-1}(g_2)\right\|_{p,\omega} \leq \frac{1}{1 - \omega\lambda}\|g_1 - g_2\|_{p,\omega}.$$

This proves that $\mathcal{S}_R^\omega$ is ω-accretive (via Proposition 3.11.1), which ends the proof for $p \in (1, +\infty)$.

Now we consider the space X_1^ω. Let $\lambda \in (0, 1/\omega)$ and consider $g_1, g_2 \in \mathsf{X}_1^\omega$. It follows from the result of the first part of the proof that there exist $\varphi_1, \varphi_2 \in D(\mathcal{S}_R^\omega)$ such that $\varphi_1 = (I + \lambda\mathcal{S}_R^\omega)^{-1}(g_1)$ and $\varphi_2 = (I + \lambda\mathcal{S}_R^\omega)^{-1}(g_2)$. Hence

$$\begin{aligned}\|\varphi_1 - \varphi_2\| &= \left\|\left(I + \lambda\mathcal{S}_R^\omega\right)^{-1}(g_1) - \left(I + \lambda\mathcal{S}_R^\omega\right)^{-1}(g_1)\right\|_{1,\omega} \\ &\leq -\lambda\int_{\ell_1}^{\ell_2}\int_0^l \operatorname{sgn}_0(\varphi_1 - \varphi_2)h_\omega\frac{\partial(\varphi_1 - \varphi_2)}{\partial a}\,da\,dl + \int_{\ell_1}^{\ell_2} h_\omega(a,l)\big|g_1(a,l) - g_2(a,l)\big|\,da\,dl \\ &\leq -\lambda\int_{\ell_1}^{\ell_2}\int_0^l \frac{\partial}{\partial a}\big|h_\omega(\varphi_1 - \varphi_2)\big|\,da\,dl + \lambda\omega\int_{l_1}^{l_2}\int_0^l \big|h_\omega(\varphi_1 - \varphi_2)\big|\,da\,dl \\ &\quad + \int_{\ell_1}^{\ell_2}\int_0^l h_\omega(a,l)\big|g_1(a,l) - g_2(a,l)\big|\,da\,dl \\ &\leq \lambda\left(\int_{\ell_1}^{\ell_2}\left(e^{-\omega l}|R(\varphi_{2|\Gamma_2}) - R(\varphi_{1|\Gamma_2}))| - \big|(\varphi_{1|\Gamma_2} - \varphi_{2|\Gamma_2})\big|\right)dl \\ &\quad + \lambda\omega\|\varphi_1 - \varphi_2\|_{1,\omega} + \lambda\omega\|g_1 - g_2\|_{1,\omega} \\ &\leq \lambda\left(e^{-\omega\ell_1}\kappa - 1\right)\|(\varphi_{1|\Gamma_2} - \varphi_{2|\Gamma_2})\|_{\mathsf{Y}_1} + \lambda\omega\|\varphi_1 - \varphi_2\|_{1,\omega} + \|g_1 - g_2\|_{1,\omega} \\ &\leq \lambda\omega\|\varphi_1 - \varphi_2\|_{1,\omega} + \|g_1 - g_2\|_{1,\omega}.\end{aligned}$$

From this inequality we conclude that

$$\|(I+\lambda\mathcal{S}_R^{\omega})^{-1}(g_1)-(I+\lambda\mathcal{S}_R^{\omega})^{-1}(g_2)\|_{1,\omega}\le\frac{1}{1-\lambda\omega}\|g_1-g_2\|_{1,\omega},$$

which completes the proof. ☐

Lemma 5.1.6. *Let* $p\in[1,+\infty)$ *and* $\omega\ge 0$. *Then we have* $\overline{D(\mathcal{S}_R^{\omega})}^{\mathsf{X}_p^{\omega}}=\mathsf{X}_p^{\omega}$ *where* $\mathcal{S}_R^0\equiv\mathcal{S}_R$ *and* $\mathsf{X}_p^0\equiv\mathsf{X}_p$ *(i. e.,* $\omega=0$*).*

Proof. It is enough to prove that $C_0^{\infty}(\overline{\Omega})\subseteq\overline{D(\mathcal{S}_R^{\omega})}^{\mathsf{X}_p^{\omega}}$. Let φ be an element of C_0^{∞}. For each $n\in\mathbb{N}$, define the function

$$u_n(a,l)=\begin{cases}\varphi(a,l), & (a,l)\in\Omega\text{ with }a\ge\frac{\ell_1}{n},\\ R(\varphi(l,l)), & (a,l)\in\Omega\text{ with }0\le a<\frac{\ell_1}{n}.\end{cases}$$

It is clear that $|u_n(a,l)|\le\max\{|\varphi(a,l)|,|R(\varphi(l,l))|\}$ and both functions belong to $L^p(\Omega)$. It is also clear that $h_{\omega}u_n\to h_{\omega}\varphi$ a. e. in Ω. Thus, by the dominated convergence theorem, we have that $\|u_n-\varphi\|_{p,\omega}\to 0$ as n tends to $+\infty$. On the other hand,

$$\frac{\partial u_n}{\partial a}(a,l)=\begin{cases}\frac{\partial\varphi}{\partial a}(a,l), & (a,l)\in\Omega,\ a\ge\frac{\ell_1}{n},\\ 0, & (a,l)\in\Omega,\ 0\le a<\frac{\ell_1}{n}.\end{cases}$$

Since $u_n\in\mathsf{W}_p^{\omega}$, $u_n(l,l)=\varphi(l,l)\in\mathsf{Y}_p$, $u_n(0,l)=\varphi(0,l)\in\mathsf{Y}_p$, and $u_n(l,0)=Ru_n(l,l)$, we conclude that $u_n\in D(\mathcal{S}_R^{\omega})$, which completes the proof. ☐

We next introduce the following hypothesis:

(H2) The function $\sigma:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function such that there exists $\zeta\in L^{\infty}(\Omega)$ such that

$$|\sigma(a,l,z_1)-\sigma(a,l,z_2)|\le|\zeta(a,l)||z_1-z_2|,$$

where $z_1,z_2\in\mathbb{R}$, and $\sigma(\cdot,0)\in L^p(\Omega)$.

Lemma 5.1.7. *If the hypothesis* (H2) *is satisfied, then* $\mathcal{N}_{\sigma}$ *maps* X_p *into itself and there exists a constant* $C>0$ *such that*

$$\|\mathcal{N}_{\sigma}(f_1)-\mathcal{N}_{\sigma}(f_2)\|_p\le C\|f_1-f_2\|_p\quad\text{for all }f_1,f_2\in\mathsf{X}_p,$$

where $\mathcal{N}_{\sigma}$ *is the Nemytskii operator associated to the function* $\sigma(\cdot,\cdot,\cdot)$.

Proof. Since $\sigma(\cdot,\cdot,\cdot)$ is a Carathéodory function, we can define the Nemytskii operator $\mathcal{N}_{\sigma}:\mathsf{X}_p\to\mathsf{X}_p$ by

$$\mathsf{X}_p\ni f\mapsto(\mathcal{N}_{\sigma}f)(a,l)=\sigma(a,l,f(a,l)).$$

It follows from Corollary 1.15.1 that $\mathcal{N}_\sigma$ is continuous and maps bounded sets into bounded sets.

For $f_1, f_2 \in \mathsf{X}_p$, we have $\mathcal{N}_\sigma(f_1) - \mathcal{N}_\sigma(f_2) = \sigma(a, l, f_1) - \sigma(a, l, f_2)$. It follows from (H2) that

$$\begin{aligned}\|\mathcal{N}_\sigma(f_1) - \mathcal{N}_\sigma(f_2)\|_p^p &= \int_{\ell_1}^{\ell_2}\int_0^l |\mathcal{N}_\sigma(f_1)(a,l) - \mathcal{N}_\sigma(f_2)(a,l)|^p \, da\, dl \\ &= \int_{\ell_1}^{\ell_2}\int_0^l |\sigma(a,l,f_1(a,l)) - \sigma(a,l,f_2(a,l))|^p \, da\, dl \\ &\le \int_{\ell_1}^{\ell_2}\int_0^l |\zeta(a,l)|^p |f_1(a,l) - f_2(a,l)|^p \, da\, dl \\ &\le \|\zeta\|_\infty^p \|f_1 - f_2\|_p^p.\end{aligned}$$

Hence,

$$\|\mathcal{N}_\sigma(f_1) - \mathcal{N}_\sigma(f_2)\|_p \le \|\zeta\|_\infty \|f_1 - f_2\|_p,$$

where $C = \|\zeta\|_\infty$. This ends the proof. □

Remark 5.1.3. Since X_p and X_p^ω are equivalent Banach spaces, it is clear that $\mathcal{N}_\sigma$ maps X_p^ω into itself and there exists a constant C_1 such that $\mathcal{N}_\sigma$ is C_1-Lipschitz on X_p^ω.

5.1.1.1 Existence and uniqueness results

In this section, we will discuss existence and uniqueness results for problem (5.4) when the initial data $f_0 \in \mathsf{X}_p$. We may consider the Banach space $\mathsf{X}_p = L^p(\Omega)$ and $u(t) := f(t,\cdot,\cdot) \in \mathsf{X}_p$ and define the operator $\mathcal{N}_\sigma : \mathsf{X}_p \to \mathsf{X}_p$ by $\mathcal{N}_\sigma(\varphi)(a,l) := \sigma(a,l,\varphi(a,l))$.

Theorem 5.1.1. *Assume that the hypotheses* (H1) *and* (H2) *are satisfied and the transition operator R is κ-contractive ($\kappa \in [0,1)$) and $\ell_1 \ge 0$. Then the problem*

$$\begin{cases} u'(t) + \mathcal{S}_R(u(t)) = \mathcal{N}_\sigma(u(t)), \\ u(0) = f_0 \in \mathsf{X}_p \end{cases} \tag{5.25}$$

has a unique mild solution. Moreover, if $p \in (1,+\infty)$ and $f_0 \in D(\mathcal{S}_R)$, then this mild solution is, in fact, a strong solution.

Proof. According to Lemma 5.1.2, the operator $\mathcal{S}_R$ is m-accretive on X_p. By Lemma 5.1.7, $\mathcal{N}_\sigma$ is $\|\zeta\|_\infty$-Lipschitz on X_p. According to Lemma 5.1.6, we have $f_0 \in \overline{D(\mathcal{S}_R)}$. By Corollary 4.5.2, we conclude that problem (5.25) has a unique integral solution and, by Theorem 4.6.3(ii), it is the unique mild solution. On the other hand, for $p \in (1,+\infty)$, X_p is a Banach space with Radon–Nikodym property, so, if $f_0 \in D(\mathcal{S}_R)$, then the use of Theorem 4.6.5 implies that this solution is a strong solution. ☐

We note that Theorem 5.1.1 remains valid whenever R is a nonexpansive mapping and $l_1 > 0$.

Theorem 5.1.2. *If the hypotheses* (H1) *and* (H2) *are satisfied and* $\ell_1 > 0$, *then the problem*

$$\begin{cases} u'(t) + \mathcal{S}_R^{\omega}(u(t)) = \mathcal{N}_{\sigma}(u(t)), \\ u(0) = f_0 \in \mathsf{X}_p \end{cases} \tag{5.26}$$

has a unique mild solution. Moreover, if $p > 1$ *and* $f_0 \in D(\mathcal{S}_R^{\omega})$, *then this mild solution is, in fact, a strong solution.*

Proof. If R is κ-contractive ($\kappa \in [0,1)$), then the result follows from Theorem 5.1.1. Thus we may assume that $\kappa > 1$. In this case, we fix $\omega > \frac{1}{\ell_1}\ln(\kappa)$ and we shall consider the problem on the Banach space X_p^{ω}. By Lemma 5.1.5, the operator $\mathcal{S}_R^{\omega}$ is ω-m-accretive in X_p^{ω}. It follows from Remark 5.1.3 that $\mathcal{N}_{\sigma} : \mathsf{X}_p^{\omega} \to \mathsf{X}_p^{\omega}$ is Lipschitz on X_p^{ω}. Hence, according to Theorem 4.6.6, the operator $\mathcal{S}_R^{\omega} - \mathcal{N}_{\sigma}$ is m-quasiaccretive on X_p^{ω}. The first part of the theorem follows from Theorem 4.6.3, definition of mild solution and the fact that X_p and X_p^{ω} are equivalent Banach spaces. The second part is a consequence of Theorem 4.6.5, the definition of a strong solution, and the fact that X_p and X_p^{ω} are equivalent Banach spaces. ☐

5.1.2 Lebowitz–Rubinow model for infinite maximum cell cycle length

This section deals with the existence and uniqueness results for the Lebowitz–Rubinow model where the cycle length l is allowed to be arbitrarily large (from a biological point of view, this could mean that there are cells that never reach the division). This case is treated separately because it presents supplementary difficulties and works only on L^1-spaces. Note that an examination of the proofs of Lemmas 5.1.2 and 5.1.5 shows that Theorems 5.1.1 and 5.1.2 are not valid for an infinite maximum cycle length ($\ell_2 = +\infty$).

Let Ω be the subset of $\mathbb{R}^2$ defined by

$$\Omega := \{(a,l) \text{ such that } 0 \le a \le l,\ \ell_1 \le l < +\infty\}$$

and consider the Banach space $L^1(\Omega) = L^1(\Omega, da\,dl)$ equipped with its natural norm. For $i = 1,2$, let $L^1(\Gamma_i, dl)$ denote the boundary spaces, where $\Gamma_1 := \{(0,l) : l > \ell_1\}$ and $\Gamma_2 := \{(l,l) : l > \ell_1\}$. As in Section 5.1, we identify $L^1(\Gamma_i, dl)$, $i = 1,2$, with the space

$$\mathsf{Y} := L^1([\ell_1, +\infty), dl).$$

Let $\omega > 0$ be an arbitrary real number and define the weighted space X^ω by

$$\mathsf{X}^\omega := L^1(\Omega, h^\omega \, da \, dl),$$

where the weight function $h^\omega(\cdot,\cdot)$ is given by

$$h^\omega(a,l) = e^{-\omega(l-a)}. \tag{5.27}$$

The space X^ω is equipped with the norm

$$\|\varphi\|_{\mathsf{X}^\omega} = \int_\Omega |\varphi h^\omega|(a,l)\, da\, dl = \int_{\ell_1}^{+\infty} \left(\int_0^l |\varphi(a,l)| h^\omega(a,l)\, da \right) dl.$$

Proposition 5.1.1. *The space X^ω is complete.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X^ω. Then, $(f_n h^\omega)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(\Omega)$. By the completeness of $L^1(\Omega)$, there exists $g \in L^1(\Omega)$ such that $\|f_n h^\omega - g\|_1 \to 0$. It is easy to check that $\|f_n - gh^{-\omega}\|_{\mathsf{X}^\omega} = \|f_n h^\omega - g\|_1$ for all $n \in \mathbb{N}$. Thus, we have $\|f_n - gh^{-\omega}\|_{\mathsf{X}^\omega} \to 0$. Moreover, $gh^{-\omega} \in \mathsf{X}^\omega$. Therefore $(f_n)_{n\in\mathbb{N}}$ is convergent in X^ω. ☐

Lemma 5.1.8. *Let $\mu > 0$ and let ω be a fixed real number of $(0,\mu)$. If $f : \Omega \to \mathbb{R}^+$ is a continuous function in the first variable, then*

$$\int_0^l \left(\int_0^a f(s,l) e^{-\mu(a-s)}\, ds \right) e^{-\omega(l-a)}\, da \le \frac{1}{\mu-\omega} \int_0^l f(a,l) e^{-\omega(l-a)}\, da \tag{5.28}$$

for all $l > \ell_1$.

Proof. Note that

$$\int_0^l \left(\int_0^a f(s,l) e^{-\mu(a-s)}\, ds \right) e^{-\omega(l-a)}\, da = e^{-\omega l} \int_0^l \left(\int_0^a f(s,l) e^{\mu s}\, ds \right) e^{-(\mu-\omega)a}\, da. \tag{5.29}$$

Using integration by parts (and the first fundamental theorem of calculus) with

$$u(a) = \int_0^a f(s,l) e^{\mu s}\, ds \quad \text{and} \quad v(a) = \frac{-1}{\mu-\omega} e^{-(\mu-\omega)a},$$

we see that (5.29) is equal to

$$e^{-\omega l}\left(\left[\frac{-1}{\mu-\omega}e^{-(\mu-\omega)a}\int_0^a f(s,l)e^{\mu s}\,ds\right]_{a=0}^{l}+\frac{1}{\mu-\omega}\int_0^l f(a,l)e^{\omega a}\,da\right)$$

$$=\frac{1}{\mu-\omega}\int_0^l f(a,l)(e^{-\omega(l-a)}-e^{\mu(a-l)})\,da$$

$$\leq\frac{1}{\mu-\omega}\int_0^l f(a,l)e^{-\omega(l-a)}\,da,$$

which ends the proof. ☐

Proposition 5.1.2. *Let μ and ω be as in Lemma* 5.1.8. *If $f:\Omega\to\mathbb{R}^+$ belongs to X^ω, then f satisfies* (5.28).

Proof. Just apply Lemma 5.1.8 and the well-known fact that continuous functions (with compact support) are dense in $L^1(\Omega)$. ☐

Let us notice that $L^1(\Omega)$ is a proper subspace of X^ω. Indeed, consider, for instance, the function $g(a,l)=l^{-2}$. It is clear that g does not belong to $L^1(\Omega)$ but $g\in\mathsf{X}^\omega$.

For a given real $\mathbf{r}>0$, we denote by $\mathbb{B}_{\mathbf{r}}$ the closed ball

$$\mathbb{B}_{\mathbf{r}}:=\{f\in\mathsf{X}^\omega \text{ such that } \|f\|_{\mathsf{X}^\omega}\leq\mathbf{r}\}.$$

Now we introduce the weighted partial Sobolev space

$$\mathsf{W}_\omega:=\left\{\varphi\in\mathsf{X}^\omega \text{ such that } \frac{\partial\varphi}{\partial a}\in\mathsf{X}^\omega\right\}$$

endowed with the norm

$$\|\varphi\|_{\mathsf{W}_\omega}:=\|\varphi\|_{\mathsf{X}^\omega}+\left\|\frac{\partial\varphi}{\partial a}\right\|_{\mathsf{X}^\omega}.$$

We note that, if $\psi\in\mathsf{W}_\omega$ and $\psi_{|\Gamma_1}\in\mathsf{Y}$, then $\psi_{|\Gamma_2}\in\mathsf{Y}$. To see this, we take $f=\psi\cdot h^\omega$. It is clear that f and $\frac{\partial f}{\partial a}$ belong to $L^1(\Omega,\,da\,dl)$ because $\psi\in\mathsf{W}_\omega$ and $\frac{\partial f}{\partial a}=\frac{\partial\psi}{\partial a}h^\omega+\omega\psi h^\omega$. Accordingly, we have

$$f(l,l)=f(0,l)+\int_0^l\frac{\partial f}{\partial a}(a,l)\,da,$$

and therefore

$$\psi(l,l)=\psi(0,l)e^{-\omega l}+\int_0^l\frac{\partial\psi}{\partial a}(a,l)h^\omega(a,l)\,da+\omega\int_0^l\psi(a,l)h^\omega(a,l)\,da.$$

Hence,

$$\|\psi_{|\Gamma_2}\|_{\mathsf{Y}} \le \int_{\ell_1}^{\infty} \psi(0,l)e^{-\omega l}\,dl + \left\|\frac{\partial\psi}{\partial a}\right\|_{\mathsf{X}^\omega} + \omega\|\psi\|_{\mathsf{X}^\omega} \le \|\psi_{|\Gamma_1}\|_{\mathsf{Y}} + \left\|\frac{\partial\psi}{\partial a}\right\|_{\mathsf{X}^\omega} + \omega\|\psi\|_{\mathsf{X}^\omega}.$$

For the converse, we shall give a counterexample. Consider the function $\psi(a,l) = e^{-\omega a}$. In this case we have

$$\|\psi\|_{\mathsf{X}^\omega} = \frac{e^{-\omega\ell_1}}{\omega}\left(\ell_1 + \frac{1}{\omega}\right), \quad \left\|\frac{\partial\psi}{\partial a}\right\|_{\mathsf{X}^\omega} = e^{-\omega\ell_1}\left(\ell_1 + \frac{1}{\omega}\right), \quad \text{and} \quad \|\psi_{|\Gamma_2}\|_{\mathsf{Y}} = \frac{e^{-\omega\ell_1}}{\omega},$$

but $\psi_{|\Gamma_1} = 1 \notin \mathsf{Y}$.

So, in this section we shall consider the following subspace of W_ω:

$$\mathsf{W}^*_\omega := \{\psi \in W_w \quad \text{such that } \psi_{|\Gamma_1} \in \mathsf{Y}\}.$$

Remark 5.1.4. Note that the space W^*_ω is dense in $L^1(\Omega)$ and any function ψ belonging to W^*_ω possesses traces $\psi_{|\Gamma_1}$ and $\psi_{|\Gamma_2}$ belonging to Y.

5.1.2.1 Local boundary conditions

We are concerned here with the existence and uniqueness of solutions to the following initial boundary value problem:

$$\begin{cases} \frac{\partial f}{\partial t}(t,a,l) + \frac{\partial f}{\partial a}(t,a,l) = -\sigma(a,l,f(t,a,l)), \\ f(0,a,l) = f_0(a,l), \\ f(t,0,l) = [Rf(t,\cdot,\cdot)](l). \end{cases} \tag{5.30}$$

In the sequel, the following hypotheses are required:

(H3) There exists $\alpha > 0$ such that $\|R(u) - R(v)\|_{\mathsf{Y}} \le \alpha\|u - v\|_{\mathsf{Y}}$, for all $u, v \in \mathsf{Y}$.

(H4) $\sigma : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $\beta_\sigma \in L^\infty(\Omega, da\,dl)$ such that

$$\left|\sigma(a,l,x) - \sigma(a,l,y)\right| \le \left|\beta_\sigma(a,l)\right||x-y| \quad \text{for all } (a,l) \in \Omega \text{ and } x,y \in \mathbb{R}.$$

(H5) The function $\sigma_0 : \Omega \to \mathbb{R}$, given by $\sigma_0(a,l) := \sigma(a,l,0)$, belongs to X^ω.

(H6) The constant ω satisfies $\omega \ge \max\{0, \ln(\alpha)/\ell_1\}$, where α is the constant appearing in the hypothesis (H3) and $\ell_1 > 0$.

We define the nonlinear operator $\mathcal{F} : \mathsf{X}^\omega \to \mathsf{X}^\omega$ by

$$\mathcal{F}(\psi)(a,l) := -\sigma\big(a,l,\psi(a,l)\big). \tag{5.31}$$

The operator $\mathcal{F}$ is well defined. Indeed, we have

$$\begin{aligned}\big|\mathcal{F}(\psi)(a,l)\big| &\le \big|\sigma(a,l,\psi(a,l))-\sigma(a,l,0)\big|+\big|\sigma_0(a,l)\big|\\ &\le \big|\beta_\sigma(a,l)\big|\big|\psi(a,l)\big|+\big|\sigma_0(a,l)\big|,\end{aligned}$$

and therefore,

$$\big\|\mathcal{F}(\psi)\big\|_{\mathsf{X}^\omega}\le \|\beta_\sigma\|_\infty\|\psi\|_{\mathsf{X}^\omega}+\|\sigma_0\|_{\mathsf{X}^\omega}.$$

Let $\mathcal{S}_R$ be the operator defined by

$$\begin{cases}\mathcal{S}_R: D(\mathcal{S}_R)\subset \mathsf{X}^\omega\to\mathsf{X}^\omega,\\ \psi\mapsto(\mathcal{S}_R\psi)(a,l):=\frac{\partial\psi}{\partial a}(a,l),\\ D(\mathcal{S}_R):=\{\psi\in\mathsf{W}^*_\omega \text{ such that } \psi_{|\Gamma_1}=R(\psi_{|\Gamma_2})\}.\end{cases}$$

Taking $u(t):=\psi(t,\cdot,\cdot)\in\mathsf{X}^\omega$ and assuming that $f_0\in\mathsf{X}^\omega$, we can rewrite problem (5.30) as follows:

$$\begin{cases}u'(t)+\mathcal{S}_R(u(t))=\mathcal{F}(u(t)),\\ u(0)=f_0\in\mathsf{X}^\omega.\end{cases}\tag{5.32}$$

To establish the well-posedness of problem (5.32), we shall first prove some preparatory results. Before going further, we first introduce the following linear operators:

$$\begin{cases}A_\lambda:\mathsf{Y}\longrightarrow\mathsf{Y},\\ u\mapsto(A_\lambda u)(l):=u(l)e^{-\frac{l}{\lambda}},\end{cases}\qquad \begin{cases}B_\lambda:\mathsf{Y}\longrightarrow\mathsf{X}^\omega,\\ u\mapsto(B_\lambda u)(a,l):=u(l)e^{-\frac{a}{\lambda}},\end{cases}$$

$$\begin{cases}C_\lambda:\mathsf{X}^\omega\longrightarrow\mathsf{Y},\\ g\mapsto(C_\lambda g)(l):=\frac{1}{\lambda}\int_0^l e^{\frac{s-l}{\lambda}}g(s,l)\,ds,\end{cases}\qquad \begin{cases}D_\lambda:\mathsf{X}^\omega\longrightarrow\mathsf{X}^\omega,\\ g\mapsto(D_\lambda g)(a,l):=\frac{1}{\lambda}\int_0^a e^{\frac{s-a}{\lambda}}g(s,l)\,ds,\end{cases}$$

where λ is a real number, $\lambda\in(0,\frac{1}{\omega})$.

It is clear that the operators A_λ and B_λ are bounded and their norms satisfy the estimates

$$\|A_\lambda\|_{\mathcal{L}(\mathsf{X})}\le e^{-\frac{\ell_1}{\lambda}},\tag{5.33}$$

$$\|B_\lambda\|_{\mathcal{L}(\mathsf{X},\mathsf{X}^\omega)}\le\lambda.\tag{5.34}$$

Next, bearing in mind that $\lambda\in(0,\frac{1}{\omega})$, for any $g\in\mathsf{X}^\omega$, we can write

$$\begin{aligned}\|C_\lambda g\|_{\mathsf{Y}}&\le\int_{\ell_1}^{+\infty}\frac{1}{\lambda}\int_0^l e^{\frac{s-l}{\lambda}}\big|g(s,l)\big|\,ds\,dl=\frac{1}{\lambda}\int_{\ell_1}^{+\infty}\int_0^l e^{\frac{s-l}{\lambda}}e^{\omega(l-s)}e^{-\omega(l-s)}\big|g(s,l)\big|\,ds\,dl\\ &=\frac{1}{\lambda}\int_{\ell_1}^{+\infty}e^{l(\omega-\frac{1}{\lambda})}\int_0^l e^{s(\frac{1}{\lambda}-\omega)}e^{-\omega(l-s)}\big|g(s,l)\big|\,ds\,dl\end{aligned}$$

$$\leq \frac{1}{\lambda} \int_{\ell_1}^{+\infty} e^{l(\omega - \frac{1}{\lambda})} \int_0^l e^{l(\frac{1}{\lambda} - \omega)} e^{-\omega(l-s)} |g(s,l)|\, ds\, dl = \frac{1}{\lambda} \|g\|_{\mathsf{X}^\omega},$$

and therefore

$$\|C_\lambda\|_{\mathcal{L}(\mathsf{X}^\omega, \mathsf{X})} \leq \frac{1}{\lambda}. \tag{5.35}$$

In the same way we can write

$$\begin{aligned} \|D_\lambda g\|_{\mathsf{X}^\omega} &= \int_{\ell_1}^{+\infty} \int_0^l \left| \frac{1}{\lambda} \int_0^a e^{+\frac{s-a}{\lambda}} g(s,l)\, ds \right| e^{-\omega(l-a)}\, da\, dl \\ &\leq \frac{1}{\lambda} \int_{\ell_1}^{+\infty} \int_0^l \left(\int_0^a e^{\frac{s-l}{\lambda}} |g(s,l)|\, ds \right) e^{-\omega(l-a)}\, da\, dl \end{aligned}$$

But, according to Lemma 5.1.8, for each $\mu > \omega$, we have

$$\int_0^l \left(\int_0^a e^{\frac{s-l}{\lambda}} |g(s,l)|\, ds \right) e^{-\omega(l-a)}\, da \leq \frac{1}{\mu - \omega} \int_0^l g(a,l) e^{-\omega(l-a)}\, da.$$

Because $\frac{1}{\lambda} > \omega$, the above inequality yields

$$\|D_\lambda g\|_{\mathsf{X}^\omega} \leq \frac{1}{\lambda} \int_{\ell_1}^{+\infty} \frac{1}{\frac{1}{\lambda} - \omega} \int_0^l |g(a,l)| e^{-\omega(l-a)}\, da\, dl \leq \frac{1}{1 - \lambda\omega} \|g\|_{\mathsf{X}^\omega},$$

and consequently,

$$\|D_\lambda\|_{\mathcal{L}(\mathsf{X}^\omega)} \leq \frac{1}{1 - \lambda\omega}. \tag{5.36}$$

Lemma 5.1.9. *If the hypothesis* (H6) *is satisfied, then the operator* $\mathcal{S}_R$ *is m-quasiaccretive on* X^ω.

Proof. Let $\lambda \in (0, \frac{1}{\omega})$ (for convenience, if $\omega = 0$, $\frac{1}{\omega}$ means $+\infty$).

First we shall see that $\mathsf{R}(I + \lambda\mathcal{S}_R) = \mathsf{X}^\omega$. In order to do this, given $g \in \mathsf{X}^\omega$, we have to find a function $\psi \in D(\mathcal{S}_R)$ such that $\psi + \lambda\mathcal{S}_R\psi = g$. Thus, we seek the solution of the following differential equation:

$$\psi(a,l) + \lambda \frac{\partial \psi}{\partial a}(a,l) = g(a,l). \tag{5.37}$$

The above equation is linear with respect to the variable a, so the general solution is formally given by

$$\psi(a,l) = \delta(l)e^{-a/\lambda} + \frac{1}{\lambda}\int_0^a g(s,l)e^{-(a-s)/\lambda}\,ds,$$

where $\delta : [\ell_1, +\infty) \to \mathbb{R}$ is an unknown function. By using the linear operators introduced above, the latter equation may be written as

$$\psi(a,l) = (B_\lambda\delta)(a,l) + (D_\lambda g)(a,l).$$

Hence, for $a = l$, we have

$$\psi(l,l) = (A_\lambda\delta)(l) + (C_\lambda g)(l).$$

Let $H_\lambda : \mathsf{Y} \to \mathsf{Y}$ be the operator defined by

$$H_\lambda\varphi(l) := \big(A_\lambda(R(\varphi))\big)(l) + (C_\lambda g)(l).$$

Note that H_λ is well defined because R maps Y into itself. Furthermore, for all $\varphi_1, \varphi_2 \in \mathsf{Y}$, we have

$$\big\|H_\lambda(\varphi_1) - H_\lambda(\varphi_2)\big\|_{\mathsf{Y}} \le e^{-\ell_1/\lambda}\big\|R(\varphi_1) - R(\varphi_2)\big\|_{\mathsf{Y}} \le \alpha e^{-\ell_1/\lambda}\|\varphi_1 - \varphi_2\|_{\mathsf{Y}}.$$

The operator H_λ is then $\alpha e^{-\ell_1/\lambda}$-contractive (the conditions $\omega \ge \ln(\alpha)/\ell_1$ and $\lambda \in (0, \frac{1}{\omega})$ imply $\alpha e^{-\ell_1/\lambda} < 1$). Hence, by the Banach contraction principle (Theorem 6.2.1), there exists a unique function u in Y such that

$$u(l) = \big(A_\lambda(R(u))\big)(l) + (C_\lambda g)(l).$$

Now, define the function $\psi : \Omega \to \mathbb{R}$ by

$$\psi(a,l) := \big(B_\lambda(Ru)\big)(a,l) + (D_\lambda g)(a,l).$$

It is clear that ψ satisfies equation (5.37) and that $\psi \in \mathsf{X}^\omega$. Furthermore, $\frac{\partial\psi}{\partial a} \in \mathsf{X}^\omega$ because ψ satisfies (5.37), that is, $\frac{\partial\psi}{\partial a} = \frac{1}{\lambda}(g - \psi)$, and $g, \psi \in \mathsf{X}^\omega$. Therefore, $\psi \in \mathsf{W}_\omega$. Bearing in mind the definition of ψ, it is clear that $\psi_{|\Gamma_2} \in \mathsf{Y}$ and $R(\psi_{|\Gamma_2}) = \psi_{|\Gamma_1}$. Accordingly, $\psi \in D(\mathcal{S}_R)$ and therefore, $\mathsf{R}(I + \lambda\mathcal{S}_R) = \mathsf{X}^\omega$.

Finally, we are going to apply Proposition 3.11.1 to see that $\mathcal{S}_R$ is ω-accretive. We shall prove that the resolvent operator $J_\lambda := (I + \lambda\mathcal{S}_R)^{-1} : \mathsf{X}^\omega \to D(\mathcal{S}_R)$ is a single-valued $\frac{1}{1-\omega\lambda}$-Lipschitz mapping. We have to check that, for all $\varphi_1, \varphi_2 \in \mathsf{X}^\omega$,

$$\|J_\lambda\varphi_1 - J_\lambda\varphi_2\|_{\mathsf{X}^\omega} \le \frac{1}{1-\lambda\omega}\|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega}.$$

Let $\varphi_1, \varphi_2 \in \mathsf{X}^\omega$. Since $\mathsf{X}^\omega = \mathsf{R}(I + \lambda\mathcal{S}_R)$, there exist $\psi_i \in D(\mathcal{S}_R)$, $i = 1, 2$, such that $(I + \lambda\mathcal{S}_R)^{-1}\varphi_i = \psi_i$, that is, $\varphi_i = \psi_i + \lambda\mathcal{S}_R\psi_i$. Thus, we have

$$\begin{aligned}
\|\psi_1-\psi_2\|_{X^\omega} &= \left\|\varphi_1-\varphi_2-\lambda\left(\frac{\partial\psi_1}{\partial a}-\frac{\partial\psi_2}{\partial a}\right)\right\|_{X^\omega}\\
&= -\lambda\int_{\ell_1}^{+\infty}\int_0^l\left(\frac{\partial\psi_1}{\partial a}-\frac{\partial\psi_2}{\partial a}\right)\operatorname{sgn}_0(\psi_1-\psi_2)h_\omega\,da\,dl\\
&\quad+\int_{\ell_1}^{+\infty}\int_0^l(\varphi_1-\varphi_2)\operatorname{sgn}_0(\psi_1-\psi_2)h_\omega\,da\,dl\\
&= -\lambda\int_{\ell_1}^{+\infty}\int_0^l\frac{\partial}{\partial a}(|\psi_1-\psi_2|h_\omega)\,da\,dl+\omega\lambda\int_{\ell_1}^{+\infty}\int_0^l|\psi_1-\psi_2|h_\omega\,da\,dl\\
&\quad+\int_{\ell_1}^{+\infty}\int_0^l(\varphi_1-\varphi_2)\operatorname{sgn}_0(\psi_1-\psi_2)h_\omega\,da\,dl\\
&\le -\lambda\int_{\ell_1}^{+\infty}\left(\left|\psi_1(l,l)-\psi_2(l,l)\right|-e^{-\omega l}\left|\psi_1(0,l)-\psi_2(0,l)\right|\right)dl\\
&\quad+\omega\lambda\|\psi_1-\psi_2\|_{X^\omega}+\|\varphi_1-\varphi_2\|_{X^\omega}\\
&=\lambda\int_{\ell_1}^{+\infty}\left(e^{-wl}\left|K(\psi_1(l,l))-K(\psi_2(0,l))\right|-\left|\psi_1(l,l)-\psi_2(l,l)\right|\right)dl\\
&\quad+\omega\lambda\|\psi_1-\psi_2\|_{X^\omega}+\|\varphi_1-\varphi_2\|_{X^\omega}\\
&\le\lambda(ae^{-\omega\ell_1}-1)\int_{\ell_1}^{+\infty}\left|\psi_1(l,l)-\psi_2(l,l)\right|dl\\
&\quad+\omega\lambda\|\psi_1-\psi_2\|_{X^\omega}+\|\varphi_1-\varphi_2\|_{X^\omega}\\
&=\lambda(ae^{-\omega\ell_1}-1)\|\psi_{1|\Gamma_2}-\psi_{2|\Gamma_2}\|_{Y}+\omega\lambda\|\psi_1-\psi_2\|_{X^\omega}+\|\varphi_1-\varphi_2\|_{X^\omega}.
\end{aligned}$$

Then,

$$(1-\omega\lambda)\|\psi_1-\psi_2\|_{X^\omega}\le\lambda(ae^{-\omega\ell_1}-1)\|\psi_{1|\Gamma_2}-\psi_{2|\Gamma_2}\|_{Y}+\|\varphi_1-\varphi_2\|_{X^\omega}.$$

Since $ae^{-\omega\ell_1}-1\le 0$, we obtain $(1-w\lambda)\|\psi_1-\psi_2\|_{X^\omega}\le\|\varphi_1-\varphi_2\|_{X^\omega}$. □

Remark 5.1.5. If the transition operator R is nonexpansive ($\kappa\le 1$) and $\omega=0$, then $\mathcal{S}_R$ is m-accretive on $L^1(\Omega, da\,dl)$.

Lemma 5.1.10. $\overline{D(\mathcal{S}_R)}^{X^\omega}=X^\omega$.

Proof. Let $\mathcal{C}_0^\infty(\overline{\Omega})$ be the set of all $\mathcal{C}^\infty$-functions with compact support. We first show that $\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in X^ω. Let $f\in X^\omega$. Notice that $fh^\omega\in L^1(\Omega)$. Bearing in mind that $\mathcal{C}_0^\infty(\overline{\Omega})$ is dense in $L^1(\Omega)$, there exists $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_0^\infty(\overline{\Omega})$ such that $\|u_n - fh^\omega\|_1 \to 0$ as $n \to +\infty$. For each $n \in \mathbb{N}$, $u_n h^{-\omega} \in \mathcal{C}_0^\infty(\overline{\Omega})$ and

$$\begin{aligned}\|f - u_n h^{-\omega}\|_{\mathrm{X}^\omega} &= \int_{\ell_1}^{+\infty}\int_0^l \left|f(a,l) - u_n(a,l)e^{\omega(l-a)}\right|e^{-\omega(l-a)}\,da\,dl\\ &= \int_{\ell_1}^{+\infty}\int_0^l \left|f(a,l)e^{-\omega(l-a)} - u_n(a,l)\right|\,da\,dl\\ &= \left\|fh^\omega - u_n\right\|_1.\end{aligned}$$

Then f can be approximated by $\mathcal{C}^\infty(\overline{\Omega})$-functions, which proves our claim.

Secondly, let us see that $\mathcal{C}^\infty(\overline{\Omega}) \subseteq \overline{D(\mathcal{S}_R)}^{\mathrm{X}^\omega}$. Let $v \in \mathcal{C}^\infty(\overline{\Omega})$. For each $n \in \mathbb{N}$, we define the function

$$u_n(a,l) = \begin{cases} v(a,l) & \text{if } (a,l) \in \Omega, \text{ with } a \geq \frac{\ell_1}{n},\\ R(v(l,l)) & \text{if } (a,l) \in \Omega, \text{ with } 0 \leq a < \frac{\ell_1}{n}.\end{cases}$$

Note that $u_n, \frac{\partial u_n}{\partial a} \in \mathrm{X}^\omega$ because

$$\frac{\partial u_n}{\partial a}(a,l) = \begin{cases} \frac{\partial v}{\partial a}(a,l) & \text{if } (a,l) \in \Omega, \text{ with } a \geq \frac{\ell_1}{n},\\ 0 & \text{if } (a,l) \in \Omega, \text{ with } 0 \leq a < \frac{\ell_1}{n}.\end{cases}$$

Therefore $u_n \in \mathrm{W}_\omega$. Moreover, since $R : \mathrm{Y} \to \mathrm{Y}$, we have $u_n \in \mathrm{W}_\omega^*$. Using the fact that $u_{n|\Gamma_1} = R(v(l,l)) = R(u_{n|\Gamma_2})$, we obtain that $u_n \in D(\mathcal{S}_R)$.

We now define a map $f : \Omega \to \mathbb{R}$ by $f(a,l) = \max\{v(a,l), R(v(l,l))\}$. Thus, $|u_n h_w| \leq f$ a. e. in Ω and $f \in \mathrm{X}^\omega$, because $v \in \mathrm{X}^\omega$ and

$$\int_{\ell_1}^{+\infty}\int_0^l R(v(l,l))h_\omega(a,l)\,da\,dl = \int_{\ell_1}^{+\infty} R(v(l,l))\frac{1}{\omega}(1 - e^{-\omega l})\,dl \leq \frac{1}{\omega}\int_{\ell_1}^{+\infty} R(v(l,l))\,dl < \infty.$$

Since $u_n h_\omega \to v h_\omega$ a. e. in Ω, by the Lebesgue dominated convergence theorem, we get

$$\|u_n - v\|_{\mathrm{X}^\omega} \to 0 \quad \text{as} \quad n \to +\infty,$$

which concludes the proof. ☐

Lemma 5.1.11. *If the hypotheses* (H3)–(H6) *are satisfied, then there exists a constant* $L_{\mathcal{F}} > 0$ *such that, for all* $\psi_1, \psi_2 \in \mathrm{X}^\omega$, *we have*

$$\left\|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)\right\|_{\mathrm{X}^\omega} \leq L_{\mathcal{F}}\|\psi_1 - \psi_2\|_{\mathrm{X}^\omega},$$

where $\mathcal{F}$ *is defined in* (5.31) *and* $L_{\mathcal{F}} = \|\beta_\sigma\|_\infty$.

Proof. Let $\psi_1, \psi_2 \in X^\omega$. For each $(a,l) \in \Omega$, we have

$$\begin{aligned}|\mathcal{F}(\psi_1)(a,l) - \mathcal{F}(\psi_2)(a,l)| &\le |\sigma(a,l,\psi_1(a,l)) - \sigma(a,l,\psi_2(a,l))|\\ &\le |\beta_\sigma(a,l)||\psi_1(a,l) - \psi_2(a,l)|,\end{aligned}$$

which implies that $\|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)\|_{X^\omega} \le \|\beta_\sigma\|_\infty \|\psi_1 - \psi_2\|_{X^\omega}$. ☐

Now we are ready to prove

Theorem 5.1.3. *If the hypotheses* (H3)–(H6) *are satisfied, then problem* (5.32) *has a unique mild solution in* X^ω *for each initial data* $f_0 \in X^\omega$.

Proof. By Lemma 5.1.9, we know that the operator $\mathcal{S}_R$ is m-quasiaccretive on X^ω. Because $\mathcal{F}$ is globally Lipschitz (cf. Lemma 5.1.11), the use of Theorem 4.6.6 shows that $\mathcal{S}_R - \mathcal{F}$ is also m-quasiaccretive on X^ω. Moreover, by Lemma 5.1.10, the domain of $\mathcal{S}_R$ is dense in X^ω, hence the domain of $\mathcal{S}_R - \mathcal{F}$ is also dense in X^ω. Now, the use of Theorem 4.6.3 allows us to conclude that, for each initial data $f_0 \in X^\omega$, problem (5.32) admits a unique mild solution. ☐

5.1.2.2 Nonlocal boundary conditions

In the sequel we shall discuss the existence and uniqueness of solutions to the following initial boundary value problem:

$$\begin{cases}\frac{\partial\psi}{\partial t}(t,a,l) + \frac{\partial\psi}{\partial a}(t,a,l) = -\sigma_1(a,l,\langle\psi\rangle)\psi(t,a,l),\\ \psi(0,a,l) = \psi_0,\\ \psi(t,0,l) = \int_{\ell_1}^{+\infty} \zeta(l,l',\langle\psi\rangle,\psi(t,l',l'))\,dl' + \sigma_2(l,\langle\psi\rangle,\psi(t,l,l)),\end{cases} \tag{5.38}$$

where

$$\langle\psi\rangle := \int_{\ell_1}^{+\infty}\int_{\ell_1}^{l} \psi(t,a,l)h^\omega(a,l)\,da\,dl.$$

The function $\sigma_1(\cdot,\cdot,\cdot)$ is a nonlinear function of $\langle\psi\rangle$ while $\zeta(\cdot,\cdot,\cdot,\cdot)$ and $\sigma_2(\cdot,\cdot,\cdot)$ are nonlinear functions of both the density of population ψ and $\langle\psi\rangle$. Here ψ_0 denotes the initial data.

We shall now introduce the following hypotheses:

(H7) The function $\sigma_1(\cdot,\cdot,\cdot)$ is measurable and for any $r > 0$, there exists $\beta_r \in L^\infty(\Omega, da\,dl)$ such that

$$|\sigma_1(a,l,u_1) - \sigma_1(a,l,u_2)| \le \beta_r(a,l)|u_1 - u_2|,$$

for all $(a,l) \in \Omega, u_1, u_2 \in [-r, r]$.

Furthermore, we suppose that $\sigma_1(\cdot,\cdot,0) \in L^\infty(\Omega, dl\,da)$ and there exists $\overline{\sigma}_r \geq |\sigma_1(a,l,u)|$ for a. e. $(a,l) \in \Omega$ and $|u| \leq r$.

(H8) The functions $\zeta(\cdot,\cdot,\cdot,\cdot)$ and $\sigma_2(\cdot,\cdot,\cdot)$ are measurable and there exist a constant C and four functions $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, $\gamma_3(\cdot)$, $\gamma_4(\cdot)$ belonging to Y such that

$$\left|\zeta(l,l',u_1,v_1) - \zeta(l,l',u_2,v_2)\right| \leq \left|\gamma_1(l)\right|\left|\gamma_2(l')\right||u_1-u_2| + \left|\gamma_3(l)\right||v_1-v_2|$$

and

$$\left|\sigma_2(l,\eta,\tau) - \sigma_2(l,\tilde{\eta},\tilde{\tau})\right| \leq \gamma_4(l)|\eta-\tilde{\eta}| + C|\tau-\tilde{\tau}|,$$

for all $l, l' \in [\ell_1,\infty)$ and $\eta, \tilde{\eta}, \tau, \tilde{\tau} \in \mathbb{R}$.
Suppose further that $\zeta(\cdot,\cdot,0,0) \in L^1([\ell_1,+\infty)\times[\ell_1,+\infty))$ and $\sigma_2(\cdot,0,0) \in \mathsf{Y}$.

We now define the operator $\mathfrak{U}$ by

$$\begin{cases} \mathfrak{U} : \mathsf{X}^\omega \to \mathsf{X}^\omega, \\ (\mathfrak{U}\psi)(a,l) = -\sigma_1(a,l,\langle\psi\rangle)\psi(a,l). \end{cases}$$

Hence, problem (5.38) may be written in the form

$$\begin{cases} \frac{\partial\psi}{\partial t}(t,a,l) + \frac{\partial\psi}{\partial a}(t,a,l) = \mathfrak{U}(\psi)(t,a,l), \\ \psi(0,\cdot,\cdot) = \psi_0(\cdot,\cdot) \in \mathsf{X}^\omega, \\ \psi(t,0,l) = [\mathcal{R}(\psi_{|\Gamma_2}(t,\cdot),\psi(t,\cdot,\cdot))](l), \end{cases} \tag{5.39}$$

where $\mathcal{R} : \mathsf{Y}\times\mathsf{X}^\omega \longrightarrow \mathsf{Y}$ denotes the following nonlocal boundary operator:

$$\mathcal{R}(v,\psi)(l) = \int_{\ell_1}^{+\infty} \zeta(l,l',\langle\psi\rangle,v(l'))\,dl' + \sigma_2(l,\langle\psi\rangle,v(l)).$$

Evidently, the space $\mathsf{Y}\times\mathsf{X}^\omega$ is equipped with the norm

$$\left\|(v,\psi)\right\|_* = \|v\|_{\mathsf{Y}} + \|\psi\|_{\mathsf{X}^\omega}, \quad \forall (v,\psi) \in \mathsf{Y}\times\mathsf{X}^\omega.$$

Let us first check that the operator $\mathfrak{U}$ is well defined on X^ω. Indeed, for $f \in \mathsf{X}^\omega$, we have

$$\begin{aligned} \|\mathfrak{U}(f)\|_{\mathsf{X}^\omega} &= \int_{\ell_1}^{+\infty}\int_0^l |-\sigma_1(a,l,\langle f\rangle)f(a,l)|e^{-\omega(l-a)}\,da\,dl \\ &\leq \int_{\ell_1}^{+\infty}\int_0^l |\sigma_1(a,l,\langle f\rangle) - \sigma_1(a,l,0)||f(a,l)|e^{-\omega(l-a)}\,da\,dl \end{aligned}$$

$$
\begin{aligned}
&+\int_{\ell_1}^{+\infty}\int_0^l |\sigma_1(a,l,0)||f(a,l)|e^{-\omega(l-a)}\,da\,dl \\
&\le \int_{\ell_1}^{+\infty}\int_0^l |\beta_r(a,l)||\langle f\rangle||f(a,l)|e^{-\omega(l-a)}\,da\,dl \\
&\quad+\int_{\ell_1}^{+\infty}\int_0^l |\sigma_1(a,l,0)||f(a,l)|e^{-\omega(l-a)}\,da\,dl \\
&\le \|\beta_r\|_\infty\|f\|^2_{\mathsf{X}^\omega}+\big\|\sigma_1(\cdot,\cdot,0)\big\|_\infty\|f\|_{\mathsf{X}^\omega}<+\infty.
\end{aligned}
$$

This proves that $\mathfrak{U}$ maps X^ω into itself.

The operator $\mathcal{R}$ is also well defined on $\mathsf{Y}\times\mathsf{X}^\omega$. To see this, let $(v,\psi)\in\mathsf{Y}\times\mathsf{X}^\omega$. The use of the hypothesis (H8) allows us to write

$$
\begin{aligned}
\big\|\mathcal{R}(v,\psi)\big\|_{\mathsf{Y}} &\le \int_{\ell_1}^{+\infty}\int_{\ell_1}^{+\infty}\big|\zeta(l,l',\langle\psi\rangle,v(l'))\big|\,dl'\,dl+\int_{\ell_1}^{+\infty}\big|\sigma_2(l,\langle\psi\rangle,v(l))\big|\,dl \\
&\le \int_{\ell_1}^{+\infty}\int_{\ell_1}^{+\infty}\big|\zeta(l,l',\langle\psi\rangle,v(l'))-\zeta(l,l',0,0)\big|\,dl'\,dl+\int_{\ell_1}^{+\infty}\int_{\ell_1}^{+\infty}\big|\zeta(l,l',0,0)\big|\,dl'\,dl \\
&\quad+\int_{\ell_1}^{+\infty}\big|\sigma_2(l,\langle\psi\rangle,v(l))-\sigma_2(l,0,0)\big|\,dl+\int_{\ell_1}^{+\infty}\big|\sigma_2(l,0,0)\big|\,dl \\
&\le \|\gamma_1\|_{\mathsf{Y}}\|\gamma_2\|_{\mathsf{Y}}\|\psi\|_{\mathsf{X}^\omega}+\|\gamma_3\|_{\mathsf{Y}}\|v\|_{\mathsf{Y}} \\
&\quad+\big\|\zeta(\cdot,\cdot,0,0)\big\|+\|\gamma_4\|_{\mathsf{Y}}\|\psi\|_{\mathsf{X}^\omega}+C\|v\|_{\mathsf{Y}}+\big\|\sigma_2(\cdot,0,0)\big\| \\
&\le \max\big(\|\gamma_3\|_{\mathsf{Y}}+C,\|\gamma_1\|_{\mathsf{Y}}\|\gamma_2\|_{\mathsf{Y}}+\|\gamma_4\|_{\mathsf{Y}}\big)\big\|(v,\psi)\big\|_* \\
&\quad+\big\|\zeta(\cdot,\cdot,0,0)\big\|+\big\|\sigma_2(\cdot,0,0)\big\|<+\infty.
\end{aligned}
$$

Therefore $\mathcal{R}$ is well defined.

Taking $u(t):=\psi(t,\cdot)\in\mathsf{X}^\omega$, problem (5.39) can be expressed as

$$
\begin{cases}
u'(t)+\mathcal{S}_{\mathcal{R}}(u(t))=\mathfrak{U}(u(t)),\\
u(0)=\psi_0,
\end{cases}
\tag{5.40}
$$

where $\mathcal{S}_{\mathcal{R}}$ stands for the operator

$$
\begin{cases}
\mathcal{S}_{\mathcal{R}}:D(\mathcal{S}_{\mathcal{R}})\subset\mathsf{X}^\omega\to\mathsf{X}^\omega,\\
\psi\to\mathcal{S}_{\mathcal{R}}\psi(a,l):=\frac{\partial\psi}{\partial a}(a,l),\\
D(\mathcal{S}_{\mathcal{R}})=\{\psi\in\mathsf{W}^*_\omega\text{ such that }\psi_{|\Gamma_1}=\mathcal{R}(\psi_{|\Gamma_2},\psi)\}.
\end{cases}
$$

Lemma 5.1.12. *Let* $r > 0$ *and assume that conditions* (H7)–(H8) *are satisfied. Then there exist two constants* $C_r > 0$ *and* $\vartheta > 0$ *such that*

(i) $\|\mathfrak{U}(\varphi_1) - \mathfrak{U}(\varphi_2)\|_{\mathsf{X}^\omega} \leq C_r \|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega}$ *for all* $\varphi_1, \varphi_2 \in \mathbb{B}_r$,

(ii) $\|\mathcal{R}(v_1, \varphi_1) - \mathcal{R}(v_2, \varphi_2)\|_{\mathsf{Y}} \leq \vartheta \|(v_1, \varphi_1) - (v_2, \varphi_2)\|_*$ *for all* $(v_i, \varphi_i) \in \mathsf{Y} \times \mathsf{X}^\omega$, $i = 1, 2$.

Proof. (i) For all $\varphi_1, \varphi_2 \in \mathbb{B}_r$, we have

$$\begin{aligned}
\mathfrak{U}(\varphi_1)(a,l) - \mathfrak{U}(\varphi_2)(a,l) &= -\sigma_1(a,l,\langle\varphi_1\rangle)\varphi_1(a,l) + \sigma_1(a,l,\langle\varphi_2\rangle)\varphi_2(a,l) \\
&\leq \big|\sigma_1(a,l,\langle\varphi_1\rangle)\big|\big|\varphi_1(a,l) - \varphi_2(a,l)\big| \\
&\quad + \big|\sigma_1(a,l,\langle\varphi_1\rangle) - \sigma_1(a,l,\langle\varphi_2\rangle)\big|\big|\varphi_2(a,l)\big| \\
&\leq \overline{\sigma_r}\big|\varphi_1(a,l) - \varphi_2(a,l)\big| + \beta_r(a,l)\big|\langle\varphi_1\rangle - \langle\varphi_2\rangle\big|\big|\varphi_2(a,l)\big|,
\end{aligned}$$

and therefore

$$\begin{aligned}
\big\|\mathfrak{U}(\varphi_1) - \mathfrak{U}(\varphi_2)\big\|_{\mathsf{X}^\omega} &\leq \overline{\sigma_{\mathbf{r}}} \int_{\ell_1}^{+\infty}\int_0^l \big|\varphi_1(a,l) - \varphi_2(a,l)\big| e^{-\omega(l-a)}\,da\,dl \\
&\quad + \|\beta_r\|_\infty \big|\langle\varphi_1\rangle - \langle\varphi_2\rangle\big| \int_{\ell_1}^{+\infty}\int_0^l \big|\varphi_2(a,l)\big| e^{-\omega(l-a)}\,da\,dl \\
&\leq \overline{\sigma_r}\|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} + \|\beta_r\|_\infty \|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega}\|\varphi_2\|_{\mathsf{X}^\omega} \\
&\leq (\overline{\sigma_r} + r\|\beta_r\|_\infty)\|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega}.
\end{aligned}$$

Hence,

$$\big\|\mathfrak{U}(\varphi_1) - \mathfrak{U}(\varphi_2)\big\|_{\mathsf{X}^\omega} \leq C_r \|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega},$$

where $C_r = \overline{\sigma_r} + r\|\beta_r\|_\infty$.

(ii) Now we will check the second estimate. For $(v_1, \varphi_1), (v_2, \varphi_2) \in \mathsf{X} \times \mathsf{X}^\omega$, we can write

$$\begin{aligned}
\big\|\mathcal{R}(v_1, \varphi_1) - \mathcal{R}(v_2, \varphi_2)\big\|_{\mathsf{Y}} &\leq \int_{\ell_1}^{+\infty}\int_{\ell_1}^{+\infty} \big|\zeta(l, l', \langle\varphi_1\rangle, v_1(l')) - \zeta(l, l', \langle\varphi_2\rangle, v_2(l'))\big|\,dl'\,dl \\
&\quad + \int_{\ell_1}^{+\infty} \big|\sigma_2(l, \langle\varphi_1\rangle, v_1(l)) - \sigma_2(l, \langle\varphi_2\rangle, v_2(l))\big|\,dl \\
&= J_1 + J_2,
\end{aligned}$$

where

$$J_1 = \int_{\ell_1}^{+\infty}\int_{\ell_1}^{+\infty} |\gamma_1(l)||\gamma_2(l')||\langle\varphi_1\rangle - \langle\varphi_2\rangle|\,dl\,dl'$$

and

$$J_2 = \int_{\ell_1}^{+\infty} \big|\sigma_2(l, \langle\varphi_1\rangle, v_1(l)) - \sigma_2(l, \langle\varphi_2\rangle, v_2(l))\big|\, dl.$$

Using assumption (H8), we may write

$$\begin{aligned} J_1 &\le \int_{\ell_1}^{+\infty}\int_{\ell_1}^{+\infty} |\gamma_1(l)||\gamma_2(l')||\langle\varphi_1\rangle - \langle\varphi_2\rangle|\, dl\, dl' + \int_{\ell_1}^{+\infty}\int_{\ell_1}^{+\infty} |\gamma_3(l||v_1(l') - v_2(l')|)\, dl\, dl' \\ &\le \|\gamma_1\|_{\mathsf{Y}}\|\gamma_2\|_{\mathsf{Y}}\|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} + \|\gamma_3\|_{\mathsf{Y}}\|v_1 - v_2\|_{\mathsf{Y}} \\ &\le \max(\|\gamma_1\|_{\mathsf{Y}}\|\gamma_2\|_{\mathsf{Y}}, \|\gamma_3\|_{\mathsf{Y}})\big\|(\varphi_2, v_2) - (\varphi_1, v_1)\big\|_*. \end{aligned}$$

Similarly, using (H8), we have

$$\begin{aligned} J_2 &= \int_{\ell_1}^{+\infty} \big|\sigma_2(l, \langle\varphi_1\rangle, v_1(l)) - \sigma_2(l, \langle\varphi_2\rangle, v_2(l))\big|\, dl \\ &\le \int_{\ell_1}^{+\infty} \big(|\gamma_4(l)||\langle\varphi_1\rangle - \langle\varphi_2\rangle| + C|v_1(l) - v_2(l)|\big)\, dl \\ &\le \|\gamma_4\|_{\mathsf{Y}}\|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} + C\|v_1 - v_2\|_{\mathsf{Y}} \\ &\le \max(\|\gamma_4\|, C)\big\|(\varphi_2, v_2) - (\varphi_1, v_1)\big\|_*. \end{aligned}$$

Putting $\vartheta = \max(\|\gamma_1\|_{\mathsf{Y}}\|\gamma_2\|_{\mathsf{Y}}, \|\gamma_3\|_{\mathsf{Y}}) + \max(\|\gamma_4\|_{\mathsf{Y}}, C)$, we obtain

$$\big\|\mathcal{R}(v_1, \varphi_1) - \mathcal{R}(v_2, \varphi_2)\big\|_{\mathsf{Y}} \le \vartheta\big\|(v_1, \varphi_1) - (v_2, \varphi_2)\big\|_*. \qquad \square$$

In the following lemma, we will show that $\mathcal{S}_{\mathcal{R}}$ is a quasiaccretive operator on X^ω whenever $\omega \ge \max\{0, \frac{\ln(\vartheta)}{\ell_1}\}$ where ϑ is the constant appearing in Lemma 5.1.12(ii).

Lemma 5.1.13. *Let ω be an arbitrary real number satisfying $\omega \ge \max\{0, \ln(\vartheta)/\ell_1\}$. If the hypothesis* (H7) *is satisfied, then there exists a constant $\delta > 0$ such that, for all $g_1, g_2 \in \mathsf{X}^\omega$, the inequality*

$$\big\|(I + \lambda\mathcal{S}_{\mathcal{R}})^{-1}(g_1) - (I + \lambda\mathcal{S}_{\mathcal{R}})^{-1}(g_2)\big\|_{\mathsf{X}^\omega} \le \frac{1}{1 - \lambda\delta}\|g_1 - g_2\|_{\mathsf{X}^\omega}$$

holds for all $\lambda \in (0, \frac{1}{\delta})$.

Proof. Put $\delta := \omega + 1$ and let $g_1, g_2 \in \mathsf{R}(I + \lambda\mathcal{S}_{\mathcal{R}})$ be such that $\varphi_1 = (I + \lambda\mathcal{S}_{\mathcal{R}})^{-1}(g_1)$ and $\varphi_2 = (I + \lambda\mathcal{S}_{\mathcal{R}})^{-1}(g_2)$. Hence we have

$$
\begin{aligned}
\|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} &= \left\|(I + \lambda \mathcal{S}_{\mathcal{R}})^{-1}(g_1) - (I + \lambda \mathcal{S}_{\mathcal{R}})^{-1}(g_2)\right\|_{\mathsf{X}^\omega} \\
&\leq -\lambda \int_\Omega e^{-\omega(l-a)} \frac{\partial(\varphi_1 - \varphi_2)}{\partial a} \operatorname{sgn}_0(\varphi_1 - \varphi_2)\, da\, dl \\
&\quad + \int_\Omega e^{-\omega(l-a)} (g_1 - g_2) \operatorname{sgn}_0(\varphi_1 - \varphi_2)\, da\, dl \\
&\leq -\lambda \int_{\ell_1}^{+\infty} \left(\int_0^l \frac{\partial}{\partial a} \left| e^{-\omega(l-a)} (\varphi_1 - \varphi_2) \right| da \right) dl \\
&\quad + \lambda \int_{\ell_1}^{+\infty} \left(\int_0^l |\varphi_1 - \varphi_2| \left(\frac{\partial e^{-\omega(l-a)}}{\partial a} \right) da \right) dl + \int_{\ell_1}^{+\infty} \left(\int_0^l e^{-\omega(l-a)} |g_1 - g_2|\, da \right) dl \\
&\leq -\lambda \int_{\ell_1}^{+\infty} \left(\left|\varphi_{1|\Gamma_2}(l) - \varphi_{2|\Gamma_2}(l)\right| - e^{-\omega l} \left|\mathcal{R}(\varphi_{1|\Gamma_2}, \varphi_1)(l) - \mathcal{R}(\varphi_{2|\Gamma_2}, \varphi_2)(l)\right| \right) dl \\
&\quad + \lambda\omega \|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} + \|g_1 - g_2\|_{\mathsf{X}^\omega}.
\end{aligned}
$$

Since $\varphi_1, \varphi_2 \in D(S_{\mathcal{R}})$, applying Lemma 5.1.12(ii), it is clear that

$$
\begin{aligned}
\|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} &\leq \lambda \left(e^{-\omega \ell_1} \left\|\mathcal{R}(\varphi_{1|\Gamma_2}, \varphi_1) - \mathcal{R}(\varphi_{2|\Gamma_2}, \varphi_2)\right\|_{\mathsf{X}} - \|\varphi_{1|\Gamma_2} - \varphi_{2|\Gamma_2}\|_{\mathsf{X}}\right) \\
&\quad + \lambda\omega \|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} + \|g_1 - g_2\|_{\mathsf{X}^\omega} \\
&\leq \lambda \vartheta e^{-\omega \ell_1} \|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} + \lambda(\vartheta e^{-\omega \ell_1} - 1) \|\varphi_{1|\Gamma_2} - \varphi_{2|\Gamma_2}\|_{\mathsf{Y}} \\
&\quad + \lambda\omega \|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} + \|g_1 - g_2\|_{\mathsf{X}^\omega}.
\end{aligned}
$$

Because $\vartheta e^{-\omega \ell_1} \leq 1$, we get

$$
\begin{aligned}
\|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} &\leq \lambda(\omega + 1) \|\varphi_1 - \varphi_2\|_{\mathsf{X}^\omega} + \|g_1 - g_2\|_{\mathsf{X}^\omega} \\
&\leq \frac{1}{1 - \lambda(\omega + 1)} \|g_1 - g_2\|_{\mathsf{X}^\omega} \\
&= \frac{1}{1 - \lambda\delta} \|g_1 - g_2\|_{\mathsf{X}^\omega}.
\end{aligned}
$$

This ends the proof. ☐

The above result yields that $\mathcal{S}_{\mathcal{R}}$ is quasiaccretive on X^ω. Next, we will prove that $\mathcal{S}_{\mathcal{R}}$ is in fact m-quasiaccretive.

Indeed, let $g \in \mathsf{X}^\omega$ and consider the problem

$$
(I + \lambda \mathcal{S}_{\mathcal{R}}) f = g, \tag{5.41}
$$

where f must be sought in $D(\mathcal{S}_{\mathcal{R}})$.

Let ω be an arbitrary real number satisfying $\omega \geq \max\{0, \frac{\ln(\vartheta)}{\ell_1}\}$. For any $\lambda \in (0, \frac{1}{\omega})$, the solution of problem (5.41) is given by

$$f(a,l) = e^{-\frac{a}{\lambda}}\mathcal{R}(f_{|\Gamma_2},f)(l) + \frac{1}{\lambda}\int_0^a e^{\frac{s-a}{\lambda}} g(s,l)\,ds. \tag{5.42}$$

For $a = l$, we get

$$f(l,l) := f_{|\Gamma_2} = e^{-\frac{l}{\lambda}}\mathcal{R}(f_{|\Gamma_2},f)(l) + \frac{1}{\lambda}\int_0^l e^{\frac{s-l}{\lambda}} g(s,l)\,ds. \tag{5.43}$$

Using the operators A_λ, B_λ, C_λ, and D_λ and the fact that f must satisfy the boundary conditions, equations (5.42) and (5.43) can be expressed as

$$\begin{cases} f = B_\lambda\mathcal{R}(f_{|\Gamma_2},f) + D_\lambda g, \\ f_{|\Gamma_2} = A_\lambda\mathcal{R}(f_{|\Gamma_2},f) + C_\lambda g. \end{cases}$$

Accordingly, $(f_{|\Gamma_2},f)$ is a solution to the fixed point problem $\mathsf{F}_\lambda(v,w) = (v,w)$ on the product space $\mathsf{Y}\times\mathsf{X}^\omega$ where

$$\mathsf{F}_\lambda(v,w) = (\mathsf{F}_\lambda^1,\mathsf{F}_\lambda^2)(v,w) = \big(A_\lambda\mathcal{R}(v,w) + C_\lambda g, B_\lambda\mathcal{R}(v,w) + D_\lambda g\big). \tag{5.44}$$

Lemma 5.1.14. *If the assumption* (H8) *is satisfied, then there exists a constant* $\rho > 0$ *such that, for any* $\lambda \in (0,\rho)$ *and* $g \in \mathsf{X}^\omega$, *there exists a unique function* $f \in D(\mathcal{S}_\mathcal{R})$ *such that*

$$f + \lambda\mathcal{S}_\mathcal{R}(f) = g. \tag{5.45}$$

Proof. As we have seen above, in order to solve problem (5.45), it suffices to prove that the fixed point problem $\mathsf{F}_\lambda(u,f) = (u,f)$ has a unique solution in $\mathsf{Y}\times\mathsf{X}^\omega$ where the operator F_λ is defined by (5.44). Let ρ be the real number defined by

$$\rho := \sup\{\lambda > 0 \text{ such that } (e^{\frac{-\ell_1}{\lambda}} + \lambda)\vartheta < 1\},$$

where ϑ is the constant given in Lemma 5.1.12(ii).

Let $(u_1,f_1),(u_2,f_2) \in \mathsf{X}\times\mathsf{X}^\omega$. Using Lemma 5.1.12(ii), together with estimates (5.33)–(5.36), we obtain

$$\begin{aligned} \|\mathsf{F}_\lambda(u_1,f_1) - \mathsf{F}_\lambda(u_2,f_2)\|_* &= \|\mathsf{A}_\lambda\mathcal{R}(u_1,f_1) - A_\lambda\mathcal{R}(u_2,f_2)\|_* + \|B_\lambda\mathcal{R}(u_1,f_1) - B_\lambda\mathcal{R}(u_2,f_2)\|_* \\ &\le e^{-\frac{\ell_1}{\lambda}}\|\mathcal{R}(u_1,f_1) - \mathcal{R}(u_2,f_2)\|_* + \lambda\|\mathcal{R}(u_1,f_1) - \mathcal{R}(u_2,f_2)\|_* \\ &\le (e^{-\frac{\ell_1}{\lambda}} + \lambda)\vartheta\|(u_1,f_1) - (u_2,f_2)\|_*, \end{aligned}$$

which implies

$$\|\mathsf{F}_\lambda(u_1,f_1) - \mathsf{F}_\lambda(u_2,f_2)\|_* \le (e^{-\frac{l_1}{\lambda}} + \lambda)\vartheta\|(u_1,f_1) - (u_2,f_2)\|_*.$$

According to the definition of the real number ρ, it is clear that, for each $\lambda \in (0,\rho)$, the operator F_λ is a contractive mapping. So, applying the Banach contraction principle, we conclude that, for such λ, problem (5.41) has a unique solution in $D(\mathcal{S}_{\mathcal{R}})$. ☐

Remark 5.1.6. Lemma 5.1.13 shows that $\mathcal{S}_R$ is a quasiaccretive operator in X^ω. Further, Lemma 5.1.13, along with Lemma 5.1.14, allows us to conclude that the operator $\mathcal{S}_{\mathcal{R}}$ is m-quasiaccretive on X^ω. Moreover, using the same argument as in the proof of Lemma 5.1.10, we obtain the density of $D(\mathcal{S}_{\mathcal{R}})$, that is, $\overline{D(\mathcal{S}_{\mathcal{R}})}^{\mathsf{X}^\omega} = \mathsf{X}^\omega$.

Theorem 5.1.4. *Assume that the hypotheses* (H7)–(H8) *are satisfied. Then problem* (5.40) *has a local mild solution for each* $\psi_0 \in \mathsf{X}^\omega$. *Moreover, suppose*

(a) $\zeta(\cdot,\cdot,0,0) = 0$ *and* $\sigma_2(\cdot,0,0) = 0$,

(b) *there exists a positive number* σ *such that* $\sigma \le \sigma_1(a,l,u)$ *for all* $(a,l,u) \in \Omega \times \mathbb{R}$.

Then problem (5.40) *has a unique global mild solution for each* $\psi_0 \in \mathsf{X}^\omega$.

Proof. We are going to see that problem (5.40) admits a local mild solution. Let $\psi_0 \in \mathsf{X}^\omega$ and consider $\alpha > \beta > 1$ such that $\|\psi_0\|_{\mathsf{X}^\omega} \le \beta - 1$ and $T < \frac{1}{\delta}\ln(\alpha/\beta)$. Define the radial retraction $P_\alpha(\cdot)$ by

$$P_\alpha(\psi) = \begin{cases} \psi & \text{if } \|\psi\|_{\mathsf{X}^\omega} \le \alpha, \\ \alpha\frac{\psi}{\|\psi\|_{\mathsf{X}^\omega}} & \text{if } \|\psi\|_{\mathsf{X}^\omega} \ge \alpha. \end{cases}$$

According to Lemma 5.1.12(i) and the estimate (1.10), the operator $\mathfrak{U}(P_\alpha(\cdot))$ is $2C_\alpha$-Lipschitz. Hence, by Theorem 4.6.6, $\mathcal{S}_{\mathcal{R}}(\cdot) - \mathfrak{U}(P_\alpha(\cdot))$ is an m-quasiaccretive operator. Because $D(\mathcal{S}_{\mathcal{R}})$ is dense in X^ω, Theorem 4.6.3 guarantees that the problem

$$\begin{cases} u'(t) + \mathcal{S}_{\mathcal{R}}(u(t)) = \mathfrak{U}(P_\alpha(u(t))), \\ u(0) = \psi_0 \end{cases} \tag{5.46}$$

has a unique mild solution $u(\cdot)$.

Next, since $D(\mathcal{S}_{\mathcal{R}})$ is dense in X^ω, there exists $y_0 \in D(\mathcal{S}_{\mathcal{R}})$ such that $\|y_0 - \psi_0\|_{\mathsf{X}^\omega} < \frac{1}{2}$, which means that $\|y_0\|_{\mathsf{X}^\omega} < \alpha - \frac{1}{2}$. Put $z_0 = \mathcal{S}_{\mathcal{R}}(y_0) - \mathfrak{U}(P_\alpha(y_0)) = \mathcal{S}_{\mathcal{R}}(y_0) - \mathfrak{U}(y_0)$ (because $P_\alpha(y_0) = y_0$). Since $\mathcal{S}_{\mathcal{R}}$ is m-quasiaccretive, it follows from the definition of integral solution (cf. Definition 4.6.1 and Theorem 4.6.2) that

$$\begin{aligned} \|u(t) - y_0\|^2_{\mathsf{X}^\omega} \le{}& e^{2\delta t}\|\psi_0 - y_0\|^2_{\mathsf{X}^\omega} \\ &+ 2e^{2\delta t}\int_0^t e^{-2\delta\tau}\langle \mathfrak{U}(P_\alpha(u(\tau))) - z_0 - \mathfrak{U}(P_\alpha(y_0)), u(\tau) - y_0\rangle_+ \, d\tau. \end{aligned}$$

Therefore, Lemma 4.5.1 yields

$$\|u(t)-y_0\|_{\mathsf{X}^{\omega}} \le e^{\delta t}\Big(\|\psi_0-y_0\|_{\mathsf{X}^{\omega}} + \int_0^t e^{-\delta\tau}\|\mathfrak{U}(P_a(u(\tau))) - z_0 - \mathfrak{U}(P_a(y_0))\|_{\mathsf{X}^{\omega}}\,d\tau\Big).$$

Because the operator $\mathfrak{U}$ is C_a-Lipschitz on $\mathbb{B}_a$ and $P_a(u), P_a(y_0) \in \mathbb{B}_a$, we infer that

$$\|u(t)-y_0\|_{\mathsf{X}^{\omega}} \le e^{\delta t}\Big(\|\psi_0-y_0\|_{\mathsf{X}^{\omega}} + (2aC_a + \|z_0\|_{\mathsf{X}^{\omega}})\frac{1}{\delta}(1-e^{-\delta t})\Big).$$

This implies that $\|u(t)\|_{\mathsf{X}^{\omega}} \le \|y_0\|_{\mathsf{X}^{\omega}} + \frac{1}{2} \le a$, for $0 < t < T_a$ with $T_a > 0$ suitably chosen. This completes the proof of the first assertion.

Now suppose that assumptions (a) and (b) are satisfied. In this case, assumption (a) implies that $0 \in D(\mathcal{S}_{\mathcal{R}})$, and thus $\mathcal{S}_{\mathcal{R}}(0) = 0$. On the other hand, if we consider the normalized duality map on X^{ω}, then assumption (b) guarantees that, for each $x^* \in J(\psi)$, the inequality

$$\langle -\sigma_1(\cdot,\cdot,\langle\psi\rangle)\psi, x^*\rangle \le -\sigma\|\psi\|^2_{\mathsf{X}^{\omega}}$$

holds true. This means that $\langle\mathfrak{U}(P_a(\psi)), x^*\rangle \le 0$. Finally, since $0 = \mathcal{S}_{\mathcal{R}}(0)$, the integral solution of (5.46) satisfies

$$\|u(t)\|^2 e^{-2\delta t} \le \|u(0)\|^2 + 2\int_0^t e^{-2\delta\tau}\langle\mathfrak{U}(P_a(u(\tau)), u(\tau)\rangle_+\,d\tau \le \|u(0)\|^2,$$

and therefore

$$\|u(t)\| \le e^{\delta T}\beta \le a \quad \text{for all } t \in [0,T].$$

The above fact means that u is the solution of problem (5.40). ☐

5.2 Rotenberg's model

In 1983, R. Rotenberg [205] presented a model for growing cell population in which each cell is distinguished by two parameters. The first is the degree of maturity μ and a cell in the process evolution has a degree maturity $\mu \in [0,1]$. Thus, during each cell mitosis, the degree of maturity of a mother cell is $\mu = 1$ and that of its daughter cells is $\mu = 0$. The second is the velocity of maturation v. The positivity of velocities comes from the fact that a cell may not become less mature with time and then $v \in (a,b)$ $(0 < a < b < +\infty)$. If the function $\psi(t,\mu,v)$ represents the density of the population with respect to the maturation μ and the velocity of maturation v at time t, Rotenberg derived the following partial differential equation:

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,\mu,v) + v\frac{\partial}{\partial \mu}\psi(t,\mu,v) = -\sigma(\mu,v)\psi(t,\mu,v) + \int_a^b \kappa(\mu,v,v')\psi(t,\mu,v')\,dv', \\ \psi(0,\mu,v) = \psi_0(\mu,v). \end{cases} \tag{5.47}$$

The kernel $\kappa(\mu,v,v')$ is the transition rate. It specifies the transition of cells from the maturation velocity v' to v while $\sigma(\mu,v) = \int_a^b \kappa(\mu,v',v)\,dv'$ denotes the total transition cross-section. It is the rate of cell mortality or cell loss due to other causes than division. During each cell division, it can be assumed that there is a correlation between the maturation velocity of mother cells v' and that of daughter cells v. This correlation is governed by a transition biological rule, mathematically described by the boundary condition

$$v\psi(t,0,v) = \beta \int_a^b k(v,v')\psi(t,1,v')v'\,dv',$$

where β is the average number of daughter cells viable at mitosis (see [205]).

Rotenberg discussed essentially the Fokker–Planck approximation of problem (5.47) for which he obtained numerical solutions. Using the eigenfunction technique, C. van der Mee and P. Zweifel [233] obtained analytical solutions of (5.47) for a variety of boundary conditions. Using Lebowitz and Rubinow's boundary conditions, it was established in [121, 3, Chapter XIII, Section 5] that the associated Cauchy problem to (5.47) is governed by a positive C_0-semigroup and an estimate of its type was derived on the space L^1 which allowed describing the time asymptotic behavior of the solution of the Cauchy problem. Similar results were obtained for various boundary conditions (see, for example, [39, 130]). In [76] a detailed spectral analysis of problem (5.47) supplemented with general (linear) transition rule relating mother and daughter cells at mitosis, covering, in particular, all classical ones considered in [121, 153, 205, 231–233], was given.

In the introduction of his paper, R. Rotenberg has pointed out that an adequate formulation of the model (5.47) seems to be a nonlinear one. Actually, the cells under consideration are in contact with a nutrient environment which is not part of the mathematical formulation. Fluctuations in nutrient concentration and other density-dependent effects such as contact inhibition of growth make the transition rates $\sigma(\cdot,\cdot)$ and $\kappa(\cdot,\cdot,\cdot)$ functions of the population density, thus creating a nonlinear problem. On the other hand, the biological boundaries at $\mu = 0$ and $\mu = 1$ are fixed and tightly coupled throughout mitosis. The conditions present at the boundaries are felt throughout the system and cannot be removed. This phenomenon suggests that at mitosis daughter and parent cells are related by a nonlinear nonlocal reproduction rule.

In this section we study the initial boundary value problem

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,\mu,v) + v\frac{\partial}{\partial \mu}\psi(t,\mu,v) = -\sigma(\mu,v,\psi(t,\mu,v)) + \int_a^b \kappa(\mu,v,v',\psi(t,\mu,v'))\,dv', \\ \psi(0,\mu,v) = \psi_0(\mu,v), \\ \psi(t,0,v) = [R\psi(t,1,\cdot)](v), \end{cases} \tag{5.48}$$

where $\sigma(\cdot,\cdot,\cdot)$ and $\kappa(\cdot,\cdot,\cdot,\cdot)$ are nonlinear functions while R denotes a nonlinear operator on suitable trace spaces modeling the transition biological rule. As in the previous sections we shall discuss existence results for problem (5.48) for local and nonlocal boundary conditions.

5.2.1 The functional setting of the problem and preliminaries

Let a, b be two real numbers such that $0 < a < b < +\infty$ and for each $p \in [1, +\infty)$ let the real number q denote the conjugate exponent of p, i. e., $q = \frac{p}{p-1}$ (if $p = 1$, then $q = +\infty$). Let Ω be the set

$$\Omega := \{(\mu, v) \text{ such that } 0 \le \mu \le 1,\ a \le v \le b\}.$$

For each $p \in [1, +\infty)$, we consider the Banach space

$$\mathsf{X}_p := L^p(\Omega; d\mu\, dv)$$

with its natural norm

$$\|\varphi\|_p = \left(\int_\Omega |\varphi(\mu, v)|^p\, d\mu\, dv\right)^{\frac{1}{p}} = \left(\int_0^1 \left(\int_a^b |\varphi(\mu, v)|^p\, dv\right) d\mu\right)^{\frac{1}{p}}.$$

We consider the partial Sobolev space

$$\mathsf{W}_p := \left\{\varphi \in \mathsf{X}_p : v\frac{\partial}{\partial\mu}\varphi \in \mathsf{X}_p \text{ and } v\varphi \in \mathsf{X}_p\right\}$$

equipped with the norm

$$\|\varphi\|_{\mathsf{W}_p} = \|v\varphi\|_p + \left\|v\frac{\partial\varphi}{\partial\mu}\right\|_p,$$

which is a Banach space. Let Y_p be the space

$$\mathsf{Y}_p := L^p([a, b], v\, dv)$$

endowed with the norm

$$\|\psi\|_{(*,p)} := \left(\int_a^b |\psi(v)|^p v\, dv\right)^{\frac{1}{p}}.$$

It is well known (see, for example, [56, 57] or [121, Proposition 3.3, p. 379]) that any function $\varphi \in \mathsf{W}_p$ has traces on the space Y_p. Moreover, the trace mappings $\gamma_0 : \varphi \longmapsto \varphi(0, \cdot)$ and $\gamma_1 : \varphi \longmapsto \varphi(1, \cdot)$ are linear and continuous from W_p into Y_p.

For any $\delta > 0$, we introduce the weighted space

$$\mathsf{X}_{p,\delta} = L^p(\Omega, h^\delta \, d\mu \, dv)$$

equipped with the norm

$$\|\varphi\|_{p,\delta} = \Bigl(\int_\Omega |(\varphi h^\delta)(\mu,v)|^p \, d\mu \, dv\Bigr)^{\frac{1}{p}} = \Bigl(\int_0^1 \Bigl(\int_a^b |\varphi h^\delta|^p(\mu,v) \, dv\Bigr) d\mu\Bigr)^{\frac{1}{p}},$$

where h^δ is given by

$$h^\delta(\mu,v) = e^{-\delta\frac{1-\mu}{v}}.$$

The next lemma shows that the norms $\|\cdot\|_{p,\delta}$ and $\|\cdot\|_p$ are equivalent on X_p.

Lemma 5.2.1. *Let $\delta \geq 0$ and $\psi \in \mathsf{X}_p$. Then,*

$$\|\psi\|_1 \leq e^{\frac{\delta}{a}}(b)^{\frac{1}{q}}\|\psi\|_{p,\delta}, \quad \|\psi\|_{p,\delta} \leq \|\psi\|_p \leq e^{\frac{\delta}{a}}\|\psi\|_{p,\delta}.$$

Proof. Using Hölder inequality, we may write

$$\begin{aligned}
\|\psi\|_1 &= \int_0^1 \int_a^b |\psi(\mu,v)| \, d\mu \, dv \\
&= \int_0^1 \int_a^b e^{\frac{\delta(1-\mu)}{v}} \exp\Bigl(-\frac{\delta(1-\mu)}{v}\Bigr)|\psi(\mu,v)| \, d\mu \, dv \\
&\leq \Bigl(\int_0^1 \int_a^b \exp\Bigl(\frac{q\delta(1-\mu)}{v}\Bigr) d\mu \, dv\Bigr)^{\frac{1}{q}} \|\psi\|_{\delta,p} \\
&= \Bigl(\int_a^b \frac{v}{\delta q}\Bigl(\exp\Bigl(\frac{q\delta}{v}\Bigr) - 1\Bigr) dv\Bigr)^{\frac{1}{q}} \|\psi\|_{\delta,p}.
\end{aligned}$$

Now, using the estimate

$$\exp\Bigl(\frac{q\delta}{v}\Bigr) - 1 = \sum_{k=1}^{\infty} \frac{1}{k!}\Bigl(\frac{q\delta}{v}\Bigr)^k = \frac{q\delta}{v}\sum_{k=1}^{\infty} \frac{1}{k!}\Bigl(\frac{q\delta}{v}\Bigr)^{k-1} \leq \frac{q\delta}{v}\exp\Bigl(\frac{q\delta}{v}\Bigr)$$

gives the first inequality. The second inequality is trivial. □

We denote by $\mathsf{X}^*_{p,\delta}$ the dual space of $\mathsf{X}_{p,\delta}$ and by $\langle\cdot,\cdot\rangle_\delta$ the pairing between $\mathsf{X}_{p,\delta}$ and $\mathsf{X}^*_{p,\delta}$. By F_δ we denote the duality map defined by

$$F_\delta(0) = \{f \in \mathsf{X}_{p,\delta}^* : \|f\|_{\mathsf{X}_{p,\delta}^*} = 1\}$$

and, for $\psi \neq 0$,

$$F_\delta(\psi) = \{f \in \mathsf{X}_{p,\delta}^* : \langle \psi, f\rangle_\delta = \|\psi\|_{\mathsf{X}_{p,\delta}}, \|f\|_{\mathsf{X}_{p,\delta}^*} = 1\}.$$

We recall that, if $1 < p < +\infty$ and $\psi \in \mathsf{X}_{p,\delta}$, then

$$F_\delta(\psi) = \{\|\psi\|_{p,\delta}^{1-p} |\psi|^{p-2}\psi\},$$

and for $p = 1$ we have

$$\{\operatorname{sgn}_0 \psi\} \in F_\delta(\psi),$$

where

$$\operatorname{sgn}_0 x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Now define the partial Sobolev space

$$\mathsf{W}_{p,\delta} := \left\{\varphi \in \mathsf{X}_{p,\delta} : v\frac{\partial \varphi}{\partial \mu} \in \mathsf{X}_{p,\delta},\ v\varphi \in \mathsf{X}_{p,\delta}\right\}.$$

Next, we introduce the following linear operators depending on the parameter λ:

$$\begin{cases} P_\lambda : \mathsf{Y}_p \longrightarrow \mathsf{Y}_p, \\ u \mapsto (P_\lambda u)(v) := u(v)e^{-\frac{1}{\lambda v}}, \end{cases} \qquad \begin{cases} Q_\lambda : \mathsf{Y}_p \longrightarrow \mathsf{X}_p, \\ u \mapsto (Q_\lambda u)(\mu, v) := u(v)e^{-\frac{\mu}{\lambda v}}, \end{cases}$$

$$\begin{cases} \Pi_\lambda : \mathsf{X}_p \longrightarrow \mathsf{Y}_p, \\ (\Pi_\lambda g)(v) := \frac{1}{\lambda v}\int_0^1 e^{-\frac{1-\mu'}{\lambda v}} g(\mu', v)\, d\mu', \end{cases} \qquad \begin{cases} \Xi_\lambda : \mathsf{X}_p \longrightarrow \mathsf{X}_p, \\ (\Xi_\lambda g)(\mu, v) := \frac{1}{\lambda v}\int_0^\mu e^{-\frac{\mu-\mu'}{\lambda v}} g(\mu', v)\, d\mu'. \end{cases}$$

One checks readily that, for $\lambda > 0$, the operators P_λ and Q_λ are bounded, positive (in the lattice sense), and satisfy

$$\|P_\lambda\|_{\mathcal{L}(\mathsf{Y}_p)} \le e^{-\frac{1}{\lambda b}}, \tag{5.49}$$

$$\|Q_\lambda\|_{\mathcal{L}(L^p([a,b], v\,dv); L^p(\Omega))} \le \lambda^{\frac{1}{p}}. \tag{5.50}$$

Moreover, using Hölder inequality, we obtain

$$\begin{aligned}\|\Pi_\lambda g\|^p_{(*,p)} &= \int_a^b \frac{1}{(\lambda v)^p}\left|\int_0^1 e^{-\frac{1-\mu'}{\lambda v}} g(\mu', v)\, d\mu'\right|^p v\, dv \\ &\leq \int_a^b \frac{1}{(\lambda v)^p}\left(\int_0^1 e^{-\frac{1-\mu'}{\lambda v}}\, d\mu'\right)^{p-1}\left(\int_0^1 e^{-\frac{1-\mu'}{\lambda v}} |g(\mu', v)|^p\, d\mu'\right) v\, dv \\ &= \int_a^b \frac{1}{\lambda}[1 - e^{-\frac{1}{\lambda v}}]^{p-1}\int_0^1 e^{-\frac{1-\mu'}{\lambda v}} |g(\mu', v)|^p\, d\mu' v\, dv \leq \frac{1}{\lambda}\|g\|_p,\end{aligned}$$

which implies that Π_λ is bounded and

$$\|\Pi_\lambda\|_{\mathcal{L}(L^p(\mathsf{X}_p;\mathsf{Y}_p)} \leq \lambda^{-\frac{1}{p}}. \tag{5.51}$$

Similar calculations show the boundedness of Ξ_λ and

$$\|\Xi_\lambda\|_{\mathcal{L}(\mathsf{X}_p)} \leq \left(\frac{1}{a\lambda}\right)^{\frac{1}{p}}. \tag{5.52}$$

5.2.2 Local boundary conditions

The aim of this subsection is to discuss the existence and uniqueness results to the following initial boundary value problem:

$$\begin{cases}\frac{\partial}{\partial t}\psi(t,\mu,v) + v\frac{\partial}{\partial\mu}\psi(t,\mu,v) = & -\sigma(\mu, v, \psi(t,\mu,v)) + \int_a^b \kappa(\mu, v, v', \psi(t,\mu,v'))\, dv', \\ \psi(0,\mu,v) = \psi_0(\mu,v), \\ \psi(t,0,v) = [R\psi(t,1,\cdot)](v). \end{cases} \tag{5.53}$$

We introduce the hypotheses:

(H9) The operator $R : \mathsf{Y}_p \to \mathsf{Y}_p$ satisfies a Lipschitz condition, i. e., there exists $L_R > 0$ such that

$$\|R(f_1) - R(f_2)\|_{(*,p)} \leq L_R\|f_1 - f_2\|_{(*,p)},$$

for all $f_1, f_2 \in \mathsf{Y}_p$. (In general, L_R will be considered bigger than or equal to 1).

(H10) The function $\sigma(\cdot,\cdot,\cdot)$ is measurable and there exists $\Lambda > 0$ such that

$$|\sigma(\mu, v, \eta_1) - \sigma(\mu, v, \eta_2)| \leq \Lambda|\eta_1 - \eta_2|.$$

(H11) The function $\kappa(\cdot,\cdot,\cdot,\cdot)$ is measurable and satisfies

$$|\kappa(\mu,v,v',\psi_1(\mu,v')) - \kappa(\mu,v,v',\psi_2(\mu,v'))| \leq |\rho(\mu,v,v')||\psi_1(\mu,v') - \psi_2(\mu,v')|,$$

where $\psi_1, \psi_2 \in X_p$ and $\rho \in L^\infty([0,1]\times[a,b]\times[a,b], d\mu\, dv\, dv)$.

Next, define the nonlinear operator

$$\mathcal{B} : \mathsf{X}_p \to \mathsf{X}_p, \quad \psi \mapsto \int_a^b \kappa(\mu,v,v',\psi(\mu,v'))\, dv'$$

and, for all $\psi \in \mathsf{X}_p$, we set

$$\mathcal{F}(\psi) := \mathcal{B}\psi - \sigma(\cdot,\cdot,\psi).$$

Let $\mathbf{T_R}$ denote the operator

$$\begin{cases} \mathbf{T_R} : D(\mathbf{T_R}) \subseteq \mathsf{X}_p \to \mathsf{X}_p, \\ \varphi \mapsto (\mathbf{T_R}\varphi)(\mu,v) = v\frac{\partial\varphi}{\partial\mu}(\mu,v), \\ D(\mathbf{T_R}) := \{\varphi \in \mathsf{W}_p : \gamma_0(\varphi) = R(\gamma_1(\varphi))\}. \end{cases}$$

Remark 5.2.1. We may consider the Banach space $\mathsf{X}_{p,\delta}$ where $\delta = 0$ if R is a nonexpansive mapping and δ is a fixed number bigger than $b\ln(L_R)$ if R is an L_R-Lipschitz mapping with $L_R > 1$. Thus, in what follows we shall take δ a fixed real number satisfying $\delta \geq \max\{0, b\ln(L_R)\}$.

Proposition 5.2.1. *If the condition* (H9) *is satisfied, then* $\mathbf{T_R}$ *is an m-quasiaccretive operator on* $\mathsf{X}_{p,\delta}$.

Proof. Consider $g \in \mathsf{X}_{p,\delta}$ and $\lambda \in (0, \frac{1}{\delta})$ (here if $\delta = 0$, $\frac{1}{\delta}$ means $+\infty$). We wish to prove that there exists a function $\varphi \in D(\mathbf{T_R})$ such that $\varphi + \lambda\mathbf{T_R}(\varphi) = g$. Thus, we have to seek the solution of the following differential equation:

$$\varphi(\mu,v) + \lambda v\frac{\partial\varphi}{\partial\mu}(\mu,v) = g(\mu,v). \tag{5.54}$$

Because equation (5.54) is linear with respect to the variable μ, the general solution is given by

$$\varphi(\mu,v) = e^{-\frac{\mu}{v\lambda}}\chi(v) + \frac{1}{v\lambda}\int_0^\mu g(s,v)e^{-\frac{\mu-s}{v\lambda}}\, ds. \tag{5.55}$$

Using the operators Q_λ and Ξ_λ, we get

$$\varphi(\mu, v) = Q_\lambda \chi(v) + (\Xi_\lambda g)(\mu, v).$$

Let us show that if $\chi \in \mathsf{Y}_p$, then $\varphi \in \mathsf{W}_{p,\delta}(\Omega)$. Indeed, Hölder inequality and the estimate $0 < \lambda\delta < 1$, imply

$$\begin{aligned}
\|v\varphi\|_{p,\delta} &= \left(\int_0^1 \int_a^b e^{-\delta \frac{1-\mu}{v} p} v^p \left| e^{-\frac{\mu}{v\lambda}} \chi(v) + \frac{1}{v\lambda} \int_0^\mu g(s, v) e^{\frac{s-\mu}{v\lambda}}\, ds \right|^p dv\, d\mu \right)^{\frac{1}{p}} \\
&\leq \left(\int_0^1 \left(\int_a^b v^{p-1} v |\chi(v)|^p\, dv \right) d\mu \right)^{\frac{1}{p}} + \left(\frac{1}{\lambda^p} \int_0^1 \int_a^b e^{-\delta \frac{1-\mu}{v} p} \int_0^\mu |g(s, v)|^p e^{\frac{s-\mu}{v\lambda} p}\, ds\, dv\, d\mu \right)^{\frac{1}{p}} \\
&\leq b^{\frac{1}{q}} \|\chi\|_{(*,p)} + \frac{1}{\lambda} \left(\int_a^b \int_0^1 \int_0^\mu |g(s, v)|^p e^{-p\delta \frac{1-s}{v}}\, ds\, d\mu\, dv \right)^{\frac{1}{p}} \\
&\leq b^{\frac{1}{q}} \|\chi\|_{(*,p)} + \frac{1}{\lambda} \|g\|_{p,\delta}. \qquad (5.56)
\end{aligned}$$

On the other hand, we find that

$$\left\| v \frac{\partial \varphi}{\partial \mu} \right\|_{p,\delta} = \frac{1}{\lambda} \|g - \varphi\|_{p,\delta} \leq \frac{1}{\lambda} \left(\|g\|_{p,\delta} + \frac{1}{a} \|v\varphi\|_{p,\delta} \right).$$

It is clear that if $\chi \in \mathsf{Y}_p$, then the latter estimate, together with (5.56), implies that $\varphi \in \mathsf{W}_{p,\delta}$.

Now we will check that there exists a function $\varphi \in D(\mathbf{T_R})$ satisfying (5.55). In this case we have $\gamma_0(\varphi) = \chi$. The following condition should be satisfied:

$$\gamma_1(\varphi) = \gamma_1(Q_\lambda(\chi)) + \gamma_1(\Xi_\lambda(g)) = e^{-\frac{1}{v\lambda}} \gamma_0(\varphi) + \gamma_1(\Xi_\lambda(g)).$$

Because $\varphi \in D(\mathbf{T_R})$, we know that $\gamma_0(\varphi) = R(\gamma_1(\varphi))$, and therefore

$$\gamma_1(\varphi)(v) = P_\lambda(R(\gamma_1(\varphi)))(v) + \gamma_1(\Xi_\lambda(g))(v).$$

Let $S_\lambda : \mathsf{Y}_p \to \mathsf{Y}_p$ be the operator defined by

$$S_\lambda(\rho)(v) = P_\lambda(R(\rho))(v).$$

We claim that S_λ is $e^{-\frac{1}{b\lambda}} L_R$-contractive. Indeed, using (5.49) and the hypothesis (H9), we obtain

$$\|S_\lambda(\rho_1) - S_\lambda(\rho_2)\|_{(*,p)} = \|P_\lambda(R(\rho_1) - R(\rho_2))\|_{(*,p)} \leq e^{-\frac{1}{b\lambda}} L_R \|\rho_1 - \rho_2\|_{(*,p)}.$$

Since $\delta \geq b \ln(L_R)$ and $\lambda \in (0, \frac{1}{\delta})$, it is clear that $e^{-\frac{1}{b\lambda}} L_R < 1$, which proves our claim.

Hence the operator $I - S_\lambda : \mathsf{Y}_p \to \mathsf{Y}_p$ is bijective and $(I - S_\lambda)^{-1}$ is continuous. Consequently, given $\gamma_1(\Xi_\lambda(g)) \in \mathsf{Y}_p$ there exists a unique function $\psi \in \mathsf{Y}_p$ such that $(I - S_\lambda)(\psi) = \gamma_1(\Xi_\lambda(g))$. So we can define the function

$$\varphi(\mu, v) := (I + \lambda T_K)^{-1}(g) = Q_\lambda\big(K(I - S_\lambda)^{-1}(\gamma_1(\Xi_\lambda(g)))\big)(v) + \Xi_\lambda(g)(\mu, v) \in \mathsf{W}_{p,\delta}.$$

This yields that $\mathsf{R}(I + \lambda \mathbf{T_R}) = \mathsf{X}_{p,\delta}$ whenever $\lambda \in (0, \frac{1}{\delta})$.

Next, we will prove that $\mathbf{T_R}$ is a δ-accretive operator on $\mathsf{X}_{p,\delta}$. In order to see this, we first prove this result in the space $\mathsf{X}_{1,\delta}$. Indeed, letting $g_1, g_2 \in \mathsf{X}_{1,\delta}$, by the above arguments there exist $\varphi_1, \varphi_2 \in D(\mathbf{T_R})$ such that

$$\varphi_1 = (I + \lambda \mathbf{T_R})^{-1}(g_1), \quad \varphi_2 = (I + \lambda \mathbf{T_R})^{-1}(g_2).$$

Hence, using (5.54), (H16), and the fact that $\delta \geq b \ln(L_R)$, we can write

$$\begin{aligned}
\|\varphi_1 - \varphi_2\|_{1,\delta} &= \int_\Omega \operatorname{sgn}_0(\varphi_1 - \varphi_2)\Big[(g_1 - g_2) - \lambda v \frac{\partial}{\partial \mu}(\varphi_1 - \varphi_2)\Big] h_\delta(\mu, v)\, d\mu\, dv \\
&\leq -\lambda \int_\Omega \operatorname{sgn}_0(\varphi_1 - \varphi_2) h_\delta v \frac{\partial(\varphi_1 - \varphi_2)}{\partial \mu}\, d\mu\, dv \\
&\quad + \int_\Omega \operatorname{sgn}_0(\varphi_1 - \varphi_2) h_\delta (g_1 - g_2)\, d\mu\, dv \\
&\leq -\lambda \int_a^b \int_0^1 v \frac{\partial}{\partial \mu}\big|h_\delta(\varphi_1 - \varphi_2)\big|\, d\mu\, dv + \delta\lambda \int_\Omega \big|h_\delta(\varphi_1 - \varphi_2)\big|\, d\mu\, dv \\
&\quad + \int_\Omega h_\delta(\mu, v)\big|g_1(\mu, v) - g_2(\mu, v)\big|\, d\mu\, dv \\
&\leq \lambda \int_a^b \big(e^{-\delta/v} v \big|R(\gamma_1(\varphi_1)) - R(\gamma_1(\varphi_2))\big| - v\big|\gamma_1(\varphi_1) - \gamma_1(\varphi_2)\big|\big)\, dv \\
&\quad + \delta\lambda \|\varphi_1 - \varphi_2\|_{1,\delta} + \|g_1 - g_2\|_{1,\delta} \\
&\leq \lambda(e^{-\delta/b} a - 1)\big\|\gamma_1(\varphi_1) - \gamma_1(\varphi_2)\big\|_{Y_1} + \delta\lambda \|\varphi_1 - \varphi_2\|_{1,\delta} + \|g_1 - g_2\|_{1,\delta} \\
&\leq \delta\lambda \|\varphi_1 - \varphi_2\|_{1,\delta} + \|g_1 - g_2\|_{1,\delta},
\end{aligned}$$

and therefore

$$\big\|(I + \lambda \mathbf{T_R})^{-1}(g_1) - (I + \lambda \mathbf{T_R})^{-1}(g_2)\big\|_{1,\delta} = \|\varphi_1 - \varphi_2\|_{1,\delta} \leq \frac{1}{1 - \delta\lambda} \|g_1 - g_2\|_{1,\delta}.$$

Now, we consider the space $\mathsf{X}_{p,\delta}$ with $p > 1$. We claim that $\mathbf{T_R}$ is δ-accretive. To see this, let φ_1, φ_2 be two elements of $D(\mathbf{T_R})$ and set $\psi := \varphi_1 - \varphi_2$. Hence

$$
\begin{aligned}
[\psi, \mathbf{T}_\mathbf{R}(\varphi_1) - \mathbf{T}_\mathbf{R}(\varphi_2)]_s &\geq \|\psi\|_{p,\delta}^{1-p} \int_a^b \left(\int_0^1 h_\delta^p |\psi|^{p-1} \nu \frac{\partial}{\partial \mu}(\psi) \operatorname{sgn}_0(\psi)\, d\mu \right) d\nu \\
&= \frac{\|\psi\|_{p,\delta}^{1-p}}{p} \int_a^b \left(\int_0^1 h_\delta^p \nu \left(\frac{\partial}{\partial \mu}(|\psi|^p) \right) d\mu \right) d\nu \\
&= \frac{\|\psi\|_{p,\delta}^{1-p}}{p} \left(\int_a^b \left(\int_0^1 \nu \left(\frac{\partial}{\partial \mu}(h_\delta |\psi|)^p \right) d\mu \right) d\nu \right) \\
&\quad - \frac{\|\psi\|_{p,\delta}^{1-p}}{p} \left(\int_a^b \left(\int_0^1 \nu |\psi|^p \left(\frac{\partial}{\partial \mu} h_\delta^p \right) d\mu \right) d\nu \right).
\end{aligned} \tag{5.57}
$$

Because trace mappings are linear and continuous, the last term of inequality (5.57) is equal to

$$
\frac{\|\psi\|_{p,\delta}^{1-p}}{p} \int_a^b \nu \left(|\gamma_1(\varphi_1)(\nu) - \gamma_1(\varphi_2)(\nu)|^p - e^{-\frac{\delta p}{\nu}} |\gamma_0(\varphi_1)(\nu) - \gamma_0(\varphi_2)(\nu)|^p \right) d\nu - \delta \|\psi\|_{p,\delta}. \tag{5.58}
$$

Since $\varphi_1, \varphi_2 \in D(\mathbf{T}_\mathbf{R})$, expression (5.58) is greater than or equal to

$$
\frac{\|\psi\|_{p,\delta}^{1-p}}{p} \int_a^b \nu \left(|\gamma_1(\varphi_1)(\nu) - \gamma_1(\varphi_2)(\nu)|^p - e^{-\frac{\delta p}{\nu}} |R(\gamma_1(\varphi_1))(\nu) - R(\gamma_1(\varphi_2))(\nu)|^p \right) d\nu - \delta \|\psi\|_{p,\delta}.
$$

Finally, remembering that R is L_R-Lipschitz on Y_p and using the fact that $\delta \geq b \ln(L_R)$, we conclude that

$$
\begin{aligned}
[\psi, \mathbf{T}_\mathbf{R}(\varphi_1) - \mathbf{T}_\mathbf{R}(\varphi_2)]_s &\geq \frac{\|\psi\|_{p,\delta}^{1-p}}{p} (1 - L_R^p e^{-\frac{\delta p}{b}}) \|\gamma_1(\varphi_1) - \gamma_1(\varphi_2)\|_{\mathsf{Y}_p}^p - \delta \|\psi\|_{p,\delta} \\
&\geq -\delta \|\psi\|_{p,\delta}.
\end{aligned}
$$

This proves that $\mathbf{T}_\mathbf{R}$ is δ-accretive, as claimed. □

Remark 5.2.2. It should be noticed that, if R is nonexpansive, i. e., $L_R \leq 1$, then we can take $\delta = 0$ and therefore we can work with the spaces X_p without using any weight.

Proposition 5.2.2. $\overline{D(\mathbf{T}_\mathbf{R})}^{\mathsf{X}_p} = \mathsf{X}_p$.

Proof. To obtain this result, it is enough to prove that $C_0^\infty(\overline{\Omega}) \subseteq \overline{D(\mathbf{T}_\mathbf{R})}^{\mathsf{X}_p}$. Let u be an element of $C_0^\infty(\overline{\Omega})$. For each $n \in \mathbb{N}$, define the function

$$
u_n(\mu, \nu) = \begin{cases} u(\mu, \nu), & (\mu, \nu) \in \Omega, \mu \in [2/n, 1], \\ u(\mu, \nu)(\mu - \frac{1}{n})n + R(u(1, \nu))(\frac{2}{n} - \mu)n, & (\mu, \nu) \in \Omega,\ \mu \in [\frac{1}{n}, \frac{2}{n}[, \\ R(u(1, \cdot))(\nu), & (\mu, \nu) \in \Omega,\ \mu \in [0, 1/n[. \end{cases}
$$

It is clear that $|u_n(\mu, v)| \leq |u(\mu, v)| + |R(u(1, v)|$ and both functions belong to X_p. It is also clear that $u_n^p \to u^p$ a. e. in Ω. Thus, by the dominated convergence theorem, we have that $\|u_n - u\|_p \to 0$ as n tends to $+\infty$. On the other hand,

$$\frac{\partial u_n}{\partial \mu} = \begin{cases} \frac{\partial u}{\partial \mu}(\mu, v), & (\mu, v) \in \Omega,\ \mu \in [2/n, 1], \\ \frac{\partial u}{\partial \mu}(\mu, v)(\mu - \frac{1}{n})n + u(\mu, v) - nR(u(1, v)), & (\mu, v) \in \Omega,\ \mu \in [\frac{1}{n}, \frac{2}{n}[, \\ 0, & (\mu, v) \in \Omega,\ \mu \in [0, 1/n[. \end{cases}$$

Since $u \in \mathcal{C}_0^\infty(\overline{\Omega})$, it is an easy matter to check that $u_n \in \mathsf{W}_p$. Moreover, $\gamma_1(u_n)(v) = u(1, v)$ and $\gamma_0(u_n)(v) = R(u(1, v)) = R(\gamma_1(u_n)(v))$, which means that $u_n \in D(\mathcal{S}_R)$. ☐

Lemma 5.2.2. *Let* $p \in [1, +\infty)$ *and assume that hypotheses* (H10)–(H11) *are satisfied. Then there exists a constant* $C > 0$ *such that, for all* $\psi_1, \psi_2 \in \mathsf{X}_p$, *we have*

$$\|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)\|_p \leq C\|\psi_1 - \psi_2\|_p.$$

That is, $\mathcal{F}$ *is a globally Lipschitz mapping.*

Proof. For all $\psi_1, \psi_2 \in \mathsf{X}_p$, we have

$$\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2) = \mathcal{B}(\psi_1) - \mathcal{B}(\psi_2) - (\sigma(\cdot, \cdot, \psi_1) - \sigma(\cdot, \cdot, \psi_2)).$$

It follows from (H18) that

$$|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)| \leq \|\rho\|_{L^\infty} \int_a^b |\psi_1(\mu, v') - \psi_2(\mu, v')|\, dv' + \Lambda|(\psi_1 - \psi_2)|.$$

Using Hölder inequality, we get

$$|\mathcal{F}(\psi_1) - \mathcal{F}(\psi_2)| \leq \|\rho\|_{L^\infty} b^{\frac{1}{q}} \left(\int_a^b |\psi_1(\mu, v') - \psi_2(\mu, v')|^p\, dv' \right)^{\frac{1}{p}} + \Lambda|(\psi_1 - \psi_2)|.$$

A simple calculation using (H10) and (H11) gives

$$\begin{aligned} \|\mathcal{F}\psi_1 - \mathcal{F}\psi_2\|_p &\leq b\|\rho\|_{L^\infty}\|\psi_1 - \psi_2\|_p + \Lambda\|\psi_1 - \psi_2\|_p \\ &\leq C\|\psi_1 - \psi_2\|_p, \end{aligned}$$

where $C := b\|\rho\|_{L^\infty} + \Lambda$. ☐

Remark 5.2.3. We note that the use of Lemmas 5.2.2 and 5.2.1 implies

$$\begin{aligned}\|\mathcal{F}(\varphi_1) - \mathcal{F}(\varphi_2)\|_{p,\delta} &\le \|\mathcal{F}(\varphi_1) - \mathcal{F}(\varphi_2)\|_p \\ &\le C\|\varphi_1 - \varphi_2\|_p \\ &\le e^{\frac{\delta}{a}} C\|\varphi_1 - \varphi_2\|_{p,\delta},\end{aligned}$$

which means that $\mathcal{F} : \mathsf{X}_{p,\delta} \to \mathsf{X}_{p,\delta}$ is a globally Lipschitz mapping.

Taking $u(t) := \psi(t, \cdot) \in \mathsf{X}_p$, we may rewrite problem (5.48) as follows:

$$\begin{cases} u'(t) + \mathbf{T_R}(u(t)) = \mathcal{F}(u(t)), \\ u(0) = \psi_0 \in \mathsf{X}_p. \end{cases} \tag{5.59}$$

Theorem 5.2.1. *If assumptions* (H9)–(H11) *are satisfied, then*
(a) *problem* (5.59) *has a unique mild solution,*
(b) *if $p > 1$, this mild solution is, in fact, a weak solution,*
(c) *if $p > 1$ and the initial data $\psi_0 \in D(\mathbf{T_R})$, then the solution is a strong solution.*

Proof. Propositions 5.2.1 and 5.2.2 show that the operator $\mathbf{T_R} : D(\mathbf{T_R}) \subset \mathsf{X}_{p,\delta} \to \mathsf{X}_{p,\delta}$ is m-quasiaccretive. As Lemma 5.2.2 and Remark 5.2.3 yield that the operator $\mathcal{F} : \mathsf{X}_{p,\delta} \to \mathsf{X}_{p,\delta}$ is globally Lipschitz, by Theorem 4.6.6, we conclude that $\mathbf{T_R} - \mathcal{F}$ is also m-quasiaccretive on $\mathsf{X}_{p,\delta}$. Thus, assertion (a) follows from Theorem 4.6.3(a).

(b) To prove assertion (b), it suffices to observe that, for $1 < p < \infty$, the space $\mathsf{X}_{p,\delta}$ has Radon–Nikodym property. Hence the use of Corollary 4.6.1 gives the required result.

(c) If further $\psi_0 \in D(\mathbf{T_R})$, using again Corollary 4.6.1, we conclude that the weak solution is a strong solution. □

The next result shows that the solution depends continuously on the initial data. To this end, we introduce the Banach space $C([0, T], \mathsf{X}_{p,\delta})$ endowed with its usual norm

$$\|u\|_\infty := \sup\{\|u(t)\|_{p,\delta} : 0 \le t \le T\}.$$

Proposition 5.2.3. *Let $u_1, u_2 \in C([0, T], \mathsf{X}_{p,\delta})$, with $p \ge 1$, be two mild solutions of problem* (5.59). *Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|u_1(0) - u_2(0)\|_{p,\delta} \le \delta$, then $\|u_1 - u_2\|_\infty \le \varepsilon$.*

Proof. Since $\mathbf{T_R}$ is a δ-m accretive operator on $\mathsf{X}_{p,\delta}$ and $\mathcal{F} : \mathsf{X}_{p,\delta} \to \mathsf{X}_{p,\delta}$ is r-Lipschitz, where $r := e^{\frac{\delta}{a}} C$ (see Remark 5.2.3), it is easy to see that $\mathbf{T_R} - \mathcal{F}$ is a $(\delta + r)$-m-accretive operator on $\mathsf{X}_{p,\delta}$ (use Theorem 4.6.6). So, if $u_1, u_2 \in C([0, T], \mathsf{X}_{p,\delta})$ are two mild solutions of problem

$$\begin{cases} u'(t) + \mathbf{T_R}(u(t)) - \mathcal{F}(u(t)) = 0, \\ u(0) = u_i(0) \in \mathsf{X}_{p,\delta}, \end{cases}$$

then, by equation (4.11), we conclude that

$$\|u_1(t) - u_2(t)\|_{p,\delta} \leq e^{(\delta+r)t}\|u_1(0) - u_2(0)\|_{p,\delta}.$$

Since the above inequality is valid for every $t \in [0, T]$, we have

$$\|u_1 - u_2\|_\infty \leq e^{(\delta+r)T}\|u_1(0) - u_2(0)\|_{p,\delta}.$$

To end the proof, it suffices to take $\delta = \frac{\varepsilon}{e^{(\delta+r)T}}$. ☐

5.2.3 Nonlocal boundary conditions

In this section we are concerned with the existence and uniqueness result for problem

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,\mu,v) + v\frac{\partial}{\partial \mu}\psi(t,\mu,v) = -\sigma_1(\mu,v,\langle\psi\rangle)\psi + \int_a^b \kappa_1(\mu,v,v',\psi(t,\mu,v'))\,dv', \\ \psi(0,\mu,v) = \psi_0(\mu,v), \\ \psi(t,0,v) = \int_a^b \kappa_2(v,v',\langle\psi\rangle,\psi(t,1,v'))\,dv' + \sigma_2(v,\langle\psi\rangle,\psi(t,1,v)), \end{cases} \tag{5.60}$$

where

$$\langle\psi\rangle(t) := \int_0^1\int_a^b \psi(t,\mu,v)\,d\mu\,dv.$$

The quantity $\langle\psi\rangle$ denotes the mass of the population.

We now introduce the following hypotheses:

(H12) The function $\sigma_1(\cdot,\cdot,\cdot)$ is measurable and such that for any $\alpha > 0$, there exists $\Lambda_\alpha > 0$ such that

$$|\sigma_1(\mu,v,\eta_1) - \sigma_1(\mu,v,\eta_2)| \leq \Lambda_\alpha|\eta_1 - \eta_2|,$$

for a. e. $(\mu,v) \in [0,1] \times [a,b]$, $\eta_1,\eta_2 \in [-\alpha,\alpha]$.

(H13) The functions $\sigma_2(\cdot,\cdot,\cdot)$ and $\kappa_2(\cdot,\cdot,\cdot,\cdot)$ are measurable and such that there exists a constant $\Lambda > 0$ for which

$$|\sigma_2(v,\eta_1,\xi_1) - \sigma_2(v,\eta_2,\xi_2)| \leq \Lambda(|\eta_1 - \eta_2| + |\xi_1 - \xi_2|),$$
$$|\kappa_2(v,v',\eta_1,\xi_1) - \kappa_2(v,v',\eta_2,\xi_2)| \leq \Lambda(|\eta_1 - \eta_2| + |\xi_1 - \xi_2|),$$

for a. e. $(v,v') \in [a,b] \times [a,b]$, $\eta_1,\eta_2 \in \mathbb{R}$ and $\xi_1,\xi_2 \in \mathbb{R}$.

(H14) There exist two constants $\underline{\sigma}_1 \in \mathbb{R}$ and $\overline{\sigma}_1 > 0$ such that

$$\underline{\sigma}_1 \leq \sigma_1(\mu,v,\eta) \leq \overline{\sigma}_1,$$

for a. e. $(\mu,v) \in [0,1] \times [a,b]$, $\eta \geq 0$.

(H15) The function $\kappa_1(\cdot,\cdot,\cdot,\cdot)$ is measurable and satisfies

$$|\kappa_1(\mu,v,v',\psi_1(\mu,v')) - \kappa_1(\mu,v,v',\psi_2(\mu,v'))| \leq |\rho(\mu,v,v')||\psi_1 - \psi_2|(\mu,v'),$$

where $\psi_1, \psi_2 \in \mathsf{X}_p$ and $\rho(\cdot,\cdot,\cdot,) \in L^\infty([0,1]\times[a,b]\times[a,b], d\mu\,dv\,dv)$.

Let $\overline{\mathcal{B}}$ be the operator defined by

$$\overline{\mathcal{B}} : \mathsf{X}_p \to \mathsf{X}_p, \quad \psi \mapsto \int_a^b \kappa_1(\mu,v,v',\psi(\mu,v'))\,dv'$$

and set

$$\overline{\mathcal{F}}(\psi) := \mathcal{B}^1(\psi) - \sigma_1(\cdot,\cdot,\langle\psi\rangle)\psi, \quad \psi \in \mathsf{X}_p. \tag{5.61}$$

We define the free streaming operator $\mathcal{T}_{\mathcal{R}}$ with domain including the nonlocal boundary conditions

$$\begin{cases} \mathcal{T}_{\mathcal{R}} : D(\mathcal{T}_{\mathcal{R}}) \subset \mathsf{X}_p \longrightarrow \mathsf{X}_p, \\ \psi \mapsto \mathcal{T}_{\mathcal{R}}\psi(\mu,v) = v\frac{\partial\psi}{\partial\mu}(\mu,v), \\ D(\mathcal{T}_{\mathcal{R}}) = \{\psi \in \mathsf{W}_p \text{ such that } \gamma_0(\psi) = \mathcal{R}(\gamma_1\psi,\psi)\}, \end{cases}$$

where $\mathcal{R}$ denotes the boundary operator

$$\begin{cases} \mathcal{R} : \mathsf{Y}_p \times L^p(\Omega) \to \mathsf{Y}_p, \\ (u,\psi) \to \mathcal{R}(u,\psi), \end{cases}$$

and

$$\mathcal{R}(u,\psi)(v) = \int_a^b \kappa_2(v,v',\langle\psi\rangle,u(v'))v'\,dv' + \sigma_2(v,\langle\psi\rangle,u(v)). \tag{5.62}$$

It is well known that $-\mathcal{T}_0$ (i. e., $\kappa_2(\cdot,\cdot,\cdot) \equiv 0$ and $\sigma_2(\cdot,\cdot,\cdot) \equiv 0$) generates a C_0-semigroup of contractions on X_p and that, for all $\lambda > 0$, the operator $(I + \lambda\mathcal{T}_0)^{-1}$ is positive (in the lattice sense).

Lemma 5.2.3. *Let $p \in [1,+\infty)$ and assume that the hypotheses* (H12)–(H14) *are satisfied. Then,*

(i) *for each $\alpha > 0$, there exists a constant $C_\alpha > 0$ such that*

$$\|\overline{\mathcal{F}}(\psi_1) - \overline{\mathcal{F}}(\psi_2)\|_p \leq C_\alpha\|\psi_1 - \psi_2\|_p,$$

for all $\psi_1, \psi_2 \in \mathbb{B}_\alpha^{p,\delta} := \{u \in \mathsf{X}_{p,\delta} : \|u\|_{p,\delta} \leq \alpha\}$;

(ii) *there exists a constant* $C := C(p)$ *such that*

$$\big\|\mathcal{R}(u_1, \psi_1) - \mathcal{R}(u_2, \psi_2)\big\|_{(*,p)} \leq C(\|u_1 - u_2\|_{(*,p)} + \|\psi_1 - \psi_2\|_p),$$

for all $\psi_1, \psi_2 \in \mathsf{X}_p$ *and* $u_1, u_2 \in \mathsf{Y}_p$.

Proof. (i) For all $(\psi_1, \psi_2) \in (\mathbb{B}_a^{p,\delta})^2$, we have

$$\overline{\mathcal{F}}(\psi_1) - \overline{\mathcal{F}}(\psi_2) = \overline{\mathcal{B}}(\psi_1 - \psi_2) - (\sigma_1(\cdot, \cdot, \langle\psi_1\rangle)\psi_1 - \sigma_1(\cdot, \cdot, \langle\psi_2\rangle)\psi_2).$$

It follows from (H14) that

$$\begin{aligned}\big|\overline{\mathcal{F}}(\psi_1) - \overline{\mathcal{F}}(\psi_2)\big| &\leq \|\rho\|_{L^\infty} \int_a^b \big|\psi_1(\mu, v') - \psi_2(\mu, v')\big|\, dv' + \big|\sigma_1(\mu, v, \langle\psi_1\rangle)(\psi_1 - \psi_2)\big| \\ &\quad + \big|(\sigma_1(\mu, v, \langle\psi_1\rangle) - \sigma_1(\mu, v, \langle\psi_2\rangle))\psi_2\big|.\end{aligned}$$

Using Lemma 5.2.1, one sees that $|\langle\psi_i\rangle| \leq \|\psi_i\|_1 \leq \alpha e^{\frac{\delta}{a}}(b)^{\frac{1}{q}}$, for $i = 1, 2$, and $\|\psi_2\|_p \leq e^{\frac{\delta}{a}}\alpha$. So, a simple calculation using (H12) and (H14) gives

$$\begin{aligned}\big\|\overline{\mathcal{F}}(\psi_1) - \overline{\mathcal{F}}(\psi_2)\big\|_p &\leq b\|\rho\|_{L^\infty}\|\psi_1 - \psi_2\|_p + \overline{\sigma}_1\|\psi_1 - \psi_2\|_p + \alpha\Lambda_\beta e^{2\frac{\delta}{a}}\|\psi_1 - \psi_2\|_p \\ &\leq C_\alpha\|\psi_1 - \psi_2\|_p,\end{aligned}$$

where $C_\alpha := b\|\rho\|_{L^\infty} + \overline{\sigma}_1 + \alpha\Lambda_\beta e^{\frac{\delta}{a}}$ and $\beta = \alpha e^{\frac{\delta}{a}}(b)^{\frac{1}{q}}$.

(ii) Taking into account (5.62), we may write

$$\begin{aligned}&\big\|\mathcal{R}(u_1, \psi_1) - \mathcal{R}(u_2, \psi_2)\big\|_{(*,p)} \\ &\quad\leq \left(\int_a^b \left[\int_a^b |\kappa_2(v, v', \langle\psi_1\rangle, u_1(v')) - \kappa_2(v, v', \langle\psi_2\rangle, u_2(v'))| v'\, dv'\right]^p v\, dv\right)^{\frac{1}{p}} \\ &\qquad + \left(\int_a^b |\sigma_2(v, \langle\psi_1\rangle, u_1(v)) - \sigma_2(v, \langle\psi_2\rangle, u_2(v))|^p v\, dv\right)^{\frac{1}{p}} \\ &\quad= I_1 + I_2.\end{aligned}$$

Using the hypothesis (H13), we get

$$\begin{aligned}I_1 &= \left(\int_a^b \left[\int_a^b |\kappa_2(v, v', \langle\psi_1\rangle, u_1(v')) - \kappa_2(v, v', \langle\psi_2\rangle, u_2(v'))| v'\, dv'\right]^p v\, dv\right)^{\frac{1}{p}} \\ &\leq \Lambda\left(\frac{b^2}{2}\right)^{\frac{1}{p}} \left(\int_a^b |\langle\psi_1\rangle - \langle\psi_2\rangle| v\, dv + \int_a^b |u_1 - u_2| v\, dv\right)\end{aligned}$$

$$\leq \Lambda\left(\frac{b^2}{2}\right)^{\frac{1}{p}}\left(\frac{b^2}{2}\|\psi_1-\psi_2\|_1+\int_a^b |u_1-u_2| v\,dv\right).$$

It follows from Hölder inequality that

$$\int_a^b \big|u_1(v)-u_2(v)\big| v\,dv = \left(\frac{b^2}{2}\right)^{\frac{1}{q}}\|u_1-u_2\|_{(*,p)}.$$

Now using Lemma 5.2.1, we get

$$\|\psi_1-\psi_2\|_1 \leq (b)^{1/q}\|\psi_1-\psi_2\|_p. \tag{5.63}$$

Hence, we conclude that

$$\begin{aligned} I_1 &\leq \Lambda\left(\frac{b^2}{2}\right)^{\frac{1}{p}}\left[\left(\frac{b^2}{2}\right)(b)^{1/q}\|\psi_1-\psi_2\|_p+\left(\frac{b^2}{2}\right)^{\frac{1}{q}}\|u_1-u_2\|_{(*,p)}\right] \\ &\leq C_1\big(\|\psi_1-\psi_2\|_p+\|u_1-u_2\|_{(*,p)}\big), \end{aligned}$$

where $C_1 = \max(\Lambda(\frac{b^2}{2})^{1+\frac{1}{p}} b^{1/q}, \frac{b^2}{2}\Lambda)$.

On the other hand, assumption (H13), Eq. (5.63), and the estimate $(|a|+|b|)^p \leq 2^p(|a|^p+|b|^p)$ allow us to write

$$\begin{aligned} I_2 &\leq \left(\int_a^b \big|\sigma_2(v,\langle\psi_1\rangle,u_1(v))-\sigma_2(v,\langle\psi_2\rangle,u_2(v))\big|^p v\,dv\right)^{\frac{1}{p}} \\ &\leq 2\Lambda\left(\int_a^b \big|\langle\psi_1\rangle-\langle\psi_2\rangle\big|^p v\,dv+\int_a^b |u_1-u_2|^p v\,dv\right)^{1/p} \\ &\leq 2\left(\frac{b^2}{2}\right)^{1/p}\Lambda\|\psi_1-\psi_2\|_1+2\Lambda\|u_1-u_2\|_{(*,p)} \\ &\leq 2\left(\frac{b^2}{2}\right)^{1/p}\Lambda b^{1/q}\|\psi_1-\psi_2\|_p+2\Lambda\|u_1-u_2\|_{(*,p)} \\ &\leq C_2\big(\|\psi_1-\psi_2\|_p+\|u_1-u_2\|_{(*,p)}\big), \end{aligned}$$

where $C_2 = \max(2\Lambda, 2(\frac{b^2}{2})^{1/p}\Lambda b^{1/q})$. Putting $C = \max(C_1, C_2)$, we get

$$\big\|\mathcal{R}(u_1,\psi_1)-\mathcal{R}(u_2,\psi_2)\big\|_{(*,p)} \leq C\big\|(u_1,\psi_1)-(u_2,\psi_2)\big\|_{(*,p)\times L^p},$$

which concludes the proof. □

Remark 5.2.4. The result (i) in Lemma 5.2.3 remains true if $\|\psi_i\|_p \leq \alpha$.

The following lemma asserts that $\mathcal{T}_{\mathcal{R}}$ is quasiaccretive on $\mathsf{X}_{p,\delta}$ whenever $\delta \geq \max(0, b\ln(2C))$, where C is the constant of Lemma 5.2.3(ii).

Lemma 5.2.4. *Let $p \in [1, +\infty)$ and assume that the hypotheses* (H12)–(H14) *are satisfied. Then, there exist two constants $\omega(p) > 0$ and $\delta \geq 0$ such that*

$$\|(\psi_1 + \lambda\mathcal{T}_{\mathcal{R}}(\psi_1)) - (\psi_2 + \lambda\mathcal{T}_{\mathcal{R}}(\psi_2))\|_{p,\delta} \geq (1 - \lambda\omega(p))\|\psi_1 - \psi_2\|_{p,\delta}$$

for all $\lambda \in (0, \omega(p)^{-1})$ and $\psi_1, \psi_2 \in D(\mathcal{T}_{\mathcal{R}})$.

Proof. Let $\psi_1, \psi_2 \in D(\mathcal{T}_{\mathcal{R}})$. It suffices to prove that

$$\langle \mathcal{T}_{\mathcal{R}}(\psi_1) - \mathcal{T}_{\mathcal{R}}(\psi_2), \varphi_\delta \rangle_\delta \geq -\omega(p)\|\psi_1 - \psi_2\|_p$$

for some $\varphi_\delta \in J_\delta(\psi_1 - \psi_2)$ (J_δ stands for the duality mapping of the space $\mathsf{X}_{p,\delta}$). To this end, we assume that $\psi_1 \neq \psi_2$ and put $\omega(p) = \delta + \frac{1}{p}$ where δ is any nonnegative real number such that $\delta \geq b(\ln(2C))$. Let C be the constant given in Lemma 5.2.3(ii). Let us write $\psi = \psi_1 - \psi_2$ and

$$\varphi_\delta(\mu, v) = \|\psi\|_{p,\delta}^{1-p} \exp\left(-\frac{p\delta(1-\mu)}{v}\right)|\psi(\mu, v)|^{p-1} \operatorname{sgn}_0(\psi(\mu, v)).$$

An integration by parts yields

$$\begin{aligned}
\langle \mathcal{T}_{\mathcal{R}}(\psi_1) - \mathcal{T}_{\mathcal{R}}(\psi_2), \varphi_\delta \rangle_\delta &= \frac{\|\psi\|_{p,\delta}^{1-p}}{p} \int_0^1 \int_a^b \exp\left(-\frac{p\delta(1-\mu)}{v}\right) v \frac{\partial}{\partial\mu}|\psi(\mu, v)|^p \, d\mu \, dv \\
&= -\frac{\|\psi\|_{p,\delta}^{1-p}}{p} \left\{ \int_0^1 \int_a^b \frac{p\delta v}{v} \exp\left(-\frac{p\delta(1-\mu)}{v}\right)|\psi(\mu, v)|^p \, d\mu \, dv \right. \\
&\quad \left. + \int_a^b \exp\left(-\frac{p\delta}{v}\right)|\gamma_0(\psi)(v)|^p v \, dv - \int_a^b |\gamma_1(\psi)(v)|^p v \, dv \right\}.
\end{aligned}$$

Using Lemma 5.2.3, we obtain

$$\begin{aligned}
\langle \mathcal{T}_{\mathcal{R}}(\psi_1) - \mathcal{T}_{\mathcal{R}}(\psi_2), \varphi_\delta \rangle_\delta \geq &- \delta\|\psi\|_{p,\delta} - \frac{\|\psi\|_{p,\delta}^{1-p}}{p} \left\{ \exp\left(-\frac{p\delta}{b}\right) \right. \\
&\left. \times \|K(\gamma_1\psi_1, \psi_1) - K(\gamma_1\psi_2, \psi_2)\|_{(*,p)}^p - \|\gamma_1\psi_1 - \gamma_1\psi_2\|_{(*,p)}^p \right\} \\
\geq &- \delta\|\psi\|_{p,\delta} - \frac{1}{p}\left(2^p C^p \exp\left(-\frac{p\delta}{b}\right)\right)\|\psi\|_{p,\delta}^{1-p}\|\psi\|_p^p \\
&- \frac{1}{p}\left(2^p C^p \exp\left(-\frac{p\delta}{b}\right) - 1\right)\|\psi\|_{p,\delta}^{1-p}\|\gamma_1\psi_1 - \gamma_1\psi_2\|_{(*,p)}^p.
\end{aligned}$$

Since $2^p C^p e^{-\frac{p\delta}{b}} \leq 1$, applying Lemma 5.2.1, we conclude that

$$\langle \mathcal{T}_{\mathcal{R}}(\psi_1) - \mathcal{T}_{\mathcal{R}}(\psi_2), \varphi_\delta \rangle_\delta \geq -\omega(p) \|\psi_1 - \psi_2\|_{p,\delta},$$

which concludes the proof. □

In the following, we seek for a solution to the resolvent equation for the operator $\mathcal{T}_{\mathcal{R}}$, $(I + \lambda\mathcal{T}_{\mathcal{R}})(\psi) = \varphi$, where φ is a given function of X_p and the unknown ψ must be sought in $D(\mathcal{T}_{\mathcal{R}})$. So, for $\lambda > 0$, by the argument developed in the proof of Proposition 5.2.1, the solution is formally given by

$$\psi(\mu, v) = e^{-\frac{\mu}{\lambda v}} \mathcal{R}(\gamma_1(\psi), \psi)(v) + \frac{1}{\lambda v} \int_0^\mu e^{-\frac{\mu - \mu'}{\lambda v}} \varphi(\mu', v)\, d\mu'. \tag{5.64}$$

In particular, for $\mu = 1$, we get

$$\psi(1, v) := \gamma_1(\psi)(v) = e^{-\frac{1}{\lambda v}} \mathcal{R}(\gamma_1(\psi), \psi)(v) + \frac{1}{\lambda v} \int_0^1 e^{-\frac{1-\mu'}{\lambda v}} \varphi(\mu', v)\, d\mu'. \tag{5.65}$$

Using the fact that ψ must satisfy the boundary conditions, Eqs. (5.64) and (5.65) may be written as

$$\begin{cases} \psi = Q_\lambda \mathcal{R}(\gamma_1 \psi, \psi) + \Xi_\lambda \varphi, \\ \gamma_1 \psi = P_\lambda \mathcal{R}(\gamma_1 \psi, \psi) + \Pi_\lambda \varphi. \end{cases}$$

Therefore, $(\gamma_1 \psi, \psi)$ is a solution of the fixed point problem

$$(u, g) = \mathcal{J}_\lambda(u, g)$$

on the product space $\mathsf{Y}_p \times \mathsf{X}_p$, where

$$\mathcal{J}_\lambda(u, g) = (P_\lambda \mathcal{R}(u, g) + \Pi_\lambda \varphi,\ Q_\lambda \mathcal{R}(u, g) + \Xi_\lambda \varphi), \quad (u, g) \in \mathsf{Y}_p \times \mathsf{X}_p.$$

Lemma 5.2.5. *Let $p \in [1, +\infty)$ and assume that the hypotheses* (H13)–(H15) *are satisfied. Then there exists a constant $\lambda_p > 0$ such that, for each $\lambda \in (0, \lambda_p)$ and $\varphi \in \mathsf{X}_p$, there exists $\psi \in D(\mathcal{T}_{\mathcal{R}})$ such that*

$$\psi + \lambda \mathcal{T}_{\mathcal{R}}(\psi) = \varphi. \tag{5.66}$$

Proof. As we have seen above, in order to solve (5.66), it suffices to prove that the fixed point problem $\mathcal{J}_\lambda(u, \psi) = (u, \psi)$ has a unique solution in $\mathsf{Y}_p \times \mathsf{X}_p$ where

$$\mathcal{J}_\lambda(u, \psi) = (\mathcal{J}_\lambda^1, \mathcal{J}_\lambda^2)(u, \psi) = \big(P_\lambda \mathcal{R}(u, \psi) + \Pi_\lambda(\varphi), Q_\lambda \mathcal{R}(u, \psi) + \Xi_\lambda(\varphi)\big).$$

To see this, we equip $\mathsf{Y}_p \times \mathsf{X}_p$ with the norm $\|(u,\psi)\| = \|u\|_{(*,p)} + \|\psi\|_p$ and set

$$\lambda_p := \sup\{\lambda > 0 : (e^{-\frac{1}{b\lambda}} + \lambda^{\frac{1}{p}})C < 1\},$$

where C is the constant given in Lemma 5.2.3(ii), and we check that the operator $\mathcal{J}_\lambda$ is a contraction mapping. Indeed, let $(u_1,\psi_1),(u_2,\psi_2) \in \mathsf{Y}_p \times \mathsf{X}_p$. Using the estimates (5.49)–(5.52) and applying Lemma 5.2.3, we have

$$\begin{aligned}\|\mathcal{J}_\lambda(u_1,\psi_1) - \mathcal{J}_\lambda(u_2,\psi_2)\| &= \|\mathcal{J}_\lambda^1(u_1,\psi_1) - \mathcal{J}_\lambda^1(u_2,\psi_2)\|_{(*,p)} + \|\mathcal{J}_\lambda^2(u_1,\psi_1) - \mathcal{J}_\lambda^2(u_2,\psi_2)\|_p \\ &\le (e^{-\frac{1}{\lambda b}} + \lambda^{\frac{1}{p}})\|\mathbf{K}(u_1,\psi_1) - \mathbf{K}(u_2,\psi_2)\|_{(*,p)} \\ &\le (e^{-\frac{1}{\lambda b}} + \lambda^{\frac{1}{p}})C(\|u_1 - u_2\|_{(*,p)} + \|\psi_1 - \psi_2\|_p).\end{aligned}$$

Hence, by the definition of λ_p, $\mathcal{J}_{\lambda_p}$ is a λ_p-contraction mapping and therefore, by the Banach contraction principle, problem (5.66) has a unique solution in $D(\mathcal{T}_\mathcal{R})$. ☐

Remark 5.2.5. Lemmas 5.2.4 and 5.2.5 guarantee the existence of $\delta > 0$ such that $\mathcal{T}_\mathcal{R}$ becomes m-quasiaccretive on $\mathsf{X}_{p,\delta}$. Moreover, using the same argument as in the proof of Proposition 5.2.2, we obtain the density of the domain of $\mathcal{T}_\mathcal{R}$, that is, $\overline{D(\mathcal{T}_\mathcal{R})} = \mathsf{X}_p$.

Taking $u(t) := \psi(t,\cdot) \in \mathsf{X}_p$, we may interpret and rewrite problem (5.60) as follows:

$$\begin{cases} u'(t) + \mathcal{T}_\mathcal{R}(u(t)) = \overline{\mathcal{F}}(u(t)), \\ u(0) = \psi_0 \in \mathsf{X}_p. \end{cases} \tag{5.67}$$

Theorem 5.2.2. *Let $p \in (1,+\infty)$ and assume that the hypotheses* (H12)–(H15) *are satisfied. Then problem* (5.67) *has a local weak solution for each $\psi_0 \in \mathsf{X}_p$. Further, if $\psi_0 \in D(\mathcal{T}_\mathcal{R})$, then problem* (5.67) *has a local strong solution. Finally, if we suppose that $\kappa_1(\cdot,\cdot,\cdot,0) = 0$, then problem* (5.67) *has a unique global strong solution for each $\psi_0 \in D(\mathcal{T}_\mathcal{R})$ and it has a unique global weak solution if $\psi_0 \notin \mathcal{T}_\mathcal{R}$.*

Proof. According to Remark 5.2.5, $\mathcal{T}_\mathcal{R}$ is an m-quasiaccretive operator in $\mathsf{X}_{p,\delta}$. Since $\mathsf{X}_{p,\delta}$ is reflexive, by Lemma 5.2.3, $\overline{\mathcal{F}}$ is locally Lipschiz. Applying Theorem 4.6.7, we obtain the existence of a local strong solution to problem (5.67) whenever the initial data ψ_0 belongs to $D(\mathcal{T}_\mathcal{R})$.

Now we are going to study the existence of local weak solution when the initial data $\psi_0 \notin D(\mathcal{T}_\mathcal{R})$. Again, by Remark 5.2.5, the domain of $\mathcal{T}_\mathcal{R}$ is dense in X_p, so we may choose a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq D(\mathcal{T}_\mathcal{R})$ such that $\psi_n \to \psi_0$ as $n \to \infty$ and set $R := \sup\{\|\psi_n\|_{p,\delta} : n = 0,1,\dots\}$. Arguing as in the proof of Theorem 4.6.7, it is easy to see that there exists $T_R > 0$ such that, for each $n \in \mathbb{N}$, the problem

$$\begin{cases} u'(t) + \mathcal{T}_\mathcal{R}(u(t)) = \overline{\mathcal{F}}(u(t)), \\ u(0) = \psi_n \end{cases}$$

admits a unique strong solution, say u_n, on the interval $(0, T_R)$. To discuss the existence of a local weak solution, it suffices to see that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{C}(0, T_R; X_{p,\delta})$, which is a trivial consequence of inequality (4.11).

Now, to check the existence of a unique global solution, we argue as follows. Consider an element $\varphi_{\delta,1}$ belonging to the set

$$J(\psi) = \{w \in X^*_{p,\delta} : \langle \psi, w\rangle_\delta = \|\psi\|^2_{p,\delta} \text{ and } \|w\|_{(X_{p,\delta})^*} = \|\psi\|_{p,\delta}\}.$$

According to (H15),

$$\langle \sigma_1(\cdot,\cdot,\langle\psi\rangle)\psi, \varphi_{\delta,1}\rangle_\delta \geq \underline{\sigma}_1\|\psi\|^2_{p,\delta} \tag{5.68}$$

and

$$\langle \overline{\mathcal{B}}\psi, \varphi_{\delta,1}\rangle_\delta \leq \|\overline{\mathcal{B}}\psi\|_{p,\delta}\|\psi\|_{p,\delta} \leq \|\psi\|_{p,\delta}\|\overline{\mathcal{B}}\psi\|_p.$$

The use of the Hölder inequality gives

$$\begin{aligned} |\overline{\mathcal{B}}\psi| &\leq \int_a^b |\rho(\mu,\nu,\nu')\psi(\mu,\nu')|\, d\nu' + \left|\int_a^b \kappa_1(\mu,\nu,\nu',0)\, d\nu'\right| \\ &\leq b^{\frac{1}{q}}\|\rho\|_\infty\left(\int_a^b |\psi(\mu,\nu')|^p\, d\nu'\right)^{\frac{1}{p}} + \left|\int_a^b \kappa_1(\mu,\nu,\nu',0)\, d\nu'\right|. \end{aligned}$$

Therefore, Fubini theorem and Minkowski inequality lead to

$$\|\overline{\mathcal{B}}\psi\|_p \leq b\|\rho\|_\infty\|\psi\|_p + \|\overline{\mathcal{B}}(0)\|_p,$$

which proves that

$$\langle \overline{\mathcal{B}}\psi, \varphi_{\delta,1}\rangle_\delta \leq b\|\rho\|_\infty e^{\frac{\delta}{a}}\|\psi\|^2_{p,\delta} + \|\overline{\mathcal{B}}(0)\|_p\|\psi\|_{p,\delta}. \tag{5.69}$$

Thus, using (5.61)–(5.69), we infer that

$$\begin{aligned} \langle -\overline{\mathcal{F}}(\psi), \varphi_{\delta,1}\rangle_\delta &= \langle \sigma_1(\cdot,\cdot,\tilde{\psi})\psi, \varphi_{\delta,1}\rangle_\delta - \langle \overline{\mathcal{B}}\psi, \varphi_{\delta,1}\rangle_\delta \\ &\geq \underline{\sigma}_1\|\psi\|^2_{p,\delta} - b\|\rho\|_\infty e^{\frac{\delta}{a}}\|\psi\|^2_{p,\delta} - \|\overline{\mathcal{B}}(0)\|_p\|\psi\|_{p,\delta} \\ &= -(b\|\rho\|_\infty e^{\frac{\delta}{a}} - \underline{\sigma}_1)\|\psi\|^2_{p,\delta} - \|\overline{\mathcal{B}}(0)\|_p\|\psi\|_{p,\delta}. \end{aligned}$$

Finally, the assumption $\kappa_1(\cdot,\cdot,\cdot,0) = 0$ yields that $\overline{\mathcal{B}}(0) = 0$. Thus, we have proved that

$$s\langle -\overline{\mathcal{F}}(\psi), \varphi_{\delta,1}\rangle_\delta \geq -\alpha\|\psi\|^2_{p,\delta},$$

where $a := (b\|\rho\|_\infty e^{\frac{\delta}{a}} - \underline{\sigma}_1)$. Now the use of Theorem 4.6.7 ensures the existence of a unique global solution. ☐

Theorem 5.2.3. *Let* $p = 1$ *and assume that the hypotheses* (H12)–(H14) *are satisfied. Then problem* (5.67) *has a local mild solution for each* $\psi_0 \in \mathsf{X}_1$. *If, further, we add the conditions*

(a) $\kappa_1(\cdot,\cdot,\cdot,0) = 0$,

(b) $\kappa_2(\cdot,\cdot,0,0) = 0$ *and* $\sigma_2(\cdot,0,0) = 0$,

(c) $b\|\rho\|_\infty e^{\frac{\delta}{a}} \le \underline{\sigma}_1$,

then problem (5.67) *has a unique global mild solution for each* $\psi_0 \in \mathsf{X}_1$.

Proof. We first establish that problem (5.67) has a local mild solution. Indeed, let $\psi_0 \in \mathsf{X}_1$ and consider $\alpha > 1$ such that $\|\psi_0\|_{1,\delta} \le \alpha - 1$. Let $\rho(\cdot)$ be the radial retraction

$$\rho(\psi) = \begin{cases} \psi & \text{if } \|\psi\|_{1,\delta} \le \alpha, \\ \frac{\alpha}{\|\psi\|_{1,\delta}}\psi & \text{if } \|\psi\|_{1,\delta} \ge \alpha. \end{cases}$$

According to Lemma 5.2.3(i), $\overline{\mathcal{F}}(\rho(\cdot))$ is $2C_\alpha$-Lipschitz. Therefore, by Theorem 4.6.6, the operator $\mathcal{T}_{\mathcal{R}} - \overline{\mathcal{F}}(\rho(\cdot))$ is m-quasiaccretive in $\mathsf{X}_{1,\delta}$. Hence, by Theorem 4.6.3, we conclude that the problem

$$\begin{cases} u'(t) + \mathcal{T}_{\mathcal{R}}(u(t)) = \overline{\mathcal{F}}(\rho(u(t))), \\ u(0) = \psi_0 \end{cases}$$

has a unique mild solution u.

Now, since $D(\mathcal{T}_{\mathcal{R}})$ is dense in X_1, we can choose an element $y_0 \in D(\mathcal{T}_{\mathcal{R}})$ with $\|y_0\|_{1,\delta} \le \alpha$ and such that $\|y_0 - \psi\|_{1,\delta} \le \frac{1}{2}$. Next put $z_0 = \mathcal{T}_{\mathcal{R}}(y_0) - \overline{\mathcal{F}}(\rho(y_0)) = \mathcal{T}_{\mathcal{R}}(y_0) - \overline{\mathcal{F}}(y_0)$ and apply (4.11) to get

$$\begin{aligned} \|u(t) - y_0\|_{1,\delta} &\le e^{wt}\|\psi_0 - y_0\|_{1,\delta} + \int_s^t e^{w(t-\tau)}[-z_0, u(\tau) - y_0]_s \, d\tau \\ &\le e^{wt}\Big(\|\psi_0 - y_0\|_{1,\delta} + \|z_0\|_{1,\delta}\frac{1}{w}(1 - e^{-wt})\Big). \end{aligned}$$

This yields

$$\|u(t)\|_{1,\delta} \le \|y_0\|_{1,\delta} + 1 \le \alpha,$$

for $0 \le t \le T_\alpha$ with $T_\alpha > 0$ suitably chosen. This allows us to get the desired conclusion because it means that u is a mild solution to problem (3.9) in the interval $[0, T_\alpha]$.

As we saw in the proof of Theorem 5.2.2, conditions (a) and (c) yield

$$\langle \overline{\mathcal{F}}(\psi), \varphi_{1,\delta} \rangle_\delta \le 0.$$

On the other hand, condition (b) means that $\mathcal{R}(0,0) = 0$ and then we have that $\mathcal{S}_{\mathcal{R}}(0) = 0$. Thus, we may use Theorem 4.6.7 in order to derive the existence of a unique global mild solution. ☐

From a biological point of view, it seems quite natural that if the density of the population of the cells at mitosis is zero, then the transition for the distribution of mothers to daughters should be also zero, this is mathematically described by assuming that the transition operator $\mathcal{R}$ satisfies $\mathcal{R}(0) = 0$. When we assume that $\mathcal{R}(0) = 0$, we can replace condition (H11) by another conditions describing the rate of cell mortality. In this sense, let us consider

(a) $\sigma_1(\cdot,\cdot,\cdot)$ is as in (H12) and (H20),
(b) $\sigma(\mu, \nu, \psi(\mu, \nu)) = \sigma_1(\mu, \nu, \langle\psi\rangle)\psi$,
(c) $\kappa(\cdot,\cdot,\cdot,\cdot)$ satisfies condition (H11), $\kappa(\cdot,\cdot,\cdot,0) = 0$ and such that $b\|\rho\|_\infty e^{\frac{\delta}{a}} \le \underline{\sigma}_1$.

Checking carefully the proof of Lemma 5.2.3(i), we may show that the operator

$$\mathcal{F}(\psi) := \mathcal{B}\psi - \sigma(\cdot,\cdot,\psi)s, \quad \psi \in \mathsf{X}_p,$$

where

$$\mathcal{B} : \mathsf{X}_p \to \mathsf{X}_p, \quad \psi \mapsto \int_a^b \kappa(\mu, \nu, \nu', \psi(\mu, \nu'))\, d\nu',$$

is locally Lipschitz as well. Using the same argument developed in the last part of the proof of Theorem 5.2.2, we may prove that there exists $\tau > 0$ such that for every $\psi \in \mathsf{X}_{p,\delta}$ with $\|\psi\|_{p,\delta} \ge \tau$, one has $[\psi, \mathcal{F}(\psi)]_s \le 0$. Therefore, it seems quite natural to study problem (5.59) under the following conditions:

(H16) For every $s > 0$, there exists $L(s) > 0$ such that

$$\left\|\mathcal{F}(\varphi) - \mathcal{F}(\psi)\right\|_{p,\delta} \le L(s)\|\varphi - \psi\|_{p,\delta}$$

whenever $\varphi, \psi \in \mathbb{B}_s^{p,\delta}(0) := \{u \in L_{p,\delta} : \|u\|_{p,\delta} \le s\}$.

(H17) There exists $\tau > 0$ such that for every $\psi \in \mathsf{X}_{p,\delta}$ with $\|\psi\|_{p,\delta} \ge \tau$, one has $[\psi, \mathcal{F}(\psi)]_s \le 0$.

Theorem 5.2.4. *Let* $p \in [1, +\infty)$. *Assume that* $0 \in D(\mathbf{T_R})$, *i. e.,* $R(0) = 0$ *and the hypotheses* (H9), (H16), *and* (H17) *are satisfied. Then problem* (5.59) *has a unique mild solution.*

Proof. We claim that problem (5.59) has a unique mild solution. Indeed, take $g \in \mathsf{X}_{p,\delta}$ and consider $\tau > 0$ such that $\|g\|_{p,\delta} \le \tau$, with (H17) satisfied for this τ. We now introduce the function

$$\rho(x) = \begin{cases} x & \text{if } \|x\|_{p,\delta} \le \tau, \\ \frac{\tau}{\|x\|_{p,\delta}} x & \text{if } \|x\|_{p,\delta} > \tau. \end{cases}$$

It follows from (H16) and [81, p. 364] that the function $\mathcal{F}(\rho(\cdot))$ is $2L(\tau)$-Lipschitz. Therefore, by Theorem 4.6.3, the problem

$$\begin{cases} u'(t) + \mathbf{T_R}(u(t)) = \mathcal{F}(\rho(u(t))), \\ u(0) = g \end{cases} \tag{5.70}$$

has a unique mild solution u. Because we assume that $0 = \mathcal{S}_R(0)$ and $\mathcal{F}$ satisfies (H24), we may apply [111, Lemma 3.1] to get that $u \in \mathbb{B}_\tau^{p,\delta}(0) = \{u \in \mathsf{X}_{p,\delta} : \|u\|_{p,\delta} \le \tau\}$. So we have $\rho(u) = u$, which yields a unique mild solution. ☐

We end this section with the following result.

Proposition 5.2.4. *Let* $p \in [1, +\infty)$. *Suppose that* (H9), (H16), *and* (H17) *hold true and, for each* $i \in \{1, 2\}$, $u_i \in C([0, T]; X_{p,\delta})$ *is a mild solution of equation* (5.59) *with initial data* $u_i(0)$. *Then, for each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $\|u_1(0) - u_2(0)\|_{p,\delta} \le \delta$, *then* $\|u_1 - u_2\|_\infty \le \varepsilon$.

Proof. Since $\mathcal{S}_R$ is a δ-m accretive operator, by (4.11) we have

$$\|u_1(t) - u_2(t)\|_{p,\delta} \le e^{\delta t}\|u_1(0) - u_2(0)\|_{p,\delta} + \int_0^t e^{\delta(t-s)}\|\mathcal{F}(u_1(s)) - \mathcal{F}(u_2(s))\|_{p,\delta}\, ds.$$

The above inequality, along with the fact that $e^{-\delta s} \le 1$ for all $s \in [0, T]$, implies

$$\|u_1(t) - u_2(t)\|_{p,\delta} \le e^{\delta T}\left(\|u_1(0) - u_2(0)\|_{p,\delta} + \int_0^t \|\mathcal{F}(u_1(s)) - \mathcal{F}(u_2(s))\|_{p,\delta}\, ds \right).$$

Since, for each $i \in \{1, 2\}$, u_i is a continuous function, there exists $k > \tau$ (τ is given in (H17)) such that $\|u_i\|_\infty \le k$. In this case, by (H16), we know that $\mathcal{F}$ is $L(k)$-Lipschitz on $\mathbb{B}_k^{p,\delta}(0)$. Consequently, putting $c = e^{\delta T}$, we get

$$\|u_1(t) - u_2(t)\|_{p,\delta} \le c\|u_1(0) - u_2(0)\|_{p,\delta} + \int_0^t cL(k)\|u_1(s) - u_2(s)\|_{p,\delta}\, ds.$$

Now, for every $t \in [0, T]$, Gronwall inequality implies

$$\|u_1(t) - u_2(t)\|_{p,\delta} \le c\|u_1(0) - u_2(0)\|_{p,\delta} e^{cL(k)T},$$

which means that

$$\|u_1 - u_2\|_\infty \leq ce^{cL(k)T}\|u_1(0) - u_2(0)\|_{p,\delta}.$$

Consequently, taking $\delta = \frac{\varepsilon}{ce^{cL(k)T}}$, we complete the proof. □

5.3 A transport equation with delayed neutrons

The kinetic neutron transport equation describes the time evolution of a neutron population in interaction with the background matter [25, 82]. It was considered in different fields of mathematical physics to describe transport processes of particles. The literature devoted to various mathematical aspects deals with transport equations in nonmultiplying medium. (see, for example, [121, 131, 136, 180] and the references therein). The derivation of neutron transport equation in nonmultiplying medium is based on the physical assumption that the production of neutrons after a collision of neutrons with a nucleus of the host material is instantaneous. However, in a multiplying medium, the absorption of neutrons by a fissile nucleus may give rise to fragments which act as delayed neutrons emitters (precursors). In fact, a small fraction of neutrons is emitted later due to certain fission products. The origin of these delayed neutrons is in general attributed to β^--decay of highly excited fission products and their decay products. To ascertain the time-dependent behavior of a nuclear reactor, one must account for the emission of these so-called delayed neutrons, for the small fraction of neutrons that are delayed make the chain reaction far more sluggish under most conditions than would first appear. Thus, in multiplying media, precursors are grouped into few decay time groups and there are as many precursor equations as there are time groups to be considered. The emitted delayed neutrons appear as a source term at the rate of the precursor decay [25, 82]. The spectral theory and Cauchy problems governed by linear transport operators with delayed neutrons were discussed by many authors. We refer, for example, to the works [43, 61, 135, 157, 162, 168, 175, 180].

In this section we shall discuss the existence and uniqueness of solutions to the problem

$$\begin{cases} \frac{df_0}{dt}(t,x,v) + v \cdot \nabla_x f_0(t,x,v) \\ \quad = -\sigma(x,v,f_0(t,x,v)) + \int_{\mathsf{V}} \kappa_0(x,v,v',f_0(t,x,v'))\,d\mu(v') \\ \qquad + \sum_{i=1}^N \lambda_i \beta_i(x,v) f_i(t,x,v), \\ \frac{df_i}{dt}(t,x,v) + \lambda_i f_i(t,x,v) = \int_{\mathsf{V}} \kappa_i(x,v,v',f_0(t,x,v'))\,d\mu(v'), \quad 1 \leq i \leq N, \\ f_i(0,x,v) = f_{i0}(x,v), \quad i = 0,1,2,\dots,N, \\ f_{0|\Gamma_-} = \mathcal{H}(f_{0|\Gamma_+}), \end{cases} \tag{5.71}$$

where $t \geq 0$, $(x,v) \in \mathsf{D} \times \mathsf{V}$. Here D is a smooth open subset of $\mathbb{R}^N$, $\mu(\cdot)$ is a positive Radon measure on $\mathbb{R}^N$ such that $\mu(0) = 0$, and we denote by V its support (V is called the space of admissible velocities). The function f_0 represents the neutron density, $f_i(t,x,v)$ represents the density of the ith group of delayed neutron emitters and $\{\lambda_i, 1 \leq i \leq N\}$ are radioactive decay constants. The functions $\sigma(\cdot,\cdot,\cdot)$ and $\kappa_i(\cdot,\cdot,\cdot,\cdot), i = 0,1,\ldots,N$, are nonlinear functions of the density of neutrons, f_0, called, respectively, the collision frequency and scattering kernels while $\beta_i(x,v), i = 1,2,\ldots,N$, are essentially bounded functions which denote physical parameters related to the ith group of delayed neutrons; $f^0 = (f_{00}, f_{01}, \ldots, f_{0N})^\perp$ stands for initial data. Here H denotes a bounded linear operator from a suitable function space on Γ_+ to a similar one on Γ_- which models the boundary conditions. The known classical boundary conditions (vacuum boundary specular reflections, diffuse reflections, periodic and mixed type boundary conditions) are special examples of this framework.

5.3.1 The functional setting of the problem and preliminaries

Let D be a smooth open subset of $\mathbb{R}^N$ and let μ be a positive Radon measure on $\mathbb{R}^N$. We denote by V the support of μ and we refer to V as the space of admissible velocities. We note that the boundary of the phase space is written as $\partial \mathrm{D} \times \mathrm{V} := \Gamma_- \cup \Gamma_+$ where

$$\Gamma_\pm = \{(x,v) \in \partial \mathrm{D} \times \mathrm{V} : \pm v \cdot \nu_x > 0\}.$$

Here ν_x stands for the outer unit normal vector at $x \in \partial \mathrm{D}$.

Letting $p \in [1, +\infty)$, we use the notation

$$L^p(\mathrm{D} \times V) := L^p(\mathrm{D} \times V;\, dx\, d\mu(v)).$$

We now introduce the following functional space:

$$\mathrm{W}_p(\mathrm{D}) = \{\psi \in L^p(\mathrm{D} \times V) \text{ such that } v \cdot \nabla_x \psi \in L^p(\mathrm{D} \times V)\}.$$

It is well known (cf. [48, 56, 57] or [121]) that any function f in $\mathrm{W}_p(\mathrm{D})$ possesses traces $f_{|\Gamma_\pm}$ on $\Gamma_\pm$ belonging to $L^\pm_{p,\mathrm{loc}}(\Gamma_\pm; |v \cdot \nu_x|\, d\gamma_x\, d\mu(v))$, where $d\gamma_x$ denotes the Lebesgue measure on $\partial \mathrm{D}$. In applications, suitable L^p-spaces for traces on $\Gamma_\pm$ are

$$L^\pm_p := L^p(\Gamma_\pm : |v \cdot \nu_x|\, d\gamma_x\, d\mu(v)).$$

So, we define the set

$$\widetilde{\mathrm{W}}_p(\mathrm{D}) = \{f \in \mathrm{W}_p : f^- \in L^-_p\}.$$

It is well known that if $f \in \mathrm{W}_p(\mathrm{D})$ and $f_{|\Gamma_-} \in L^-_p$, then $f_{|\Gamma_+} \in L^+_p$, and vice versa [56, 57, 121]. More precisely, we have the identity

$$\widetilde{\mathrm{W}}_p(\mathrm{D}) = \{f \in \mathrm{W}_p : f_{|\Gamma_-} \in L^-_p\} = \{f \in \mathrm{W}_p : f_{|\Gamma_+} \in L^+_p\}.$$

We say that an operator H is a boundary operator if

$$H(f^+) = f^-, \quad D(H) = L_p^+, \quad \text{and} \quad H(L_p^+) \subseteq L_p^-.$$

Let $H \in \mathcal{L}(L_p^+, L_p^-)$ be a positive boundary operator (the positivity is taken in the lattice sense, that is, H transforms the positive cone of L_p^+ into the positive cone of L_p^-).

Define the operator T_H

$$\begin{cases} T_H : D(T_H) \subseteq L^p(\mathsf{D} \times V) \longrightarrow L^p(\mathsf{D} \times V), \\ T_H f(x, v) = -v \cdot \nabla_x f(x, v) \end{cases}$$

with domain

$$D(T_H) = \{f \in \widetilde{\mathsf{W}}_p(\mathsf{D}) \text{ such that } f^- = H(f^+)\}.$$

Remark 5.3.1. The operator T_H is usually called the free streaming operator. It is a closed densely defined linear operator. Its resolvent set $\rho(T_H)$ contains the half-plane

$$\mathbb{C}_+ = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda > 0\}$$

(see, for example, [151]).

For $i \in \{1, 2, \dots, N\}$, let Λ_i be the bounded multiplication operators from $L^p(\mathsf{D} \times V)$ into itself defined by $\Lambda_i f_i = \lambda_i \beta_i(x, v) f_i$ and define the matrix operator

$$\Lambda = \begin{pmatrix} 0 & \Lambda_1 & \Lambda_2 & \dots & \Lambda_d \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since the functions $\beta_i(\cdot, \cdot)$, $i = 1, \dots, N$, belong to $L^\infty(\mathsf{D} \times \mathsf{V}, dx\, d\mu(v))$, the operators Λ_i, $i = 1, \dots, N$ are bounded, and therefore Λ is also bounded on $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. Here $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$ denotes the product space $L^p(\mathsf{D} \times \mathsf{V}) \times L^p(\mathsf{D} \times \mathsf{V}) \times \cdots \times L^p(\mathsf{D} \times \mathsf{V})$ ($L^p(\mathsf{D} \times \mathsf{V})$ taken $N + 1$ times) equipped with the norm

$$\|f\| = \|(f_0, \dots, f_N)^\perp\| = \sum_{i=0}^{N} \|f_i\|_p.$$

Remark 5.3.2. Let us note that the boundedness of the operator Λ implies that it is a Lipschitz operator with Lipschitz constant $\|\Lambda\| := \|\Lambda\|_{(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}}$.

Define the matrix operator

$$A_H = \begin{pmatrix} -T_H & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 I & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_N I \end{pmatrix}.$$

The domain of A_H is $D(A_H) = D(T_H) \times (L^p(\mathsf{D} \times \mathsf{V}))^N$.

Remark 5.3.3. It is well known that $D(T_H)$ is dense in $L^p(\mathsf{D} \times \mathsf{V})$, so the domain of A_H is also dense in $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$.

A general assumption

Throughout this section we shall assume that

$$\|H\| \leq 1, \quad |\mathsf{D}| < +\infty, \quad \text{and} \quad \mu(\mathsf{V}) < \infty,$$

where $|\mathsf{D}|$ denotes the Lebesgue measure of D.

Lemma 5.3.1. *If $p \in (1, +\infty)$, then, for all $f \in L^p(\mathsf{D} \times V)$, we have*

$$\|f\|_{L^1(\mathsf{D}\times V)} \leq \big(|\mathsf{D}|\mu(\mathsf{V})\big)^{\frac{1}{q}} \|f\|_{L^p(\mathsf{D}\times V)},$$

where q denotes the conjugate exponent of p.

Proof. It is an immediate consequence of Hölder inequality. □

5.3.2 Existence and uniqueness results

We first consider the following initial boundary value problem:

$$\begin{cases} \frac{df_0}{dt}(t,x,v) + v \cdot \nabla_x f_0(t,x,v) \\ \quad = -\sigma(x,v,f_0(t,x,v)) + \int_{\mathsf{V}} \kappa_0(x,v,v',f_0(t,x,v'))\, d\mu(v') \\ \qquad + \sum_{i=1}^N \lambda_i \beta_i(x,v) f_i(t,x,v), \\ \frac{df_i}{dt}(t,x,v) + \lambda_i f_i(t,x,v) = \int_{\mathsf{V}} \kappa_i(x,v,v',f_0(t,x,v'))\, d\mu(v'), \quad 1 \leq i \leq N, \\ f_i(0,x,v) = f_{0i}(x,v), \quad i = 0,1,\dots,N, \text{ with } f^0 = (f_{00}, f_{01}, \dots, f_{0N})^{\perp}, \\ f_0^- = \mathcal{H}(f_0^+), \end{cases} \tag{5.72}$$

where $f^0 = (f_{01}, f_{01}, \dots, f_{0N})^{\perp}$ is the initial data, $\sigma(\cdot,\cdot,\cdot)$ and $\kappa_0(\cdot,\cdot,\cdot,\cdot)$ are nonlinear functions of the density of neutron f_0 and, for $i \in \{1,2,\dots,N\}$, $\kappa_i(\cdot,\cdot,\cdot,\cdot)$ is a nonlinear functions of the density of the ith group of delayed neutron emitters.

We now introduce the following hypotheses:

(H18) We assume that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$ and $\beta_i(x,v)$, $i = 0,1,\dots,N$, are essentially bounded. Further, there exists a real constant $\widetilde{\beta}$ such that

$$0 \le \beta_i(x,v) \le \widetilde{\beta} \quad \text{for } i = 1,2,\dots,N.$$

(H19) The function $\sigma(\cdot,\cdot,\cdot)$ is measurable and there exists a function $\rho \in L^\infty(\mathsf{D} \times \mathsf{V}, dx\, d\mu(v))$ such that, for all $f_1, f_2 \in L^p(\mathsf{D} \times \mathsf{V})$, we have

$$\left|\sigma(x,v,f_1) - \sigma(x,v,f_2)\right| \le \left|\rho(x,v)\right| |f_1 - f_2|.$$

(H20) For each $i \in \{0,1,\dots,N\}$, the function $\kappa_i(\cdot,\cdot,\cdot,\cdot)$ is measurable and there exists a function $\rho_i \in L^\infty(\mathsf{D} \times \mathsf{V} \times \mathsf{V}, dx\, d\mu(v)\, d\mu(v'))$ such that, for all $f_1, f_2 \in L^p(\mathsf{D} \times \mathsf{V})$, we have

$$\left|\kappa_i(x,v,v',f_1(x,v')) - \kappa_i(x,v,v',f_2(x,v'))\right| \le \left|\rho_i(x,v,v')\right| \left|f_1(x,v') - f_2(x,v')\right|.$$

Set

$$\begin{aligned}
\Pi_0(f_0) &= -\sigma(x,v,f_0(t,x,v)) + \int_{\mathsf{V}} \kappa_0(x,v,v',f_0(t,x,v'))\, d\mu(v'),\\
\Pi_i(f_0) &= \int_{\mathsf{V}} \kappa_i(x,v,v',f_0(t,x,v'))\, d\mu(v'), \quad i = 1,2,\dots,N,
\end{aligned}$$

and define the matrix operator Π on $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$ by

$$\Pi(f) = \begin{pmatrix} \Pi_0 & 0 & 0 & \dots & 0 \\ \Pi_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_N & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix}$$

for all $f = (f_0, f_1, \dots, f_N)^\perp \in (L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$.

With the above assumptions, problem (5.72) may be formulated as follows:

$$\begin{cases} f'(t) + A_H f(t) = \Lambda f(t) + \Pi(f(t)), \\ f_i(0,x,v) = f_{0i}(x,v), \quad i = 0,1,\dots,N, \\ \quad \text{with } f^0 = (f_{00}, f_{01}, \dots, f_{0N})^\perp. \end{cases} \tag{5.73}$$

Remark 5.3.4. As indicated above (cf. Remark 5.3.3), we have

$$D(A_H) = D(T_H) \times \left(L^p(\mathsf{D} \times \mathsf{V})\right)^N).$$

So, the initial data $f^0 = (f_{00}, f_{01}, \dots, f_{0N})^\perp$ belongs to $D(A_H)$ if and only if $f_{00} \in D(T_H)$.

Theorem 5.3.1. *Let* $1 \le p < \infty$. *If conditions* (H18)–(H20) *hold true and* Π *maps* $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$ *into itself, then problem* (5.73) *has a unique mild solution for all initial data* f^0 *belonging to* $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. *If* $1 < p < \infty$, *it is a weak solution. Moreover, if* $f^0 \in D(A_H)$, *then it is a strong solution.*

Next, we shall consider the case where the functions $\sigma_1(\cdot,\cdot,\cdot)$ and $\kappa_i^1(\cdot,\cdot,\cdot,\cdot)$, $i = 0, 1, \ldots, N$, depend on $\langle f_0 \rangle$, the total density of neutrons, that is,

$$\langle f_0 \rangle(t) := \int\limits_{\mathsf{D}} \int\limits_{\mathsf{V}} f_0(t,x,v)\, dx\, d\mu(v).$$

Remember that the density of neutrons is a nonnegative quantity. Hence we consider the following problem:

$$\begin{cases} \frac{df_0}{dt}(t,x,v) + v \cdot \nabla_x f_0(t,x,v) \\ \quad = -\sigma_1(x,v,\langle f_0 \rangle) f_0 + \int_{\mathsf{V}} \kappa_0^1(x,v,v',\langle f_0 \rangle) f_0(t,x,v')\, d\mu(v') \\ \qquad + \sum_{i=1}^{N} \lambda_i \beta_i(x,v) f_i(t,x,v), \\ \frac{df_i}{dt}(t,x,v) + \lambda_i f_i(t,x,v) = \int_{\mathsf{V}} \kappa_i^1(x,v,v',\langle f_0 \rangle) f_0(t,x,v')\, d\mu(v'), \quad 1 \le i \le N, \\ f_i(0,x,v) = f_{0i}(x,v), \quad i = 0,1,\ldots,N \text{ with } f^0 = (f_{00}, f_{01}, \ldots, f_{0N})^{\perp}, \\ f_0^- = \mathcal{H}(f_0^+), \end{cases} \tag{5.74}$$

where $f^0 = (f_{01}, f_{01}, \ldots, f_{0N})^{\perp}$ stands for the initial data.

We now introduce the following hypotheses:

(H21) The function $\sigma_1(\cdot,\cdot,\cdot)$ is measurable and, for any $\mathbf{r} > 0$, there exists $\Lambda_{\mathbf{r}} > 0$ such that

$$\left|\sigma_1(x,v,z_1) - \sigma_1(x,v,z_2)\right| \le \Lambda_{\mathbf{r}} |z_1 - z_2|$$

for all $(x,v) \in \mathsf{D} \times \mathsf{V}$ and $z_1, z_2 \in [-\mathbf{r}, \mathbf{r}]$.

(H22) There exist two constants $\underline{\sigma}_1 \in \mathbb{R}$ and $\overline{\sigma}_1 > 0$ such that

$$\underline{\sigma}_1 \le \sigma_1(x,v,z) \le \overline{\sigma}_1, \quad \forall (x,v,z) \in \mathsf{D} \times \mathsf{V} \times \mathbb{R}.$$

(H23) The functions $\kappa_i^1(\cdot,\cdot,\cdot,\cdot)$, $i = 0, \ldots, N$, are measurable and satisfy

$$\left|\kappa_i^1(x,v,v',z_1) - \kappa_i^1(x,v,v',z_2)\right| \le \left|\rho_i(x,v,v')\right| |z_1 - z_2|,$$

where $z_1, z_2 \in [-\mathbf{r}, \mathbf{r}]$ and $\rho_i \in L^{\infty}(\mathsf{D} \times \mathsf{V} \times \mathsf{V}, dx\, d\mu(v)\, d\mu(v'))$.

(H24) For each $i \in \{0, 1, \ldots, N\}$, there exist two constants $\underline{\kappa}_i^1 \in \mathbb{R}$ and $\overline{\kappa}_i^1 > 0$ such that

$$\underline{\kappa}_i^1 \le \kappa_i^1(x,v,v',z) \le \overline{\kappa}_i^1 \quad \text{for all } (x,v,v',z) \in \mathsf{D} \times \mathsf{V} \times \mathsf{V} \times \mathbb{R}.$$

Define the operators

$$\Pi_0^1(f_0) = -\sigma_1(x,v,\langle f_0\rangle)f_0 + \int_{\mathsf{V}} \kappa_0^1(x,v,v',\langle f_0\rangle)f_0(t,x,v')\,d\mu(v'),$$
$$\Pi_i^1(f_0) = \int_{\mathsf{V}} \kappa_i^1(x,v,v',\langle f_0\rangle)f_0(t,x,v')\,d\mu(v'), \quad i = 1,2,\dots,N,$$

and consider the operator $\Pi^1 : (L^p(\mathsf{D}\times\mathsf{V}))^{N+1} \to (L^p(\mathsf{D}\times\mathsf{V}))^{N+1}$ defined by

$$(L^p(\mathsf{D}\times\mathsf{V}))^{N+1} \ni f = (f_0,f_1,\dots,f_N) \mapsto \Pi^1(f) = \begin{pmatrix} \Pi_0^1 & 0 & 0 & \dots & 0 \\ \Pi_1^1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_N^1 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix}.$$

Thus, problem (5.74) may be written as

$$\begin{cases} f'(t) + A_H f(t) = \Lambda f(t) + \Pi^1(f(t)), \\ f_i(0,x,v) = f_{0i}(x,v), \quad i = 0,1,\dots,N, \\ \quad \text{with} f^0 = (f_{00},f_{01},\dots,f_{0N})^{\perp}. \end{cases} \tag{5.75}$$

For each $\mathbf{r} > 0$, set

$$\mathbb{B}_{\mathbf{r}}^p := \{f \in (L^p(\mathsf{D}\times\mathsf{V}))^{N+1} : \|f\| \le \mathbf{r}\}.$$

Theorem 5.3.2. *Let* $\mathbf{r} > 0$ *and assume that assumptions* (H18), (H21)–(H24) *are satisfied. Then the following items hold true:*

(i) *If* $p \in (1,+\infty)$, *then, for each* $f^0 \in L^p(\mathsf{D}\times\mathsf{V})$, *problem* (5.75) *has a local weak solution. If the initial data* f^0 *belongs to* $D(A_H)$, *then it has a local strong solution. Further, problem* (5.75) *has a unique global strong solution if* $f^0 \in D(A_H)$ *and a unique global weak solution if* $f^0 \notin D(A_H)$.

(ii) *If* $p = 1$, *then, for each* $f^0 \in L^p(\mathsf{D}\times\mathsf{V})$, *problem* (5.75) *has a local mild solution.*

5.3.3 Some lemmas

We now establish some lemmas required in the proofs of Theorems 5.3.1 and 5.3.2.

Lemma 5.3.2. *The operator* A_H *is m-accretive on* $(L^p(\mathsf{D}\times\mathsf{V}))^{N+1}$.

Proof. We first prove that A_H is accretive on $(L^p(\mathsf{D}\times\mathsf{V}))^{N+1}$. Let $g_1, g_2 \in D(A_H)$ and consider $f = (f_0,\dots,f_N) \in J_1(g_1 - g_2)$. If we write $g_1 - g_2 = (g_1^0 - g_2^0, g_1^1 - g_2^1, \dots, g_1^N - g_2^N)$, then, for $i = 0,\dots,N$, we have $f_i = \|g_1^i - g_2^i\|_p^{1-p} |g_1^i - g_2^i|^{p-1} \operatorname{sgn}_0(g_1^i - g_2^i)$. So, we can write

$$
\begin{aligned}
&[f, A_H(g_1) - A_H(g_2)]_s \\
&\quad \geq \|g_1^0 - g_2^0\|_p^{1-p} \int_{\mathsf{D}} \int_{\mathsf{V}} |g_1^0 - g_2^0|^{p-1} v \cdot \nabla_x((g_1^0 - g_2^0)(x,v)) \operatorname{sgn}_0(g_1^0 - g_2^0)\, dx\, d\mu(v) \\
&\qquad + \lambda_1 \|g_1^1 - g_2^1\|_p^{1-p} \int_{\mathsf{D}} \int_{\mathsf{V}} |g_1^1 - g_2^1|^{p-1}((g_1^1 - g_2^1)(x,v)) \operatorname{sgn}_0(g_1^1 - g_2^1)\, dx\, d\mu(v) \\
&\qquad + \cdots + \lambda_N \|g_N^1 - g_N^1\|_p^{1-p} \int_{\mathsf{D}} \int_{\mathsf{V}} |g_N^1 - g_N^1|^{p-1}((g_1^1 - g_2^1)(x,v)) \operatorname{sgn}_0(g_1^1 - g_2^1)\, dx\, d\mu(v) \\
&\quad = \|g_1^0 - g_2^0\|_p^{1-p} \frac{1}{p} \int_{\mathsf{D}} \int_{\mathsf{V}} v \cdot \nabla_x(|(g_1^0 - g_2^0)(x,v)|^p)\, dx\, d\mu(v) \\
&\qquad + \lambda_1 \|g_1^1 - g_2^1\|_p^{1-p} \int_{\mathsf{D}} \int_{\mathsf{V}} (|(g_1^1 - g_2^1)(x,v)|^p)\, dx\, d\mu(v) \\
&\qquad + \cdots + \lambda_N \|g_1^N - g_2^N\|_p^{1-p} \int_{\mathsf{D}} \int_{\mathsf{V}} (|(g_1^N - g_1^N)(x,v)|^p)\, dx\, d\mu(v).
\end{aligned}
$$

Using Green formula, the right-hand side of the last equation is greater or equal to

$$
\begin{aligned}
&\|g_1^0 - g_2^0\|_p^{1-p} \frac{1}{p} \Big(\int_{\Gamma^+} |(g_1^0 - g_2^0)(x,v)|^p v \cdot v_x\, d\gamma_x\, d\mu(v) - \int_{\Gamma^-} |(g_1^0 - g_2^0)(x,v)|^p v \cdot v_x\, d\gamma_x\, d\mu(v) \Big) \\
&\qquad + \lambda_1 \|g_1^1 - g_2^1\|_p + \lambda_2 \|g_1^2 - g_2^2\|_p + \cdots + \lambda_N \|g_1^N - g_2^N\|_p \\
&\quad = \|g_1^0 - g_2^0\|_p^{1-p} \frac{1}{p} (\|(g_1^0 - g_2^0)^+\|_{L^{p,+}}^p - \|(g_1^0 - g_2^0)^-\|_{L^{p,-}}^p) \\
&\qquad + \lambda_1 \|g_1^1 - g_2^1\|_p + \lambda_2 \|g_1^2 - g_2^2\|_p + \cdots + \lambda_N \|g_1^N - g_2^N\|_p \\
&\quad = \|g_1^0 - g_2^0\|_p^{1-p} \frac{1}{p} (\|(g_1^0 - g_2^0)^+\|_{L^{p,+}}^p - \|H(g_1^0 - g_2^0)^+\|_{L^{p,+}}^p) \\
&\qquad + \lambda_1 \|g_1^1 - g_2^1\|_p + \lambda_2 \|g_1^2 - g_2^2\|_p + \cdots + \lambda_N \|g_1^N - g_2^N\|_p \\
&\quad \geq \|g_0^1 - g_0^2\|_p^{1-p} \frac{1}{p} ((1 - \|H\|^p) \|(g_0^1 - g_0^2)^+\|_{L^{p,+}}^p) \\
&\qquad + \lambda_1 \|g_1^1 - g_2^1\|_p + \lambda_2 \|g_1^2 - g_2^2\|_p + \cdots + \lambda_N \|g_1^N - g_2^N\|_p \geq 0
\end{aligned}
$$

(here we used the assumption $\|H\| \leq 1$). This proves that A_H is accretive on $(L^p(\mathsf{D}\times\mathsf{V}))^{N+1}$.

To complete the proof, it suffices to establish that $\mathsf{R}(I + A_H) = (L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. Let $(h_0, h_1, \dots, h_N) \in (L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. We search for an element $(f_0, f_1, \dots, f_N) \in D(A_H)$ such that

$$
\begin{pmatrix}
I - T_H & 0 & 0 & \dots & 0 \\
0 & I + \lambda_1 I & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & I + \lambda_N I
\end{pmatrix}
\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{pmatrix}
=
\begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_N \end{pmatrix},
$$

or equivalently, we look for a solution of the following system:

$$\begin{cases} f_0 - T_H f_0 = h_0, \\ f_i + \lambda_i f_i = h_i, \quad i = 1, \dots, N. \end{cases} \tag{5.76}$$

It is clear that, for $i = 1, \dots, N$, we have $f_i = \frac{h_i}{1+\lambda_i}$, $i = 1, \dots, N$. So, it remains to solve the equation

$$f_0(x, v) - T_H f_0(x, v) = h_0(x, v).$$

According to Remark 5.3.1(b), the latter equation has a unique solution because $1 \in \rho(T_H)$ (the resolvent set of T_H). This yields that $\mathsf{R}(I + A_H) = (L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$, which completes the proof. □

Lemma 5.3.3. *Assume that* (H19)–(H20) *are satisfied and* Π *maps* $(L^p(\mathsf{D}\times\mathsf{V}))^{N+1}$ *into itself. Then there exists a constant* $C_2 > 0$ *such that, for all* $g, h \in (L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$*, we have*

$$\|\Pi(g) - \Pi(h)\| \le C_2 \|g - h\|. \tag{5.77}$$

Proof. It is clear that, for all $g = (g_0, \dots, g_N)$ and $h = (h_0, \dots, h_N)$ in $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$, we have

$$\|\Pi(g) - \Pi(h)\| = \sum_{i=0}^{N} \|\Pi_i(g_0) - \Pi_i(h_0)\|_p. \tag{5.78}$$

So, it suffices to prove that, for each $i \in \{0, 1, \dots, N\}$, Π_i is a Lipschitz operator on $L^p(\mathsf{D}\times\mathsf{V})$.

Let $f_1, f_2 \in L^p(\mathsf{D} \times \mathsf{V})$. Using (H19) and (H20), together with Hölder inequality and the fact that $\mu(\mathsf{V}) < \infty$, we get

$$\begin{aligned} |\Pi_0(f_1) - \Pi_0(f_2)| &\le |\sigma(x, v, f_1(x, v)) - \sigma(x, v, f_2(x, v))| \\ &\quad + \int_{\mathsf{V}} |\kappa_0(x, v, v', f_1(x, v)) - \kappa_0(x, v, v', f_2(x, v'))| \, d\mu(v') \\ &\le \|\rho\|_\infty |f_1(x, v) - f_2(x, v)| + \|\rho_0\|_\infty \int_{\mathsf{V}} |f_1(x, v') - f_2(x, v')| \, d\mu(v') \\ &\le \|\rho\|_\infty |f_1(x, v) - f_2(x, v)| \\ &\quad + \|\rho_0\|_\infty \mu(\mathsf{V})^{\frac{1}{q}} \Big(\int_{\mathsf{V}} |f_1(x, v') - f_2(x, v')|^p \, d\mu(v') \Big)^{\frac{1}{p}}. \end{aligned}$$

Next, simple calculations using the estimate $(|a + b|)^p \le 2^p(a^p + b^p)$ lead to

$$
\begin{aligned}
&\int_{\mathsf{D}}\int_{\mathsf{V}} \left|\Pi_0(f_1)(x,v) - \Pi_0(f_2)(x,v)\right|^p \,dx\,d\mu(v)\\
&\quad\leq 2^p \|\rho\|_\infty^p \int_{\mathsf{D}}\int_{\mathsf{V}} |f_1(x,v) - f_2(x,v)|^p \,dx\,d\mu(v)\\
&\qquad + 2^p \|\rho_0\|_\infty^p \mu(\mathsf{V})^{\frac{p}{q}} \int_{\mathsf{D}}\int_{\mathsf{V}}\int_{\mathsf{V}} |f_1(x,v') - f_2(x,v')|^p \,dx\,d\mu(v')\,d\mu(v)\\
&\quad\leq 2^p \|\rho\|_\infty^p \|f_1 - f_2\|_p^p + 2^p \|\rho_0\|_\infty^p \mu(\mathsf{V})^{\frac{p}{q}+1} \|f_1 - f_2\|_p^p\\
&\quad\leq 2^p \left(\|\rho\|_\infty^p + \|\rho_0\|_\infty^p \mu(\mathsf{V})^p\right)\|f_1 - f_2\|_p^p.
\end{aligned}
$$

Thus,

$$
\left\|\Pi_0(f_1) - \Pi_0(f_2)\right\|_p \leq 2\left(\|\rho\|_\infty^p + \|\rho_0\|_\infty^p \mu(\mathsf{V})^p\right)^{\frac{1}{p}} \|f_1 - f_2\|_p.
$$

On the other hand, for each $i \in \{1,\dots,N\}$ and for all $f_1, f_2 \in L^p(\mathsf{D}\times\mathsf{V})$, we have

$$
\begin{aligned}
\left|\Pi_i(f_1) - \Pi_i(f_2)\right| &\leq \int_{\mathsf{V}} |\rho_i(x,v,v')|\,|f_1(x,v') - f_2(x,v')|\,d\mu(v')\\
&\leq \|\rho_i\|_\infty \mu(\mathsf{V})^{\frac{1}{q}} \left(\int_{\mathsf{V}} |f_1(x,v') - f_2(x,v')|^p \,d\mu(v')\right)^{\frac{1}{p}},
\end{aligned}
$$

and therefore,

$$
\left\|\Pi_i(f_1) - \Pi_i(f_2)\right\|_p \leq \|\rho_i\|_\infty \mu(\mathsf{V})^{1/q} \|f_1 - f_2\|_p.
$$

Hence, we obtain

$$
\left\|\Pi_j(g_0) - \Pi_j(h_0)\right\|_p \leq
\begin{cases}
2\left(\|\rho\|_\infty^p + \|\rho_0\|_\infty^p \mu(\mathsf{V})^p\right)^{\frac{1}{p}} \|g_0 - h_0\|_p, & \text{if } j = 0,\\
\|\rho_j\|_\infty \mu(\mathsf{V}) \|g_0 - h_0\|_p, & \text{if } j = 1,2,\dots,N.
\end{cases}
\tag{5.79}
$$

Now using the estimates (5.78), (5.79), and the fact that $\|g_0 - h_0\|_p \leq \|g - h\|$, we get

$$
\left\|\Pi(g) - \Pi(h)\right\| \leq C_2 \|g - h\|,
$$

where

$$
C_2 = 2\left(\|\rho\|_\infty^p + \|\rho_0\|_\infty^p \mu(\mathsf{V})^p\right)^{1/p} + \sum_{i=1}^{N} \|\rho_i\|_\infty \mu(\mathsf{V}).
$$

This ends the proof. □

Lemma 5.3.4. *Let* $\mathbf{r} > 0$, $p \in [1,+\infty)$, *and assume that* (H21)–(H22) *are satisfied. Then there exists a constant* $C_{\mathbf{r}} > 0$ *such that*

$$\|\Pi^1(g) - \Pi^1(h)\| \leq C_{\mathbf{r}} \|g - h\|$$

for all $g, h \in \mathbb{B}^p_{\mathbf{r}}$.

Proof. For all $g = (g_0, \dots, g_N)$ and $h = (h_0, \dots, h_N)$ in $B^p_{\mathbf{r}}$, we have

$$\|\Pi^1(g) - \Pi^1(h)\| = \sum_{i=0}^{N} \|\Pi^1_i(g_0) - \Pi^1_i(h_0)\|_p. \tag{5.80}$$

As in proof of the previous lemma, we just have to prove that, for each $i \in \{0, 1, \dots, N\}$, Π^1_i is a Lipschitz operator on $L^p(\mathsf{D} \times \mathsf{V})$. Let f_1 and f_2 be two elements of $L^p(\mathsf{D} \times \mathsf{V})$. Using (H21) and (H22), and the estimate $(|a + b|)^p \leq 2^p(|a|^p + |b|^p)$, we get

$$\begin{aligned}
&\left|\Pi^1_0(f_1) - \Pi^1_0(f_2)\right|^p \\
&\quad \leq 2^p \left|\sigma_1(x, v, \langle f_1\rangle)(f_1 - f_2) + (\sigma_1(x, v, \langle f_1\rangle) - \sigma_1(x, v, \langle f_2\rangle))f_2\right|^p \\
&\qquad + 2^p \Big(\int_{\mathsf{V}} \left|\kappa^1_0(x, v, v', \langle f_1\rangle)(f_1 - f_2) + (\kappa^1_0(x, v, v', \langle f_1\rangle) - \kappa^1_0(x, v, v', \langle f_2\rangle))f_2\right| d\mu(v') \Big)^p \\
&\quad \leq 2^p (2^p \left|\sigma_1(x, v, \langle f_1\rangle)(f_1 - f_2)\right|^p + 2^p \left|(\sigma_1(x, v, \langle f_1\rangle) - \sigma_1(x, v, \langle f_2\rangle))f_2\right|^p) \\
&\qquad + 2^p \Big(2^p \Big(\int_{\mathsf{V}} \left|\kappa^1_0(x, v, v', \langle f_1\rangle)(f_1 - f_2)\right| d\mu(v') \Big)^p \\
&\qquad + 2^p \Big(\int_{\mathsf{V}} \left|(\kappa^1_0(x, v, v', \langle f_1\rangle) - \kappa^1_0(x, v, v', \langle f_2\rangle))f_2\right| d\mu(v') \Big)^p \Big) \\
&\quad \leq 2^{2p} (\overline{\sigma}^p_1 |f_1 - f_2|^p + \Lambda^p_{\mathbf{r}} \left|\langle f_1\rangle - \langle f_2\rangle\right|^p |f_2|^p) \\
&\qquad + 2^{2p} (\overline{\kappa}^1_0)^p \Big(\int_{\mathsf{V}} |f_1(x, v') - f_2(x, v')| \, d\mu(v') \Big)^p \\
&\qquad + 2^{2p} \|\rho_0\|^p_\infty \Big(\int_{\mathsf{V}} |\langle f_1\rangle - \langle f_2\rangle| |f_2(x, v')| \, d\mu(v') \Big)^p.
\end{aligned}$$

Using Lemma 5.3.1 together with the assumption $\mu(V) < +\infty$ gives

$$\left|\langle f_1\rangle - \langle f_2\rangle\right| \leq \|f_1 - f_2\|_1 \leq \big(|\mathsf{D}|\mu(\mathsf{V})\big)^{\frac{1}{q}} \|f_1 - f_2\|_p. \tag{5.81}$$

Estimate (5.81) and Hölder inequality imply

$$\begin{aligned}
&\left|\Pi^1_0(f_1) - \Pi^1_0(f_2)\right|^p \\
&\quad \leq 2^{2p} (\overline{\sigma}^p_1 |f_1 - f_2|^p + \Lambda^p_{\mathbf{r}} (|\mathsf{D}|\mu(\mathsf{V}))^{\frac{p}{q}} \|f_1 - f_2\|^p_p |f_2|^p) \\
&\qquad + 2^{2p} (\overline{\kappa}^1_0)^p \Big(\int_{\mathsf{V}} d\mu(v') \Big)^{\frac{p}{q}} \int_{\mathsf{V}} |f_1(x, v') - f_2(x, v')|^p \, d\mu(v')
\end{aligned}$$

$$
\begin{aligned}
&+ 2^{2p}\|\rho_0\|_\infty^p(|\mathrm{D}|\mu(\mathrm{V}))^{\frac{p}{q}}\|f_1-f_2\|_p^p\Big(\int\limits_{\mathrm{V}} d\mu(v')\Big)^{\frac{p}{q}}\int\limits_{\mathrm{V}}|f_2(x,v')|^p\,d\mu(v')\\
&\leq 2^{2p}(\overline{\sigma}_1^p|f_1-f_2|^p+\Lambda_{\mathbf{r}}^p(|\mathrm{D}|\mu(\mathrm{V}))^{\frac{p}{q}}\|f_1-f_2\|_p^p|f_2|^p)\\
&\quad + 2^{2p}(\overline{\kappa}_0^1)^p\mu(\mathrm{V})^{\frac{p}{q}}\int\limits_{\mathrm{V}}|f_1(x,v')-f_2(x,v')|^p\,d\mu(v')\\
&\quad + 2^{2p}\|\rho_0\|_\infty^p(|\mathrm{D}|\mu(\mathrm{V}))^{\frac{p}{q}}\mu(\mathrm{V})^{\frac{p}{q}}\|f_1-f_2\|_p^p\int\limits_{\mathrm{V}}|f_2(x,v')|^p\,d\mu(v').
\end{aligned}
$$

Integrating over $\mathrm{D}\times\mathrm{V}$, one sees that

$$
\begin{aligned}
&\int\limits_{\mathrm{D}}\int\limits_{\mathrm{V}}|\Pi_0^1(f_1)(x,v')-\Pi_0^1(f_2)(x,v')|^p\,dx\,d\mu(v)\\
&\leq 2^{2p}\overline{\sigma}_1^p\int\limits_{\mathrm{D}}\int\limits_{\mathrm{V}}|f_1(x,v)-f_2(x,v)|^p\,dx\,d\mu(v)\\
&\quad + 2^{2p}\Lambda_{\mathbf{r}}^p(|\mathrm{D}|\mu(\mathrm{V}))^{\frac{p}{q}}\|f_1-f_2\|_p^p\int\limits_{\mathrm{D}}\int\limits_{\mathrm{V}}|f_2(x,v)|^p\,dx\,d\mu(v)\\
&\quad + 2^{2p}(\overline{\kappa}_0^1)^p\mu(\mathrm{V})^{\frac{p}{q}}\int\limits_{D}\int\limits_{\mathrm{V}}\Big(\int\limits_{\mathrm{V}}|f_1(x,v')-f_2(x,v')|^p\,d\mu(v')\Big)dx\,d\mu(v)\\
&\quad + 2^{2p}\|\rho_0\|_\infty^p(|\mathrm{D}|\mu(\mathrm{V}))^{\frac{p}{q}}\mu(V)^{\frac{p}{q}}\|f_1-f_2\|_p^p\int\limits_{\mathrm{D}}\int\limits_{\mathrm{V}}\Big(\int\limits_{\mathrm{V}}|f_2(x,v')|^p\,d\mu(v')\Big)dx\,d\mu(v)\\
&\leq 2^{2p}\overline{\sigma}_1^p\|f_1-f_2\|_p^p+2^{2p}\Lambda_{\mathbf{r}}^p(|\mathrm{D}|\mu(\mathrm{V}))^{\frac{p}{q}}\|f_1-f_2\|_p^p\|f_2\|_p^p\\
&\quad + 2^{2p}(\overline{\kappa}_0^1)^p\mu(\mathrm{V})^{\frac{p}{q}+1}\|f_1-f_2\|_p^p+2^{2p}\|\rho_0\|_\infty^p(|\mathrm{D}|\mu(\mathrm{V}))^{\frac{p}{q}}\mu(\mathrm{V})^p\|f_1-f_2\|_p^p\|f_2\|_p^p.
\end{aligned}
$$

Using the fact that f_2 belongs to $\mathbb{B}_{\mathbf{r}}^p$ (in fact, writing f_2 as $(0,0,f_2,0,\dots,0)$, it may be viewed as a function of $\mathbb{B}_{\mathbf{r}}^p$ with $\|(0,0,f_2,0,\dots,0)\|=\|f_2\|_p$), we get

$$
\begin{aligned}
&\int\limits_{\mathrm{D}}\int\limits_{\mathrm{V}}|\Pi_0^1(f_1)(x,v')-\Pi_0^1(f_2)(x,v')|^p\,dx\,d\mu(v)\\
&\leq 2^{2p}(\overline{\sigma}_1^p+\mathbf{r}^p\Lambda_{\mathbf{r}}^p|\mathrm{D}|^{p-1}\mu(\mathrm{V})^{p-1}+(\overline{\kappa}_0^1)^p\mu(\mathrm{V})^p+\mathbf{r}^p\|\rho_0\|_\infty^p|\mathrm{D}|^{p-1}\mu(\mathrm{V})^{2p-1})\|f_1-f_2\|_p^p.
\end{aligned}
$$

This yields that

$$\big\|\Pi_0^1(f_1)-\Pi_0^1(f_2)\big\|_p\leq\zeta_{\mathbf{r}}^0\|f_1-f_2\|_p, \tag{5.82}$$

where $\zeta_{\mathbf{r}}^0:=4(\overline{\sigma}_1^p+\mathbf{r}^p\Lambda_{\mathbf{r}}^p|\mathrm{D}|^{p-1}\mu(\mathrm{V})^{p-1}+(\overline{\kappa}_0^1)^p\mu(\mathrm{V})^p+\mathbf{r}^p\|\rho_0\|_\infty^p|\mathrm{D}|^{p-1}\mu(\mathrm{V})^{2p-1})^{\frac{1}{p}}$.

Similarly, for $i\in\{1,\dots,N\}$, the same calculations as above give

$$\big\|\Pi_i^1(f_1)-\Pi_i^1(f_2)\big\|_p\leq\zeta_{\mathbf{r}}^i\|f_1-f_2\|_p, \tag{5.83}$$

where $\zeta_{\mathbf{r}}^i=2\mu(\mathrm{V})((\overline{\kappa}_i^1)^p+\mathbf{r}^p\|\rho_i\|_\infty^p(|\mathrm{D}|\mu(\mathrm{V}))^{p-1})^{\frac{1}{p}}$.

Now, using the estimates (5.82), (5.83), and the fact that $\|g_0 - h_0\|_p \le \|g - h\|$, we conclude that, for all $j \in \{0, 1, \ldots, N\}$,

$$\left\|\Pi_j^1(g_0) - \Pi_j^1(h_0)\right\|_p \le \zeta_{\mathbf{r}}^j \|g - h\|. \tag{5.84}$$

Accordingly, the use of (5.80), gives

$$\left\|\Pi^1(g) - \Pi^1(h)\right\| \le C_{\mathbf{r}} \|g - h\|,$$

where $C_{\mathbf{r}} = \zeta_{\mathbf{r}}^0 + \sum_{i=1}^N \zeta_{\mathbf{r}}^i$. □

5.3.4 Proofs

Proof of Theorem 5.3.1. By Lemma 5.3.2 and Remark 5.3.3, we know that A_H is m-accretive on $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. On the other hand, Remark 5.3.2, together with Lemma 5.3.3, shows that $\Lambda + \Pi$ is $(\|\Lambda\| + C_2)$-Lipschitz on $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. Hence Theorem 4.6.6 guarantees that $A_H - (\Lambda + \Pi)$ is m-quasiaccretive on $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. Applying Theorem 4.6.3(ii), we conclude that problem (5.73) has a unique mild solution. Moreover, since the spaces $L^p(\mathsf{D} \times \mathsf{V})$, $1 < p < +\infty$, are Banach spaces with Radon–Nikodym property, applying Corollary 4.6.1, we infer that the obtained mild solution is a weak solution on $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. Next, if $f^0 \in D(A_H)$, then, by Theorem 4.6.5, this solution is a strong solution. □

Proof of Theorem 5.3.2. (i) Let $p \in (1, +\infty)$. We first note that, according to Remark 5.3.2 and Lemma 5.3.4, $\Lambda + \Pi^1$ is a locally Lipschitz operator on $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$. Because A_H is m-accretive on $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$ (use Lemma 5.3.2) and the spaces $L^p(\mathsf{D} \times \mathsf{V})$, $p \in (1, +\infty)$, possess Radon–Nikodym property, it follows from Theorem 4.6.7 that problem (5.75) has a local strong solution $f \in (L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$ whenever f^0 belongs to $D(A_H)$.

Next, assume that f^0 does not belong to $D(A_H)$. Since $D(A_H)$ is dense in $(L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$ (cf. Remark 5.3.3), there exists a sequence $(f_n)_{n \in \mathbb{N}}$ contained in $D(A_H)$ such that $f_n \to f^0$. Set $M := \sup\{\|f_n\|_p : n = 0, 1, \ldots\}$. A similar proof to that of Theorem 4.6.7 allows us to deduce that there exists a real $T_M > 0$ such that, for each $n \in \mathbb{N}$, the problem

$$\begin{cases} f'(t) + A_H f(t) = \Lambda f(t) + \Pi^1(f(t)), \\ f(0) = f_n \in (L^p(\mathsf{D} \times \mathsf{V}))^{N+1} \end{cases}$$

possesses a unique strong solution, say g_n, on the interval $(0, T_M)$. To prove the existence of a local weak solution, we have only to check that $(g_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $C(0, T_M; (L^p(\mathsf{D} \times \mathsf{V}))^{N+1})$. This follows immediately from the inequality (4.11).

Now we show the existence of a unique global strong solution to problem (5.75).

Let $f = (f_0, f_1, f_2, \ldots, f_N) \in (L^p(\mathsf{D} \times \mathsf{V}))^{N+1}$ and $g \in J(f)$. If we write $g = (g_0, g_1, \ldots, g^N)$, then, for $i = 0, \ldots, N$, we have $g_i = \|f\| \|f_i\|_p^{1-p} |f_i|^{p-1} \operatorname{sgn}_0(f_i)$. Further, it is clear that

$$\langle -(\Pi^1 + \Lambda)(f), g\rangle = -\sum_{i=0}^{N}\langle \Pi_i^1 f_i, g_i\rangle - \sum_{i=1}^{N}\langle \Lambda_i f_i, g_0\rangle. \tag{5.85}$$

We know that, for $i = 1, \dots, N$, Λ_i is a bounded linear operator on $L^p(\mathsf{D} \times \mathsf{V})$. Using (H18), we get $\|\Lambda_i f_i\| \le \lambda_i \|\beta_i\|_\infty \|f_i\| \le \lambda_i \widetilde{\beta}\|f_i\|$. Since $\|g_0\|_q = \|f\|$, we infer that $\langle \Lambda_i f_i, g_0\rangle \le \lambda_i \widetilde{\beta}\|f_i\|\|f\|$. Once again, by (H24), we conclude that

$$-\sum_{i=1}^{N}\langle \Lambda_i f_i, g_0\rangle \ge -\lambda_N \widetilde{\beta}\|f\|[\|f_1\| + \|f_2\| + \cdots + \|f_N\|] \ge -\lambda_N \widetilde{\beta}\|f\|^2,$$

and therefore

$$-\sum_{i=1}^{N}\langle \Lambda_i f_i, g_0\rangle \ge -\lambda_N \widetilde{\beta}\|f\|^2. \tag{5.86}$$

We recall that the operator Π_0^1 is defined by

$$\Pi_0^1(f_0) = -\sigma_1(x, v, \langle f_0\rangle)f_0 + \int_{\mathsf{V}} \kappa_0^1(x, v, v', \langle f_0\rangle) f_0(t, x, v')\, d\mu(v').$$

Using (H22), it is not difficult to check that

$$\left|\langle \sigma_1(x, v, \langle f_0\rangle)f_0, g_0\rangle\right| \le \overline{\sigma}_1\langle f_0, g_0\rangle = \overline{\sigma}_1\|f_0\|\|f\| \le \overline{\sigma}_1\|f\|^2.$$

Thus,

$$\langle \sigma_1(x, v, \langle f_0\rangle)f_0, g_0\rangle \ge -\overline{\sigma}_1\|f\|^2. \tag{5.87}$$

Putting $B_0(f(x,v)) := \int_{\mathsf{V}} \kappa_0^1(x, v, v', \langle f_0\rangle) f_0(t, x, v')\, d\mu(v')$ and using assumption (H24), we get

$$\begin{aligned} \left|B_0(f_0(x,v))\right| &\le \int_{\mathsf{V}} \left|\kappa_0^1(x, v, v', \langle f_0\rangle)\right|\left|f_0(t, x, v')\right| d\mu(v') \\ &\le \overline{\kappa_0}\int_{\mathsf{V}} \left|f_0(t, x, v')\right| d\mu(v') \\ &\le \overline{\kappa_0}\mu(\mathsf{V})^{\frac{1}{q}}\left(\int_{\mathsf{V}} \left|f_0(t, x, v')\right|^p d\mu(v')\right)^{\frac{1}{p}}, \end{aligned}$$

and therefore

$$\|B_0 f_0\|_p \le \overline{\kappa_0}\mu(\mathsf{V})\|f_0\|_p \le \overline{\kappa_0}\mu(\mathsf{V})\|f\|. \tag{5.88}$$

Since the operators Π_i^1 have the same structure as B_0, similar calculations as above give

$$\left\|\Pi_i^1 f_0\right\|_p \leq \overline{\kappa_i}\mu(\mathsf{V})\|f\| \quad \text{for } i = 1, \ldots, N. \tag{5.89}$$

Next, using (5.88) and (5.89), we get

$$\langle B_0 f_0, g_0\rangle \leq \|B_0 f_0\|_p \|g_0\| \leq \overline{\kappa_0}\mu(\mathsf{V})\|f\|^2 \tag{5.90}$$

and, for $i = 1, \ldots, N$,

$$\langle \Pi_i^1 f_0, g_i\rangle \leq \left\|\Pi_i^1 f_0\right\|_p \|g_i\| \leq \overline{\kappa_i}\mu(\mathsf{V})\|f\|^2. \tag{5.91}$$

Finally, taking into account (5.85), (5.86), (5.86), (5.90), and (5.91), we conclude that

$$\langle -(\Pi^1 + \Lambda)(f), g\rangle \geq (-\lambda_N \widetilde{\beta} - \overline{\sigma}_1)\|f\|^2 - \mu(\mathsf{V})\left(\sum_{i=0}^{N} \overline{\kappa_i}\right)\|f\|^2.$$

Thus we have proved that

$$\langle -(\Pi^1 + \Lambda)(f), g\rangle \geq -\gamma\|f\|^2,$$

where $\gamma = \lambda_N \widetilde{\beta} + \overline{\sigma}_1 + \mu(\mathsf{V})(\sum_{i=0}^{N} \overline{\kappa_i})$. Now applying Theorem 4.6.7, we conclude that f is a global strong solution.

(ii) By hypothesis, $f^0 \in (L^1(\mathsf{D} \times \mathsf{V}))^{N+1}$. Let $\mathbf{r} > 1$ be a real number such that $|||f^0|||_1 < \mathbf{r} - 1$ and consider the radial retraction mapping of $(L^1(\mathsf{D} \times \mathsf{V}))^{N+1}$ on the ball $\mathbb{B}_{\mathbf{r}}$,

$$R(g) = \begin{cases} g & \text{if } |||g|||_1 \leq \mathbf{r}, \\ \frac{\mathbf{r}}{|||g|||_1} g & \text{if } |||g|||_1 > \mathbf{r}. \end{cases}$$

We know from Lemma 5.3.2 that A_H is m-accretive. So, using Remark 5.3.2 and Lemma 5.3.4, together with the estimate (1.10), we conclude that $\Lambda(R(\cdot)) + \Pi^1(R(\cdot))$ is $2(\|\Lambda\| + C_r)$-Lipschitz on $(L^1(\mathsf{D} \times \mathsf{V}))^{N+1}$. Consequently, the operator $A_H - \Lambda(R(\cdot)) - \Pi^1(R(\cdot))$ is $2(C_\Lambda + C_r)$-m-accretive on $(L^1(\mathsf{D} \times \mathsf{V}))^{N+1}$. Now applying Theorem 4.6.3(i), we infer that the problem

$$\begin{cases} f'(t) + A_H(f(t)) = \Lambda(R(f(t))) + \Pi^1(R(f(t))), \\ f_0(0) = f^0 \end{cases}$$

has a unique mild solution f on $(L^1(\mathsf{D} \times \mathsf{V}))^{N+1}$.

Next, using the fact that $D(A_H)$ is dense in $(L^1(\mathsf{D} \times \mathsf{V}))^{N+1}$, we can choose $h^0 \in D(A_H)$ such that $\|h^0 - f^0\|_1 \leq 1/2$. It is clear that $h^0 \in \mathbb{B}^1_{\mathbf{r}}$. Set

$$\eta^0 = A_H(h^0) - \Lambda(R(h^0)) - \Pi^1(R(h^0)) = A_H(h^0) - \Lambda(h^0) - \Pi^1(h^0).$$

Now, putting $\omega := 2(\|\Lambda\| + C_r)$ and applying Lemma 4.5.1, we obtain

$$\begin{aligned}|||f(t) - h^0|||_1 &\le e^{\omega t}|||h^0 - f^0|||_1 + \int\limits_0^t e^{\omega(t-s)}[-\eta^0, f(s) - h^0]_s\,ds \\ &\le e^{\omega t}\Big(|||h^0 - f^0|||_1 + \frac{1}{\omega}|||\eta_0|||_1(1 - e^{\omega t})\Big),\end{aligned}$$

and therefore

$$|||f(t)|||_1 \le |||h^0|||_1 + 1 < \mathbf{r}$$

for $0 \le t \le T_{\mathbf{r}}$, where $T_{\mathbf{r}} > 0$ is suitably chosen. This shows that $f(\cdot)$ is a mild solution to problem (5.75) in the interval $[0, T_{\mathbf{r}}]$, which ends the proof. ☐

5.4 Bibliographical remarks

The material presented in this chapter consists essentially of the work of the authors and their collaborators during the last ten years.

In Sections 5.1–5.1.2 we discussed the existence and uniqueness of solutions to nonlinear evolution problems derived from a model describing the growth of a cell population due to J. L. Lebowitz and S. I. Rubinow. Section 5.1 deals with case where the maximum cell cycle length ℓ_2 is finite, while in Section 5.1.2 we suppose that the maximum cell cycle length ℓ_2 is infinite. The results of these two sections were taken from the papers [12, 14, 23, 100].

In Section 5.2 a nonlinear evolution problem derived from a transport equation describing the growth of a cell population due to R. Rotenberg was considered. The results of this section may be found in [104] and we followed the presentation given in this paper.

In Section 5.3 we discussed the existence and uniqueness of solutions to a nonlinear evolution problem derived from transport equations with delayed neutrons. We followed the presentation given in the paper [15].

6 Metric fixed point theorems

6.1 Introduction

In metric fixed point theory, we study results that involve properties of an essentially isometric nature. The division between the metric fixed point theory and the more general topological theory is often a vague one. The use of successive approximations to establish the existence and uniqueness of solutions of differential equations in the nineteenth century, it goes back to A. L. Cauchy, J. Liouville, G. Peano, E. I. Fredholm, and, especially, E. Picard. In 1922, S. Banach established his remarkable fixed point theorem known as the Banach contraction principle which is one of the most important results of analysis. He placed the ideas underlying the method of successive approximations into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations.

Since the 1950s, the Banach contraction principle had numerous generalizations. We quote in particular the works by E. Rakotch [198], V. M. Sehgal [215, 216], D. W. Boyd and J. S. W. Wong [44], M. Edelstein [90, 91], L. Janos [128, 129], R. Kannan [133, 134], A. Meir and E. Keeler [174], S. Reich [199], S. P. Singh [219, 220], L. B. Ćirić [65–67], Y. Liu and Z. Li [164], and T. A. Burton [54] (see also [152]). The subject is currently very dynamic and, especially after the works by T. Suzuki [225, 226] and D. Wardowski [238], it has seen many new developments.

After this major breakthrough, it is not surprising that some researches approached the study of the existence of fixed points for nonexpansive mappings (Lipschitz mappings with Lipschitz constant 1) as an extension of the corresponding problem for contractions. These mappings can obviously be viewed as a natural extension of contraction mappings. However, the fixed point problem for nonexpansive mappings differs greatly from that of the contraction mappings in the sense that additional structure of the domain set is needed to insure the existence of fixed points. The first positive results for nonexpansive mappings were obtained by F. E. Browder [50], D. Göhde [117], and W. A. Kirk [142] in 1965.

This chapter is primarily intended to serve as an introduction to metric fixed point theory and the text is self-contained. In terms of content, this chapter overlaps in many places with the following books on fixed point theory by K. Goebel and W. A. Kirk [115], by J. Dugundji and A. Granas [87], by M. A. Khamsi and W. A. Kirk [141], and by K. Latrach [152]. However, Sections 6.6–6.8 are recent.

6.2 The Banach contraction principle

As said above, in 1922 S. Banach established a remarkable fixed point theorem known as the Banach contraction principle (BCP) which is one of the most important results

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of analysis. The Banach contraction principle and most of its generalizations are very important in diverse disciplines of mathematics, statistics, engineering, and economics.

We begin with the following definition.

Definition 6.2.1. Let X be a topological space. A function $f : X \to X$ is said to have a fixed point if there exists an element z of X such that $f(z) = z$. The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$.

Definition 6.2.2. Let (X, d) be a metric space. A map $f : X \to X$ is said to be Lipschitz if there exists a constant $k \geq 0$ such that

$$d(f(x), f(y)) \leq kd(x, y) \quad \text{for all } x, y \in X. \tag{6.1}$$

We call the Lipschitz constant of f the smallest real k for which (6.1) holds, that is,

$$k(f) := \inf\{k > 0 : d(f(x), f(y)) \leq kd(x, y) \text{ for all } x, y \in X\}.$$

A Lipschitz map is called contractive (or a contraction or $k(f)$-contractive) if $k(f) < 1$. If there is no confusion, we denote $k(f)$ by k.

Obviously, all Lipschitz maps and, in particular, contractive maps are continuous.

Let X be a metric space and f a k-contractive map on X. By induction, if f^m denotes the composition of f with itself m times, then

$$d(f^m(x), f^m(y)) \leq k^m d(x, y).$$

If f is a k-contractive map, then, by the triangle inequality, we obtain

$$d(x, y) \leq d(x, f(x)) + d(f(x), f(y)) + d(f(y), y),$$

so $(1 - k)d(x, y) \leq d(x, f(x)) + d(f(y), y)$. Since $k < 1$, for all $x, y \in X$, we have

$$d(x, y) \leq \frac{1}{1 - k}(d(x, f(x)) + d(y, f(y))). \tag{6.2}$$

It is clear that if f is a k-contractive map for some $k \in [0, 1)$ on a metric space X and if x and y are two fixed points of f, then (6.2) implies that $d(x, y) = 0$ or again $x = y$, hence:

Lemma 6.2.1. *A contractive map can have at most one fixed point.*

Proposition 6.2.1. *If $f : X \to X$ is a k-contractive map (for some $k \in [0, 1)$), then, for any x in X, the sequence $(f^n(x))_{n\in\mathbb{N}}$ of iterates of x under f is a Cauchy sequence.*

Proof. Taking $x_1 = f^n(x)$ and $x_2 = f^m(x)$ in the estimate (6.2), we get

$$\begin{aligned} d(x_1, x_2) &\le \frac{1}{1-k}\big(d(f^n(x), f^n(f(x))) + d(f^m(x), f^m(f(x)))\big) \\ &\le \frac{k^n + k^m}{1-k} d(x, f(x)). \end{aligned} \tag{6.3}$$

Since $k < 1$, $k^n \to 0$ as $n \to +\infty$, one has $d(f^n(x), f^m(x)) \to 0$, as n and m tend to infinity. ☐

The following classical result is due to Banach in his PhD thesis (published in 1922). It is known as the Banach contraction principle (BCP).

Theorem 6.2.1 (Banach). *If (X, d) is a complete metric space and $f : X \to X$ is a k-contractive map (for some $k \in [0,1)$), then f has a unique fixed point z and, for any x in X, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to z. Moreover, for all $n \in \mathbb{N}$, we have*

$$d(f^n(x), z) \le \frac{k^n}{1-k} d(x, f(x)). \tag{6.4}$$

This result requires the hypothesis $k < 1$. If $k = 1$, then f does not necessarily possess a fixed point. Indeed, taking $X = \mathbb{R}$ and $f(x) = x + 2$, we have $k = 1$ but f does not possess a fixed point.

Proof. The uniqueness follows from Lemma 6.2.1. Let $x \in X$, we define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_1 = f(x), \cdots, x_{n+1} = f(x_n), \ldots$ It follows from Proposition 6.2.1 that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, $(x_n)_{n\in\mathbb{N}}$ converges to a point $z \in X$. Next, using the continuity of f, we conclude that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(z)$, so that $z = f(z)$. This proves that z is a fixed point of f. Now, by letting $m \to +\infty$ in the inequality (6.3), we obtain (6.4). ☐

Example 6.2.1. Let $f : [0,1] \longrightarrow [0,1]$ be the map defined by

$$f(x) = \begin{cases} 2x + 1/2, & x \in [0, 1/4), \\ 1/2, & x \in [1/4, 1]. \end{cases}$$

It is clear that f is not continuous on $[0,1]$ but it possesses an unique fixed point, that is, $z = 1/2$. We also notice that, for all $x \in [0,1]$, $f^2(x) = \frac{1}{2}$ and thus f^2 is contractive.

Lipschitz maps which are not necessarily contractive appear in some applications, while some power of such maps may become contractive. The following corollary takes into account this situation and gives an existence and uniqueness result of a fixed point under more general conditions than those of Theorem 6.2.1.

Corollary 6.2.1. *Let f be a map defined on a complete metric space X. If there exists an integer $n \ge 2$ such that f^n is a contraction on X, then f has a unique fixed point.*

We note that, in this corollary, the map f is not assumed to be continuous (see Example 6.2.1).

Proof. According to Theorem 6.2.1, f^n has a unique fixed point $z \in X$ and $z = \lim_{k\to+\infty}(f^n)^k(x_0)$ where x_0 is an arbitrary point in X. Since $f^n(z) = z$, we have $f(z) = f(f^n(z)) = f^{n+1}(z) = f^n(f(z))$. This proves that $f(z)$ is also a fixed point of f^n. By the uniqueness of the fixed point of f^n, we conclude that $f(z) = z$. Hence, z is a fixed point of f. To establish the uniqueness, we suppose that $z' \in X$ is another fixed point of f. By iteration we obtain $f^n(z') = z'$, which shows that z' is also a fixed point of f^n. The uniqueness of the fixed point of f^n ensures that $z' = z$. ☐

Example 6.2.2. Consider the metric space $X = C(I)$ of continuous real-valued functions defined on the compact interval $I = [a, b]$. Equipped with the supremum norm, X is a Banach space. Let $f : X \to X$ be a mapping defined by

$$f(u)(t) = \int_a^t u(s)\,ds.$$

For each $u, v \in X$, we have $\|f(u) - f(v)\| \le (b-a)\|u - v\|$. On the other hand,

$$f^2(u)(t) = \int_a^t \left(\int_0^s u(\tau)d\tau \right) ds,$$

and easy calculations show

$$\left|f^2(u)(t) - f^2(v)(s)\right| \le \|u - v\| \int_a^t (s-a)\,ds = \frac{(t-a)^2}{2}\|u - v\|.$$

Therefore

$$\left\|f^2(u) - f^2(v)\right\| \le \frac{(b-a)^2}{2}\|u - v\|.$$

Inductively, we have

$$\left\|f^n(u) - f^n(v)\right\| \le \frac{(b-a)^n}{n!}\|u - v\|.$$

Hence, f^n is a contraction map for all values of n for which $\frac{(b-a)^n}{n!} < 1$. Thus, according to Corollary 6.2.1, f has $u = 0$ as a unique fixed point.

It may happen that a mapping $f : X \to X$ is not a contraction on the whole space X, but rather a contraction on some neighborhood of a given point. In this case we have the following local version of Banach contraction principle.

Proposition 6.2.2. *Let (X,d) be a complete metric space, U an open subset of X, and $f : U \to X$ a map. Suppose*

(a) *f is k-contractive for some $k \in [0,1)$,*
(b) *there exists $u \in U$ such that $d(u, X\backslash U) > \beta$, $\beta > 0$,*
(c) *$d(u, f(u)) < \beta(1-k)$.*

Then f has a unique fixed point $z \in U$ and $d(u,z) < \beta$.

Proof. We first note that $f(u) \in U$ because $d(u,f(u)) < \beta(1-k) < \beta < d(u, X\backslash U)$. Define a sequence $(u_n)_{n\in\mathbb{N}}$ by $u_0 = u, \dots, u_{n+1} = f(u_n), \dots$ We show that, for $n \in \mathbb{N}$, $u_n \in U$. It is clear that u_0 and u_1 are in U. Assume that $u_0, \dots, u_n$ are in U. We have $d(u, f(u_n)) \le d(u,u_1) + d(u_1,u_2) + \cdots + d(u_n, u_{n+1})$. But it is clear that $d(u_p, f(u_p)) \le d(f^p(u), f^p(u_1)) < k^p\beta(1-k)$. Hence, for $0 \le p \le n$, we have

$$d(u_p, f(u_p)) = d(f^p(u), f^p(u_1)) < k^p\beta(1-k)$$

and then

$$d(u, f(u_n)) < \beta(1-k)(1 + k + k^2 + \cdots + k^n) = \beta(1-k^{n+1}) < \beta.$$

We infer that $u_{n+1} = f(u_n) \in U$ and, by induction, we have $u_n \in U$ for all $n \in \mathbb{N}$. The same reasoning shows that

$$d(u_n, u_{n+q}) \le \beta(1-k)(k^n + \cdots + k^{n+q-1}) = \beta(1-k)k^n\frac{1-k^q}{1-k} < \beta k^n.$$

Hence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in U. Because X is complete, it converges to some point $z \in X$. The limit belongs to U because, by letting q go to $+\infty$, we obtain $d(u_n, z) < \beta k^p < \beta$. Moreover, by letting n tend to $+\infty$, we get $d(u,z) < \beta$.

The point z is a fixed point of f. Indeed, since f is continuous, from the equality $u_{n+1} = f(u_n)$ we deduce by letting n towards $+\infty$ that $z = f(z)$.

If z' is also a fixed point of f, then $d(z,z') = d(f(z), f(z')) < kd(z,z')$, which is a contradiction. Hence, $z = z'$. □

As an immediate consequence of this proposition, we have

Corollary 6.2.2. *Let (X,d) be a complete metric space, τ a point of X, and $f : \mathbb{B}_r(\tau) \to X$ a mapping where $r > 0$. Suppose*

(a) *f is k-contractive ($k \in (0,1)$),*
(b) *$d(\tau, f(\tau)) < r(1-k)$.*

Then f has a unique fixed point in $\mathbb{B}_r(\tau)$.

Remark 6.2.1. It is worth to mention that the BCP does not characterize the metric completeness of (X, d). Indeed, E. H. Connell [69] gave an example of a metric space X such that X is not complete and every contractive map on X has a fixed point.

6.3 Some generalizations of the BCP

The Banach contraction principle is fundamental in fixed point theory and it is a simple tool in establishing existence and uniqueness results for functional and operator equations. This motivated researchers to try to extend and generalize Theorem 6.2.1 in such a way that its area of applications be as large as possible. It has been extended to some larger classes of mappings by replacing the contractiveness condition by weaker conditions of various type. There are thousands of theorems which assure the existence of a fixed point of a self-mapping f of a complete metric space X. A comparative study of some of these results obtained before 1977 has been made by B. E. Rhoades [202] (we also refer to [143, Chapter 1]). Nevertheless, here we shall only present two generalizations that will be used in later chapters.

6.3.1 Nonlinear contraction mappings

Definition 6.3.1. Let (X, d) be a metric space. A mapping $f : X \to X$ is called a nonlinear contraction if there exists a function $\psi : \mathbb{R}^+ \to [0, +\infty)$ such that

$$d(f(x), f(y)) \leq \psi(d(x, y)) \quad \forall x, y \in X,$$

where ψ is upper semicontinuous from the right (that is, for any sequence $t_n \downarrow t \geq 0$, one has $\limsup_{n\to+\infty} \psi(t_n) \leq \psi(t)$) and satisfies $0 \leq \psi(t) < t$ for $t > 0$.

Remark 6.3.1. We note that if $\psi(r) = ar$, $0 < a < 1$, then f is a contraction. So, any contraction is a nonlinear contraction, however, the converse is, in general, not true.

The following result, due to D. W. Boyd and J. S. W. Wong [44], is the analog of the Banach contraction principle for this kind of mappings. It is very useful for establishing existence and uniqueness of solutions for nonlinear differential and integral equations.

Theorem 6.3.1. *Let (X, d) be a complete metric space and $f : X \to X$ a map. If f is a nonlinear contraction, then f has a unique fixed point $z \in X$. In addition, for each $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to z.*

Proof. Let x_0 be an arbitrary element of X and define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_n = f(x_{n-1}) = f^n(x_0)$, for all $n \geq 1$. Set $a_n = d(x_{n-1}, x_n)$. Note that the sequence $(a_n)_{n\in\mathbb{N}}$ is monotone decreasing and bounded below, thus it is convergent and we let $\lim_{n\to+\infty} a_n = a$. If $a > 0$, then we obtain a contradiction. Indeed, $a_{n+1} \leq \psi(a_n)$, and thus $a \leq \psi(a)$, which contradicts the fact that $\psi(r) < r$ for all $r > 0$. Therefore $(a_n)_{n\in\mathbb{N}}$ converges to 0. Now, we show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that this is not the case. Then, there exists $\varepsilon > 0$ such that, for any $k \in \mathbb{N}$, there are $m_k > n_k \geq k$ such that we have the relation

$$d_k = d(x_{m_k}, x_{n_k}) \geq \varepsilon.$$

On the other hand,

$$d_k \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \leq 2a_k + \psi(d_k).$$

It follows that $\varepsilon \leq \psi(\varepsilon)$, a contradiction. Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, $x_n \to z \in X$. Thus by the continuity of f, we get $f(z) = z$. The uniqueness of z follows from the contractivity condition. ☐

As an immediate consequence of this theorem, we have

Corollary 6.3.1. *Let (X, d) be a complete metric space and $f : X \to X$ a map. If there exists an integer $n \geq 2$ such that f^n is a nonlinear contraction, then f has a unique fixed point.*

In [171], J. Matkowski extended Theorem 6.3.1 by assuming that ψ is continuous at 0 and that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \searrow 0$ and $\psi(t_n) < t_n$. The following result is another variation of Theorem 6.3.1 obtained by J. Matkowski [170].

Theorem 6.3.2. *Let (X, d) be a complete metric space and $f : X \to X$ a map satisfying*

$$d(f(x), f(y)) \leq \psi(d(x, y)) \quad \text{for all } x, y \in X,$$

where $\psi : (0, +\infty) \to (0, +\infty)$ is monotone nondecreasing and satisfies, for all $t > 0$, $\lim_{n\to+\infty} \psi^n(t) = 0$. Then f has a unique fixed point.

6.3.2 Separate contraction mappings

Definition 6.3.2. Let (X, d) be a metric space. A function $f : X \to X$ is said to be a separate contraction if there exist two functions $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying
(a) $\psi(0) = 0$, ψ is strictly increasing,
(b) $d(f(x), f(y)) \leq \varphi(d(x, y))$,
(c) $\psi(r) \leq r - \varphi(r)$ for all $r > 0$.

This class of functions was introduced in [164]. Note that each separate contraction mapping is continuous and it is not difficult to see that contraction maps are separate contractions.

Theorem 6.3.3. *Let (X, d) be a complete metric space and let $f : X \to X$ be a separate contraction mapping. Then f has a unique fixed point in X.*

Proof. Let x_0 be an arbitrary (but fixed) point in X and $(x_n)_{n\in\mathbb{N}}$ the sequence of iterates of x_0 by f. Let $(a_n)_{n\in\mathbb{N}}$ be the real sequence defined by

$$a_n = d(f^n(x_0), f^{n+1}(x_0)).$$

It is clear that $0 \le a_{n+1} \le \varphi(a_n) < a_n \le \varphi(a_{n-1})$ for any $n > 0$. From this, we see that both $(a_n)_{n\in\mathbb{N}}$ and $(\varphi(a_n))_{n\in\mathbb{N}}$ are decreasing positive sequences and $\lim_{n\to+\infty}(\varphi(a_n) - a_n) = 0$. Letting $\lim_{n\to+\infty} a_n = a$, we have $\psi(a) \le \psi(a_n) \le a_n - \varphi(a_n) \to 0$ as $n \to +\infty$, and so $a = 0$. Because, for all $r > 0$, $\varphi(r) < r$ (use condition (c) of Definition 6.3.2), we suppose that, for fixed $\varepsilon > 0$, $\varepsilon^* = \sup_{0\le r\le\varepsilon} \varphi(r) < \varepsilon$. Now we choose a sufficiently large number N such that $a_N \le \varepsilon - \varepsilon^*$. In order to show that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, we prove that the set $\{x : d(x, x_N) \le \varepsilon\}$ is invariant under f. Indeed, for all $z \in \{x : d(x, x_N) \le \varepsilon\}$, we have

$$\begin{aligned} d(f(z), x_N) &\le d(f(z), f(x_N)) + d(f(x_N), x_N) \le a_N + \varphi(d(z, x_N)) \\ &\le \varepsilon^* + \varepsilon - \varepsilon^* = \varepsilon. \end{aligned}$$

Hence $\{x : d(x, x_N) \le \varepsilon\}$ is invariant under f and then the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n\to+\infty} x_n = z$. The continuity of f ensures that z is a fixed point of f.

If z' is another fixed point of f, then $d(z, z') \le \varphi(d(z, z'))$. Since $\psi(d(z, z')) \le d(z, z') - \varphi(d(z, z'))$, we deduce that $\psi(d(z, z')) = 0$ and therefore $d(z, z') = 0$, that is, $z = z'$. ☐

6.3.3 Expansive mappings

Definition 6.3.3. Let (X, d) be a metric space and let K be a subset of X. A map $g : K \to X$ is said to be expansive, if there exists a constant $\beta > 1$ such that, for all $x, y \in K$, we have

$$d(g(x), g(y)) \ge \beta d(x, y). \tag{6.5}$$

Theorem 6.3.4. *Let (X, d) be a complete metric space, K a nonempty closed subset of X, and $g : K \to X$ an expansive mapping. If $K \subseteq g(K)$, then g has a unique fixed point in K.*

Proof. It follows from (6.5) that g is invertible and its inverse $g^{-1} : g(K) \to K$ is a $1/\beta$-contractive mapping. Since $K \subset g(K)$, we can assume $g^{-1} : K \to K$ and, by the Banach contraction principle, there exists a unique point $z \in K$ such that $g^{-1}(z) = z$. This shows that $g(z) = z$. ☐

This result is obviously valid if $g(K) = X$ (in particular, if the domain of g is the whole space X and $g(X) = X$).

Corollary 6.3.2. *Let (X, d) be a complete metric space and let g be a mapping from X into X. If there exists an integer $n \geq 2$ such that g^n is expansive and onto, then g has a unique fixed point $z \in X$.*

In this corollary, the map g is not necessarily expansive.

Proof. Since g^n is expansive and $X \subseteq g^n(X)$, according to Theorem 6.3.4, there exists a unique point $z \in X$ such that $g^n(z) = z$. In addition, we have $g^{n+1}(z) = g(g^n(z)) = g(z)$, which proves that $g(z)$ is also a fixed point of g^n. Using the fact that the fixed point of g^n is unique, we infer that $g(z) = z$, which proves that z is a fixed point of g. The uniqueness of the fixed point of g follows from the fact that any fixed point of g is also a fixed point of g^n and the uniqueness of the fixed point of g^n. □

6.4 Nonlinear expansive mappings

Definition 6.4.1. Let (X, d) be a metric space, and let K be a subset of X. The mapping $f : K \to K$ is said to be a nonlinear expansion, if there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying:

1. ϕ is nondecreasing,
2. $\phi(t) > t$ for each $t > 0$,
3. ϕ is right continuous and such that

$$d(f(x), f(y)) \geq \phi(d(x, y)), \quad \text{for all } x, y \in X. \tag{6.6}$$

We note that if we take $\phi(t) = \beta t$ with $\beta > 1$, the nonlinear expansion in (6.6) reduces to an expansion with constant β (cf. Definition 6.3.3).

For nonlinear expansion mappings, we have the following fixed point theorem.

Theorem 6.4.1. *Let K be a nonempty closed subset of a complete metric space (X, d) and let $f : K \to X$ be a nonlinear expansion. If $f(K) \supset K$, then f has a unique fixed point in K.*

Proof. Uniqueness is clear from (6.6), so we need only prove existence. Let $\alpha = \inf\{d(x, f(x)) : x \in K\}$. Now we prove $\alpha = 0$. Suppose that this is not the case. For any $\varepsilon > 0$, we can find $x \in K$ such that

$$\alpha \leq d(x, f(x)) \leq \alpha + \varepsilon.$$

Since $f(K) \supset K$, there is $y \in K$ such that $x = f(y)$, and so $\alpha \leq d(y, f(y))$. By virtue of nondecreasingness of ϕ, we have

$$\phi(\alpha) \leq \phi(d(y, f(y))) \leq d(f(y), f^2(y)) = d(x, f(x)) < \alpha + \varepsilon,$$

so $\phi(\alpha) \leq \alpha$, a contradiction. Thus $\alpha = 0$.

Now we will show that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in K satisfying $\lim_{n\to+\infty} d(x_n, f(x_n)) = 0$ and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that this is not the case. Then we can find two sequences of integers $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in N}$ with $m_k > n_k \geq k$ such that

$$\beta = \liminf_{p\to+\infty, k\geq p} d(x_{m_k}, x_{n_k}) > 0.$$

We define the sequence of real numbers $(\beta_k)_{k\in\mathbb{N}}$ by

$$\beta_k = d(x_{m_k}, x_{n_k}), \quad k = 1, 2, 3, \dots,$$

and then there is a subsequence of $(\beta_k)_{k\in\mathbb{N}}$ which is nonincreasing and converges to β. Without loss of generality, we might suppose that it is $(\beta_k)_{k\in\mathbb{N}}$. Thus,

$$\begin{aligned}
\phi(\beta_k) &= \phi(d(x_{m_k}, x_{n_k})) \\
&\leq d(f(x_{m_k}), f(x_{n_k})) \\
&\leq d(f(x_{m_k}), x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) \\
&= d(f(x_{m_k}), x_{m_k}) + d(x_{n_k}, f(x_{n_k})) + \beta_k. \qquad (6.7)
\end{aligned}$$

Letting $k \to +\infty$ in (6.7), we obtain

$$\phi(\beta) \leq \beta,$$

a contradiction. Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since K is a closed subset of a complete metric space, there exists $x \in K$ such that

$$\lim_{n\to+\infty} x_n = x.$$

Since $f(K) \supset K$, there exist $(u_n)_{n\in\mathbb{N}}$ and u in K such that $x_n = f(u_n)$ and $x = f(u)$. We will prove that u is a fixed point of f. Indeed, for any $\varepsilon > 0$, there is a nonnegative integer N such that $d(x_n, f(x_n)) < \frac{\varepsilon}{3}$ and $d(x_n, x) < \frac{\varepsilon}{3}$, provided that $n \geq N$. Since f is a nonlinear expansion, we have

$$\begin{aligned}
d(u, f(u)) &\leq d(u, u_n) + d(u_n, x_n) + d(x_n, f(u)) \\
&\leq \phi(d(u, u_n)) + \phi(d(u_n, x_n)) + d(x_n, f(u)) \\
&\leq d(f(u), f(u_n)) + d(f(u_n), f(x_n)) + d(x_n, f(u)) \\
&= 2d(x, x_n) + d(x_n, f(x_n)) < \varepsilon.
\end{aligned}$$

This implies that u is a fixed point of f and therefore the proof is complete. □

Note that, unlike the proof of Theorem 6.3.4, the proof of Theorem 6.4.1 does not rely upon the fixed point of f^{-1}.

The following result is an immediate consequence of Theorem 6.4.1.

Corollary 6.4.1. *Let K be a nonempty closed subset of a complete metric space (X, d) and let $f : K \to X$ be a surjective nonlinear expansion. Then f has a unique fixed point in K.*

6.5 Nonexpansive mappings

6.5.1 Generalities

As mentioned in the introduction of this chapter, some researchers approached the study of existence of fixed points for nonexpansive mappings because they include the weak contractions as well as all isometries. On the other hand, the study of existence of fixed points for these mappings is also interesting since, as we have seen in Chapter 3, the resolvents of accretive operators are nonexpansive mappings.

Definition 6.5.1. Let (X, d) be a metric space and let K be a nonempty subset of X. A mapping $f : K \to X$ is nonexpansive if

$$d((x), f(y)) \leq d(x, y) \quad \text{for all } x, y \in K.$$

In the context of normed spaces, this definition may be written as follows. Let X be a Banach space, and let K be a nonempty subset of X. A map $f : K \to X$ is called nonexpansive if

$$\|f(x) - f(y)\| \leq \|x - y\|, \quad \forall x, y \in K. \tag{6.8}$$

In contrast with the fact that contractive mappings always have a fixed point on complete metric spaces, in general, it is not the case for nonexpansive mappings, and if a nonexpansive map has a fixed point, it is not necessarily unique.

Example 6.5.1.

(a) Let X be a Banach space, $a \in X \setminus \{0\}$, and $f : X \to X$ be defined by $f(x) = x + a$. It is clear that f is nonexpansive but it cannot have a fixed point. However, the map $g = id_X$ is nonexpansive and, of course, every $x \in X$ is a fixed point of g.

(b) Consider $X = c_0$, the space consisting of all sequences $x = (x_n)_{n\in\mathbb{N}}$ converging to zero, with the norm $\|x\|_\infty = \max_{n\geq 1} |x_n|$. Define the map $f : \mathbb{B}_1 \to \mathbb{B}_1$ by

$$f(x) = (1 - \|x\|_\infty, x_1, x_2, \dots),$$

where $\mathbb{B}_1$ is the closed unit ball of X. For $x, y \in \mathbb{B}_1$, we have

$$\|f(x) - f(y)\|_\infty = \max\Big\{\big|\|x\|_\infty - \|y\|_\infty\big|, \max_{n\geq 1} |x_n - y_n|\Big\} \leq \|x - y\|_\infty,$$

which proves that f is nonexpansive. However, f does not have a fixed point. In fact, if $f(x) = x$, then $x_1 = x_2 = \cdots = 1 - \|x\|_\infty$. This is impossible.

(c) Let $(c_0, \|\cdot\|_\infty)$ and define the map $f : \mathbb{B}_1 \to \mathbb{B}_1$ by

$$\mathbb{B}_1 \ni (x_n)_{n\in\mathbb{N}} \mapsto f\big((x_n)_{n\in\mathbb{N}}\big) = (1, x_1, x_2, \dots).$$

It is not hard to see that $\|f((x_n)_{n\in\mathbb{N}}) - f((y_n)_{n\in\mathbb{N}})\|_\infty = \|(x_n)_{n\in\mathbb{N}} - (y_n)_{n\in\mathbb{N}}\|_\infty$. Hence f is nonexpansive and it maps $\mathbb{B}_1$ into $\mathbb{B}_1$. On the other hand, the unique possible fixed point of f is $(1, 1, 1, \dots)$ but such a point does not belong to c_0.

We have already noticed that nonexpansive mappings may be fixed point free, and obviously when such a mapping has a fixed point, this fixed point is not necessarily unique. The fixed point theory for nonexpansive mappings is different from that of contractive mappings. Even if a nonexpansive mapping has a fixed point, the iterative procedure described in the preceding sections does not work.

Example 6.5.2. Let $\mathbb{B}_1$ be the closed unit ball of the Banach space $(\mathbb{R}^2, \|\cdot\|_2)$. Define the map $f : \mathbb{B}_1 \to \mathbb{B}_1$ by $f(x,y) = (y,x)$ for every $(x,y) \in \mathbb{B}_1$. Clearly, f is nonexpansive and if $(x,x) \in \mathbb{B}_1$, then (x,x) is a fixed point of f in $\mathbb{B}_1$. However, if we consider the sequence $(f^n(1,0))_{n\in\mathbb{N}}$, then it is easy to show that such sequence does not converge.

Example 6.5.2 is interesting because it says that, for nonexpansive maps, a result like in Theorem 6.2.1 will not be possible. Most of the interest in the fixed point theory for nonexpansive maps is given within the Banach space framework. Let $(X, \|\cdot\|)$ be a real Banach space and let K be a nonempty subset of X. We recall that a mapping $f : K \to K$ is said to be nonexpansive if it satisfies (6.8).

If we intend to get positive results, then we should impose some additional conditions on K. We start by noting the following basic fact.

Lemma 6.5.1. *If X is a Banach space, K is a nonempty, bounded, closed, and convex subset of X and $f : K \to K$ is a nonexpansive mapping, then*

$$\inf\{\|x - f(x)\| : x \in K\} = 0.$$

Proof. Fix $z \in K$ and $\varepsilon \in (0,1)$, and consider the mapping $f_\varepsilon : K \longrightarrow K$ defined by

$$f_\varepsilon(x) = \varepsilon z + (1-\varepsilon) f(x).$$

Note that, for all $x, y \in K$, f_ε we have

$$\|f_\varepsilon(x) - f_\varepsilon(y)\| \le (1-\varepsilon)\|x - y\|.$$

Thus f_ε is $(1-\varepsilon)$-contractive. By the Banach contraction principle, there exists $x_\varepsilon \in K$ such that $x_\varepsilon = f_\varepsilon(x_\varepsilon)$. Hence,

$$\begin{aligned}\|x_\varepsilon - f(x_\varepsilon)\| &= \|\varepsilon z + (1-\varepsilon)f(x_\varepsilon) - f(x_\varepsilon)\| \\ &= \varepsilon\|z - f(x_\varepsilon)\| \\ &\le \varepsilon \operatorname{diam}(K).\end{aligned}$$

The result is obtained upon letting $\varepsilon \to 0$. □

Definition 6.5.2. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to+\infty}\|x_n - f(x_n)\| = 0$, is called either an approximate fixed point sequence of f or an almost fixed point sequence of f.

Lemma 6.5.1 asserts that f has approximate fixed points sequences. Thus, the question of whether f has a fixed point is equivalent to the question of whether the continuous function $\varphi : K \to \mathbb{R}^+$ defined by $\varphi(x) = \|x - f(x)\|$ achieves its minimal value zero. Obviously, the answer is positive if K is compact. On the other hand, if K is not compact, then the answer can be negative.

Proposition 6.5.1. *If K is a nonempty, compact, and convex subset of a Banach space, then any nonexpansive mapping from K into itself has a fixed point.*

We note that Proposition 6.5.1 is a special case of Schauder's fixed point theorem (cf. Theorem 7.2.3) which asserts that any continuous function f from a compact convex subset K of a Banach space into itself has a fixed point.

Definition 6.5.3. Let X be a Banach space and let K be a subset of X. We say that K has the fixed point property (in short, (FPP)) for nonexpansive mappings if, for each nonexpansive mapping $f : K \to K$, we have $\mathrm{Fix}(f) \neq \emptyset$. The space X is said to have the fixed point property for nonexpansive mappings (in short, (FPP)) if every nonempty, closed, convex, and bounded subset of X has (FPP).

The first positive results about the (FPP) for nonexpansive mappings were obtained in 1965 by F. E. Browder [50], D. Göhde [117], and W. A. Kirk [142]. Browder and Göhde showed independently that every uniformly convex Banach space has (FPP) and W. A. Kirk proved a more general result that every reflexive Banach space with normal structure enjoys (FPP) (slightly earlier Browder saw that every Hilbert space has (FPP)). Thus we may consider that the recognition of fixed point theory for nonexpansive mappings as a noteworthy avenue of research dates from 1965. Since then, several books have appeared (see, for example, [10, 115, 143] and the references therein). Therefore in this section, we shall only see the above mentioned results.

6.5.2 Hilbert spaces

Theorem 6.5.1. *Let H be a real Hilbert space. Then H enjoys (FPP).*

Proof. Let K be a nonempty, closed, convex, and bounded subset of H, and let $f : K \to K$ be a nonexpansive mapping. Lemma 6.5.1 guarantees that there exists a sequence

$(x_n)_{n\in\mathbb{N}}$ in K such that $\lim_{n\to+\infty}\|x_n - f(x_n)\| = 0$. Since H is reflexive, without loss of generality, we may suppose that $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to $\zeta \in K$. Our objective now is to check that $\zeta \in \mathrm{Fix}(f)$. To see this, let us first note that

$$\begin{aligned}\|x_n - f(\zeta)\|^2 &= \langle x_n - \zeta - f(\zeta) + \zeta, x_n - \zeta - f(\zeta) + \zeta\rangle \\ &= \langle x_n - \zeta, x_n - \zeta\rangle + \langle x_n - \zeta, \zeta - f(\zeta)\rangle \\ &\quad + \langle \zeta - f(\zeta), x_n - \zeta\rangle + \langle \zeta - f(\zeta), \zeta - f(\zeta)\rangle.\end{aligned}$$

This yields,

$$\liminf_{n\to+\infty}\|x_n - f(\zeta)\|^2 = \liminf_{n\to+\infty}\|x_n - \zeta\|^2 + \|f(\zeta) - \zeta\|^2. \tag{6.9}$$

On the other hand, since f is nonexpansive, we obtain

$$\liminf_{n\to+\infty}\|x_n - f(\zeta)\| \le \liminf_{n\to+\infty}(\|x_n - f(x_n)\| + \|x_n - \zeta\|) \le \liminf_{n\to+\infty}\|x_n - \zeta\|. \tag{6.10}$$

Finally, (6.9) and (6.10) imply $f(\zeta) = \zeta$. ☐

If we check the proof of Theorem 6.5.1 and the tools used, we may define.

Definition 6.5.4. A Banach space X is said to satisfy the Opial condition (see [184]) if for each x in X and each sequence $(x_n)_{n\in\mathbb{N}}$ weakly convergent to x, the estimate

$$\liminf_{n\to+\infty}\|x_n - x\| < \liminf_{n\to+\infty}\|x_n - y\|$$

holds for $y \neq x$.

It is clear that every Hilbert space, as well as the classical Banach spaces $(\ell_p, \|\cdot\|_p)$ with $1 \le p < \infty$, satisfy Opial condition. Moreover, by using similar arguments as in the proof of Theorem 6.5.1, we easily establish the following theorem.

Theorem 6.5.2. *Let X be a reflexive Banach space satisfying the Opial condition. Then X has (FPP).*

6.5.3 Uniformly convex Banach spaces

The proposal of this subsection is to obtain the Browder result, i. e., to show that uniformly convex Banach spaces enjoy (FPP). In order to do this, we will use an indirect method which was introduced in 1972 by M. Edelstein.

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach X, and let K be a nonempty, closed, and convex subset of X. Consider the function $r(\cdot, (x_n)_{n\in\mathbb{N}}) : X \to [0, +\infty[$ defined, for each $x \in X$, by

$$r(x,(x_n)_{n\in\mathbb{N}}) := \limsup_{n\to+\infty} \|x_n - x\|.$$

The asymptotic radius of $(x_n)_{n\in\mathbb{N}}$ in K is the number

$$r(K,(x_n)_{n\in\mathbb{N}}) = \inf\{r(x,(x_n)_{n\in\mathbb{N}}) : x \in K\}.$$

The number

$$A(K,(x_n)_{n\in\mathbb{N}}) = \{z \in K : r(z,(x_n)_{n\in\mathbb{N}}) = r(K,(x_n)_{n\in\mathbb{N}})\}$$

is the asymptotic center of $(x_n)_{n\in\mathbb{N}}$ in K.

If the sequence $(x_n)_{n\in\mathbb{N}}$ is fixed, we have that $r(\cdot,(x_n)_{n\in\mathbb{N}})$ satisfies:

(a) For all $x \in X$, $r(x,(x_n)_{n\in\mathbb{N}}) = 0 \Longleftrightarrow \lim_{n\to\infty} x_n = 0$.

(b) For all $x,y \in X$, $|r(x,(x_n)_{n\in\mathbb{N}}) - r(y,(x_n)_{n\in\mathbb{N}})| \le \|x - y\|$.

(c) For all $x,y \in X$ and for each $\alpha,\beta \ge 0$ with $\alpha + \beta = 1$,

$$r(\alpha x + \beta y,(x_n)_{n\in\mathbb{N}}) \le \alpha r(x,(x_n)_{n\in\mathbb{N}}) + \beta r(y,(x_n)_{n\in\mathbb{N}}).$$

Hence, $r(\cdot,(x_n)_{n\in\mathbb{N}})$ is a nonnegative, continuous, and convex function, and therefore, for any $a \ge 0$, the *level set*

$$A_a((x_n)_{n\in\mathbb{N}}) = \{x \in K : r(x,(x_n)_{n\in\mathbb{N}}) \le r(K,(x_n)_{n\in\mathbb{N}}) + a\}$$

is a closed, convex, and bounded subset of K. Moreover, if $(x_n)_{n\in\mathbb{N}}$ does not converge, the set $A_a((x_n)_{n\in\mathbb{N}})$ will be empty for small enough a.

Lemma 6.5.2. *Let X be a reflexive Banach space and K a nonempty, convex, closed subset of X. If $f : K \to [0,\infty)$ is a lower semicontinuous convex function such that $\lim_{\|x\|\to+\infty} f(x) = +\infty$ (if K is unbounded), then f has a minimum in K.*

Proof. Set $\lambda := \inf\{f(x) : x \in K\}$ and take $K_n := \{x \in K : f(x) \le \lambda + \frac{1}{n}\} \ne \emptyset$. By hypothesis, each K_n is closed, convex, bounded, and, since X is reflexive, it is weakly compact. Moreover, $K_{n+1} \subseteq K_n$. This implies that there exists $x_0 \in \bigcap_{n\in\mathbb{N}} K_n$, and hence $f(x_0) = \lambda$. □

Theorem 6.5.3. *If X is a uniformly convex Banach space, then X has (FPP).*

Proof. Let K be a nonempty, closed, convex, bounded subset of X and let $f : K \to K$ be a nonexpansive mapping. We have to prove that f has a fixed point in K. According to Lemma 6.5.1, f has an approximate fixed point sequence $(x_n)_{n\in\mathbb{N}}$. Consider the set $A(K,(x_n)_{n\in\mathbb{N}})$. We claim that $A(K,(x_n)_{n\in\mathbb{N}})$ is a singleton. Since the function $r(\cdot,(x_n)_{n\in\mathbb{N}}) : K \to [0,+\infty)$ satisfies the conditions of Lemma 6.5.2, we conclude that $A(K,(x_n)_{n\in\mathbb{N}}) \ne \emptyset$. Suppose that $y_0, y_1 \in K$ are such that

$$r(y_0,(x_n)_{n\in\mathbb{N}}) = r(y_1,(x_n)_{n\in\mathbb{N}}) = r_0 = r(K,(x_n)_{n\in\mathbb{N}}).$$

Since K is convex, $\frac{y_0+y_1}{2} \in K$, and $r(\cdot, (x_n)_{n\in\mathbb{N}})$ is convex, we have $r(\frac{y_0+y_1}{2}, (x_n)_{n\in\mathbb{N}}) = r_0$.

Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\max\{\|x_n - y_0\|, \|x_n - y_1\|\} \leq r_0 + \varepsilon$. Thus, for each integer $n \geq n_0$, we have $\frac{x_n - y_0}{r_0+\varepsilon}, \frac{x_n - y_1}{r_0+\varepsilon} \in \mathbb{B}_1$. Furthermore,

$$\left\| \frac{x_n - y_0}{r_0 + \varepsilon} - \frac{x_n - y_1}{r_0 + \varepsilon} \right\| = \left\| \frac{y_1 - y_0}{r_0 + \varepsilon} \right\| = u_\varepsilon < 2,$$

$$\delta(u_\varepsilon) \leq 1 - \left\| x_n - \frac{y_0 + y_1}{2} \right\| \frac{1}{r_0 + \varepsilon},$$

whenever $n \geq n_0$. Hence

$$\left\| x_n - \frac{y_0 + y_1}{2} \right\| \leq (1 - \delta(u_\varepsilon))(r_0 + \varepsilon).$$

Letting $n \to +\infty$, we get

$$r\left(\frac{y_0 + y_1}{2}, (x_n)_{n\in\mathbb{N}}\right) \leq (1 - \delta(u_\varepsilon))(r_0 + \varepsilon).$$

Now as $\varepsilon \to 0$ and bearing in mind that, if X is uniformly convex, the modulus of convexity is continuous in the interval $[0, 2]$, we conclude that

$$r\left(\frac{y_0 + y_1}{2}, (x_n)_{n\in\mathbb{N}}\right) \leq \left(1 - \delta\left(\frac{\|y_0 - y_1\|}{r_0}\right)\right) r_0 < r_0,$$

which is a contradiction. Consequently, $A(K, (x_n)_{n\in\mathbb{N}})$ is a singleton, as claimed.

Now let us see that $A(K, (x_n)_{n\in\mathbb{N}})$ is f-invariant. Indeed, if $z \in A(K, (x_n)_{n\in\mathbb{N}})$, we have

$$\begin{aligned} r(f(z), (x_n)_{n\in\mathbb{N}}) &= \limsup_{n\to+\infty} \|f(z) - x_n\| \\ &\leq \limsup_{n\to+\infty} (\|f(z) - f(x_n)\| + \|f(x_n) - x_n\|) \\ &\leq \limsup_{n\to+\infty} \|z - x_n\| = r(z, (x_n)_{n\in\mathbb{N}}). \end{aligned}$$

Consequently, the result follows easily from the fact that $A(K, (x_n)_{n\in\mathbb{N}})$ is an f-invariant singleton. □

6.5.4 Normal structure

In the previous subsections we have seen that reflexive Banach spaces with Opial property and uniformly convex Banach spaces enjoy (FPP). It is well known that normal structure (see below) is a geometrical property of Banach spaces more general than both previous conditions and, along with the reflexivity, also implies (FPP). However, in this case, the proof is of different nature because it needs Zorn lemma.

Definition 6.5.5. Let X be a Banach space and let C be a nonempty, bounded, closed, convex subset of X. A point $c \in C$ is said to be diametral if

$$\operatorname{diam}(C) = \sup_{x \in C} \|c - x\|_X.$$

We say that C has normal structure if, for any given bounded, closed, convex set $K \subset C$ containing more than one point, there exists a nondiametral $x_0 \in K$, that is,

$$\sup\{\|x_0 - x\| : x \in K\} < \operatorname{diam}(K) := \sup\{\|x - y\| : x, y \in K\}.$$

Theorem 6.5.4. *Every nonempty, compact, convex subset K of a Banach space X has normal structure.*

Proof. If K (with $\operatorname{diam}(K) > 0$) does not have normal structure, then for any $x_1 \in K$ we can find $x_2 \in K$ such that

$$\|x_1 - x_2\|_X = \operatorname{diam}(K).$$

Since K is convex, we have $\frac{x_1+x_2}{2} \in K$, so there exists $x_3 \in K$ such that

$$\left\| \frac{x_1 + x_2}{2} - x_3 \right\|_X = \operatorname{diam}(K).$$

This way we produce a sequence $(x_n)_{n \in \mathbb{N}}$ of points of K such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} x_k - x_{n+1} \right\|_X = \operatorname{diam}(K).$$

Therefore

$$\begin{aligned} \operatorname{diam}(K) &= \left\| \frac{1}{n} \sum_{k=1}^{n} x_k - x_{n+1} \right\|_X \\ &= \left\| \frac{x_1 - x_{n+1}}{n} + \frac{x_2 - x_{n+1}}{n} + \cdots + \frac{x_n - x_{n+1}}{n} \right\|_X \\ &\le \frac{1}{n} \sum_{k=1}^{n} \|x_k - x_{n+1}\|_X \le \operatorname{diam}(K). \end{aligned}$$

This implies $\|x_k - x_{n+1}\|_X = \operatorname{diam}(K)$ and therefore $(x_n)_{n \in \mathbb{N}}$ is a sequence of points of K with no convergent subsequences, a contradiction to the fact that K is compact. ☐

Theorem 6.5.5. *Every nonempty, bounded, closed, convex set K of a uniformly convex Banach space X has a normal structure.*

Proof. Without any loss of generality, we may assume that $K \subset \mathbb{B}_1$ where $\mathbb{B}_1$ denotes the closed unit ball of X.

Let K_1 be a nonempty, closed, convex subset of K with $\operatorname{diam}(K_1) > 0$ and let $x_1 \in K$ and $\varepsilon = \frac{1}{2}$. We can find $x_2 \in K_1$ such that

$$\frac{1}{2}\operatorname{diam}(K_1) \leq \|x_2 - x_1\|_X.$$

Then for any $x \in X$, we have

$$\left\|x - \frac{x_1 + x_2}{2}\right\|_X = \left\|\frac{x - x_1}{2} + \frac{x - x_2}{2}\right\|_X \leq \operatorname{diam}(K_1)\left(1 - \delta\left(\frac{1}{2}\right)\right),$$

where $\delta(\varepsilon)$ is the modulus of uniform convexity of X. Since $\delta(\frac{1}{2}) > 0$, we conclude that K has normal structure. ☐

To show the main results of this section, we need to introduce some geometric quantities associated with a set K. Let K denote a nonempty, bounded, closed, and convex subset of the Banach space X and $x \in X$. Then define $r_x(K)$, $r(K)$ and $C(K)$ as follows:

$$r_x(K) := \sup\{\|x - y\| : y \in K\}, \quad r(K) := \inf\{r_x(K) : x \in K\}.$$

If X is a reflexive Banach space, then the weak compactness of K yields that

$$C(K) := \{z \in K : r_z(K) = r(K)\}$$

is a nonempty, closed, convex subset of K. The number $r(K)$ and the set $C(K)$ are called the Chebyshev radius and Chebyshev center of K, respectively.

Theorem 6.5.6. *Let X be a reflexive Banach space and let K be a nonempty, bounded, closed, and convex subset of X with normal structure. Then K has (FPP).*

Proof. Let $f : K \to K$ be a nonexpansive map and define the set

$$\mathcal{F} := \{D \subseteq K : D \text{ is closed, convex } D \neq \emptyset \text{ and } f(D) \subseteq D\}.$$

Because the elements of $\mathcal{F}$ are weakly compact, each descendent chain in $\mathcal{F}$ has a lower bound, hence, by Zorn lemma, $\mathcal{F}$ has a minimal member, say K_0. On the other hand,

$$f(K_0) \subseteq K_0 \quad \text{implies } f(\overline{\operatorname{co}}(f(K_0))) \subseteq f(K_0) \subseteq \overline{\operatorname{co}}(f(K_0)).$$

Thus $\overline{\operatorname{co}}(f(K_0)) \in \mathcal{F}$ and, by minimality, we have $\overline{\operatorname{co}}(f(K_0)) = K_0$. Let $u \in C(K_0)$, hence $r_u(K_0) = r(K_0)$. But $\|f(u) - f(y)\| \leq \|u - y\| \leq r(K_0)$ for all $y \in K_0$ which implies that $f(K_0) \subseteq \mathbb{B}_{r(K_0)}(f(u))$. Hence

$$K_0 = \overline{\operatorname{co}}(f(K_0)) \subseteq \mathbb{B}_{r(K_0)}(f(u)),$$

which says that $r_{f(u)}(K_0) = r(K_0)$, and thus $f(u) \in C(K_0)$. Since K_0 is minimal, we obtain that $C(K_0) = K_0$. Thus $\operatorname{diam}(K_0) \le r(K_0)$, and, since K has normal structure, it must be the case that K_0 is a singleton, which is a fixed point for f. □

Remark 6.5.1. To establish (FPP) in the absence of normal structure requires the properties of minimal invariant sets that involve the mapping f in a more explicit way. One of the main ingredients was obtained independently by K. Goebel [114] and L. A. Karlovitz [137], another one is due to B. Maurey [172] who, in a brilliant series of results, first showed the usefulness of ultrapowers as a setting for such arguments. In this sense we mention that, by using Goebel–Karlovitz–Maurey arguments, in [106] it was proved that every nonsquare uniformly convex Banach space has (FPP) (recall that a Banach space X is nonsquare uniformly convex if there exists $\varepsilon_0 \in (0,2)$ such that the modulus of convexity $\delta_X(\varepsilon_0) > 0$).

6.5.5 Fixed points in unbounded sets

It is easy to see that although a Banach space X enjoys (FPP) for nonexpansive mappings, we cannot guarantee that if a nonexpansive mapping has an unbounded closed and convex domain such a mapping has a fixed point (for instance, every translation mappings with no null vector is a fixed point free nonexpansive mapping). In this subsection we will give a characterization, in the framework of Banach spaces with the (FPP), for the existence of fixed points for nonexpansive mappings defined on an unbounded, closed, and convex subset of a Banach space (see [109]).

Lemma 6.5.3. *Let X be a Banach space and let C be a nonempty, closed, and convex subset of X. If $f : C \to X$ is a nonexpansive mapping, then for every $x, y \in C$ and $j \in J(x-y)$, the following inequality holds:*

$$\|x-y\|^2 = \langle x-y, j\rangle \ge \langle f(x)-f(y), j\rangle.$$

Proof. Let $j \in J(x-y)$. Since f is a nonexpansive mapping, we have

$$\langle f(x)-f(y), j\rangle \le \|f(x)-f(y)\|\|x-y\| \le \|x-y\|^2 = \langle x-y, j\rangle.$$

Therefore,

$$\langle f(x)-f(y), j\rangle \le \langle x-y, j\rangle = \|x-y\|^2.$$ □

Remark 6.5.2. In particular, Lemma 6.5.3 shows that if $f : C \to X$ is a nonexpansive mapping, then $I - f : C \to X$ is an accretive mapping. Indeed, for all $x, y \in C$ and for all $j \in J(x-y)$, we have $\langle x - f(x) - (y - f(y)), j\rangle \ge 0$.

Proposition 6.5.2. *Let C be a closed, convex, and unbounded subset of a Banach space X, and let $f : C \to X$ be a nonexpansive mapping with a bounded approximate fixed point sequence $(x_n)_{n\in\mathbb{N}}$ in C. Then for each $x_0 \in C$,*

$$\limsup_{n\to+\infty} \langle f(x) - x_0, x - x_n\rangle_+ \le \limsup_{n\to\infty} \langle x - x_0, x - x_n\rangle_+$$

for all $x \in C$.

Proof. It is clear that there is $j_n \in J(x - x_n)$ such that $\langle f(x) - x_0, x - x_n\rangle_+ = \langle f(x) - x_0, j_n\rangle$. Therefore, by Lemma 6.5.3,

$$\begin{aligned}\langle f(x) - x_0, x - x_n\rangle_+ &= \langle f(x) - x_0, j_n\rangle \\ &= \langle f(x) - f(x_n), j_n\rangle + \langle f(x_n) - x_0, j_n\rangle \\ &\le \langle x - x_n, j_n\rangle + \langle f(x_n) - x_0, j_n\rangle \\ &= j_n(f(x_n) - x_n) + j_n(x - x_0) \\ &\le \|j_n\| \|f(x_n) - x_n\| + \langle x - x_0, x - x_n\rangle_+ \\ &= \|x - x_n\| \|f(x_n) - x_n\| + \langle x - x_0, x - x_n\rangle_+.\end{aligned}$$

Hence, $\limsup_{n\to+\infty} \langle f(x) - x_0, x - x_n\rangle_+ \le \limsup_{n\to+\infty} \langle x - x_0, x - x_n\rangle_+$. ☐

Proposition 6.5.3. *Let C be a closed, convex, and unbounded subset of a Banach space X, and let $f : C \to C$ be a nonexpansive mapping. The following statements are equivalent:*

(a) *There exists a bounded approximate fixed point sequence for f in C.*

(b) *There exist $x_0 \in C$ and $R > 0$ such that $f(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for all $x \in C \cap \partial\mathbb{B}_R(x_0)$.*

Proof. First, we will prove (a) ⇒ (b). By hypotheses, f has a bounded approximate fixed point sequence, say $(x_n)_{n\in\mathbb{N}}$. Let x_0 be an arbitrary but fixed element in C. Since $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence, there exists $R > 0$ such that $\langle x - x_0, x - x_n\rangle_+ > 0$ for all $x \in X$ such that $\|x - x_0\| \ge R$.

Now, suppose that there exist $x \in C \cap S_R(x_0)$ and $\lambda > 1$ such that $f(x) - x_0 = \lambda(x - x_0)$. In this case, we have

$$\begin{aligned}\limsup_{n\to+\infty} \langle f(x) - x_0, x - x_n\rangle_+ &= \lim_{n\to+\infty} \langle \lambda(x - x_0), x - x_n\rangle_+ \\ &= \lambda \limsup_{n\to+\infty} \langle x - x_0, x - x_n\rangle_+ \\ &> \limsup_{n\to+\infty} \langle x - x_0, x - x_n\rangle_+,\end{aligned}$$

which is in contradiction to Proposition 6.5.2.

To prove (b) ⇒ (a), we argue as follows. Letting $A := I - f$, by Lemma 6.5.3, we know that A is an accretive operator; moreover, A satisfies the range condition since $f : C \to C$. Thus for each $\lambda > 0$ there exists $x_\lambda \in C$ such that $x_0 = x_\lambda + \lambda A x_\lambda$. Denote the resolvent of A by $J_\lambda := (I + \lambda A)^{-1}$, thus $x_\lambda = J_\lambda x_0$. We will show that the set $\{J_\lambda x_0 : \lambda > 0\}$

is bounded. Indeed, otherwise there exists $\lambda_1 > 0$ such that $\|J_{\lambda_1}x_0 - x_0\| > R$. Since the mapping $\lambda \to J_\lambda x_0$ is continuous and $\lim_{\lambda\to 0^+} J_\lambda x_0 = x_0$, we can find $\lambda_2 > 0$ such that $\|J_{\lambda_2}x_0 - x_0\| = R$. However,

$$f(J_{\lambda_2}x_0) - x_0 = \frac{\lambda_2 + 1}{\lambda_2}(J_{\lambda_2}x_0 - x_0),$$

which is a contradiction since we are assuming that (b) holds.

Now, if for each $n \in \mathbb{N}$, we call $x_n := J_n x_0$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Moreover, since $A(x_n) = \frac{1}{n}(x_0 - x_n)$, we get that $\|Ax_n\| \to 0$, which means that $(x_n)_{n\in\mathbb{N}}$ is a bounded approximate fixed point sequence. □

Theorem 6.5.7. *Let X be a Banach space with (FPP) for nonexpansive mappings. Let C be a closed, convex, unbounded subset of X, and let $f : C \to C$ be a nonexpansive mapping. Then f has a fixed point if and only if there exist $x_0 \in C$ and $R > 0$ such that $f(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for all $x \in C \cap S_R(x_0)$.*

Proof. Suppose that f has a fixed point, say $x_0 \in C$. In this case, the constant sequence $x_n = x_0$ for all $n \in \mathbb{N}$ is a bounded approximate fixed point sequence and, by Proposition 6.5.3, we obtain the conclusion.

If we assume that there exist $x_0 \in C$ and $R > 0$ such that $f(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for all $x \in C \cap S_R(x_0)$, then Proposition 6.5.3 says that f has a bounded approximate fixed point sequence (x_n). In this case, let $R := \limsup_{n\to+\infty} \|x_n - x_1\| < \infty$ and consider the following set:

$$K := \Big\{y \in C : \limsup_{n\to+\infty} \|x_n - y\| \le R\Big\}.$$

It is clear that $K \neq \emptyset$ because $x_1 \in K$. Moreover, K is bounded, closed, and convex. Let us see that K is also f-invariant. Indeed, if $y \in K$ then

$$\limsup_{n\to\infty}\|f(y) - x_n\| \le \limsup_{n\to+\infty}\|f(y) - f(x_n)\| + \limsup_{n\to+\infty}\|f(x_n) - x_n\| \le \limsup_{n\to+\infty} \|x_n - y\| \le R.$$

Finally, since X has (FPP) and $f : K \to K$ is nonexpansive, then f has a fixed point in K. □

6.6 Fixed points for ϕ-expansive mappings

Definition 6.6.1. A mapping $f : D(f) \subseteq X \to X$ is said to be ϕ-expansive if there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying

(a) $\phi(0) = 0$,
(b) $\phi(r) > 0$ for $r > 0$,
(c) ϕ is either continuous or nondecreasing,

and such that, for all $x, y \in D(f)$, the inequality $\|f(x) - f(y)\| \ge \phi(\|x - y\|)$ holds.

Definition 6.6.2. Let X be a Banach space. A mapping $g : X \to X$ is said to be a Φ-contraction if there exists a continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(r) < r$ for any $r > 0$ and, for all $x, y \in X$, we have

$$\|g(x) - g(y)\| \leq \psi(\|x - y\|).$$

The function ψ is called the Φ-function of g. If, further, ψ is nondecreasing, then g is called a nonlinear contraction (see Definition 6.3.1).

Remark 6.6.1. We note that if $g : D(g) \subseteq X \to X$ is a ψ-contraction, then the map $f := I - g : D(g) \subseteq X \to X$ is ϕ-expansive where $\phi(r) = r - \psi(r)$. Indeed,

$$\|f(x) - f(y)\| \geq \|x - y\| - \|g(x) - g(y)\| \geq \|x - y\| - \psi(\|x - y\|) = \phi(\|x - y\|).$$

When g is a separate contraction (cf. Definition 6.3.2), it is not difficult to see that $I - g$ is ϕ-expansive with ϕ strictly increasing.

On the other hand, it is easy to see that the mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x, y) = (y, -x)$ is a nonexpansive mapping which is not a ψ-contraction for any ψ but, nevertheless, $I - g$ is ϕ-expansive with $\phi(t) = \sqrt{2}t$.

Lemma 6.6.1. *Let X be a Banach space and let K be a nonempty, closed, convex, bounded subset of X. Suppose that $g : K \to K$ is a nonexpansive mapping such that*

$$\phi(\|g(x) - g(y)\|) \leq \|x - g(x) - (y - g(y))\|, \quad \forall x, y \in K, \tag{6.11}$$

where ϕ is a function satisfying the conditions of Definition 6.6.1. *Then g has a unique fixed point in K.*

Proof. By Lemma 6.5.1, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset K$ such that $\lim_{n\to+\infty} \|x_n - g(x_n)\| = 0$. We claim that $(g(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.

(a) We assume that ϕ is nondecreasing. If $(g(x_n))_{n\in\mathbb{N}}$ is not a Cauchy sequence, then there exist $\varepsilon_0 > 0$ and two increasing sequences $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, we have

$$\varepsilon_0 < \|g(x_{n_k}) - g(x_{m_k})\|.$$

Because $(x_n - g(x_n)) \to 0$ as $n \to +\infty$, $(x_n - g(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Putting $\varepsilon = \phi(\varepsilon_0) > 0$, we have

$$\varepsilon = \phi(\varepsilon_0) \leq \phi(\|g(x_{n_k}) - g(x_{m_k})\|) \leq \|(x_{n_k} - g(x_{n_k})) - (x_{m_k} - g(x_{m_k}))\| < \varepsilon,$$

which contradicts our assumption. So, $(g(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.

(b) Suppose now that ϕ is continuous. It is clear that the latter inequality implies that $(g(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Finally, because K is closed, $(g(x_n))_{n\in\mathbb{N}}$ converges to a point z of K. Using the continuity of g, we conclude that z is a fixed point g.

If z' is a fixed point of g distinct of z, then by the estimate (6.11), we have

$$\phi(\|g(z) - g(z')\|) = \phi(\|z - z'\|) \leq \|z - g(z) - (z' - g(z'))\| = 0.$$

In view of the properties of ϕ, we obtain $z = z'$. □

It is well known that there exist Banach spaces without the fixed point property for nonexpansive mappings. The next result shows that it is not the case for a nonexpansive mapping f which satisfies the condition that $I - f$ is ϕ-expansive. This family of mappings happens to be a wider family of mappings than the class of Φ-contractions. Indeed, consider the Hilbert space $\mathbb{R}^2$ with the usual inner product and define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (y, -x)$. It is clear that f is an isometry, so it is nonexpansive, $I - f$ is ϕ-expansive with $\phi(t) = \sqrt{2}t$, but it is not a Φ-contraction for any ϕ.

Let X be a Banach and let $f : D(f) \subset X \to X$ be a mapping. We recall that f satisfies *the range condition* (cf. Definition 3.10.1) if $\overline{D(f)} \subseteq \bigcap_{\lambda>0} \mathrm{R}(I + \lambda f)$.

Proposition 6.6.1. *Let X be a Banach space and let K be a nonempty, closed, bounded, convex subset of X. If $g : K \to K$ is a nonexpansive mapping such that $I - g$ is ϕ-expansive, then g has a unique fixed point in K.*

Proof. We note that under the hypotheses, the operator $h = I - g : K \to X$ is accretive and satisfies the range condition. This shows that $J_1^h = (I + h)^{-1}$ maps K into itself. We define the Yosida approximant A_1 by $A_1(x) = x - J_1^h(x)$. Using the fact that $A_1(x) \in h(J_1^h(x))$ (see Proposition 3.4.1(a)), we get

$$\phi(\|J_1^h(x) - J_1^h(y)\|) \leq \|A_1(x) - A_1(y)\| = \|(x - J_1^h(x)) - (y - J_1^h(y))\|. \tag{6.12}$$

Since J_1^h is nonexpansive and satisfies (6.12), by Lemma 6.6.1, there exists a unique point $z \in K$ such that $z = J_1^h(z)$ and therefore $0 = z - g(z)$. Hence, z is the unique fixed point of g. □

Proposition 6.6.2. *Let X be a Banach space and K a nonempty, closed subset of X. Let $f : K \to X$ be an injective and continuous mapping such that $f^{-1} : \mathrm{R}(f) \to K$ is uniformly continuous. Then $\mathrm{R}(f)$ is a closed subset of X.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathrm{R}(f)$ such that $x_n \to x_0$ as $n \to +\infty$. We have to prove that $x_0 \in \mathrm{R}(f)$. Indeed, since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and f^{-1} is uniformly continuous, it is easy to see that $(f^{-1}(x_n))$ is also a Cauchy sequence of K. Hence we may assume that $(f^{-1}(x_n))$ converges to some $y_0 \in K$ because K is a closed subset of X. It follows from the continuity of f that $(x_n)_{n\in\mathbb{N}}$ converges to $f(y_0)$, which means that $x_0 = f(y_0) \in \mathrm{R}(f)$. □

Lemma 6.6.2. *Let K be a nonempty, bounded, closed subset of a Banach space X, and let $f : K \to X$ be a ϕ-expansive mapping. Then f is injective and $f^{-1} : \mathrm{R}(f) \to K$ is uniformly continuous.*

Proof. If $x, y \in K$, $x \neq y$, then $\|f(x) - f(y)\| \geq \phi(\|x - y\|) > 0$ and thus f is injective. To prove that f^{-1} is uniformly continuous, we argue as follows. Suppose that there exists $\varepsilon_0 > 0$ such that, for each $n \in \mathbb{N}$, we can find $x_n, y_n \in R(f)$ satisfying both $\|x_n - y_n\| < \frac{1}{n}$ and $\|f^{-1}(x_n) - f^{-1}(y_n)\| > \varepsilon_0$. Since f is ϕ-expansive, it is clear that

$$\phi(\|f^{-1}(x_n) - f^{-1}(y_n\|)) \leq \|x_n - y_n\| < \frac{1}{n}. \tag{6.13}$$

Now assume that ϕ is nondecreasing. In this case, we have that $0 < \phi(\epsilon_0) \leq \phi(k)$ for every $k \geq \epsilon_0$. Hence, we get a contradiction because, on the one hand, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \phi(\epsilon_0)$ for all $n \geq n_0$ and, on the other hand, by (6.13), we have that

$$\phi(\epsilon_0) \leq \phi(\|f^{-1}(x_n) - f^{-1}(y_n\|)) \leq \|x_n - y_n\| < \frac{1}{n}.$$

Otherwise, ϕ will be a continuous function. In this case, since K is a bounded subset, we know that the sequence $(f^{-1}(x_n) - f^{-1}(y_n))_{n\in\mathbb{N}}$ is bounded and therefore we may, without loss of generality, assume that

$$\lim_{n\to+\infty} \|f^{-1}(x_n) - f^{-1}(y_n)\| = r \geq \varepsilon_0.$$

Consequently, $\lim_{n\to+\infty} \phi(\|f^{-1}(x_n) - f^{-1}(y_n)\|) = \phi(r) > 0$, which is a contradiction since the estimate (6.13) yields $\lim_{n\to+\infty} \phi(\|f^{-1}(x_n) - f^{-1}(y_n)\|) = 0$. ☐

As an easy consequence of Proposition 6.6.2 and Lemma 6.6.2, we have

Corollary 6.6.1. *Let K be a nonempty, closed, bounded, subset of a Banach space X, and let $f : K \to X$ be a continuous ϕ-expansive mapping. If there exists $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{R}(f)$ such that $(x_n)_{n\in\mathbb{N}}$ converges to zero, then $0 \in \mathrm{R}(f)$.*

Lemma 6.6.3. *Let K be a nonempty, closed subset of a Banach space X and let $f : K \to X$ be a continuous mapping. If there exists an almost fixed point sequence $(x_n)_{n\in\mathbb{N}}$ of f in K and if the inverse mapping $(I - f)^{-1} : \mathrm{R}(I - f) \to K$ exists and is uniformly continuous, then f has a unique fixed point $x_0 \in K$. Furthermore, $x_n \to x_0$ as $n \to +\infty$.*

Proof. Since $I - f : K \to X$ is continuous injective and $(I - f)^{-1} : \mathrm{R}(I - f) \to K$ is uniformly continuous, according to Proposition 6.6.2, we know that $\mathrm{R}(I - f)$ is closed. Hence, since $x_n - f(x_n) \to 0$ as $n \to +\infty$, we obtain that $0 \in \mathrm{R}(I - f)$. Consequently, there exists an $x_0 \in K$ such that $(I - f)(x_0) = 0$, that is, $f(x_0) = x_0$. This fixed point is unique because $I - f$ is injective. Finally, because $(I - f)^{-1}$ is continuous and $(I - f)(x_n) \to (I - f)(x_0)$, we obtain that $x_n \to x_0$. ☐

The next example shows the importance of the fact that $(I - f)^{-1}$ is uniformly continuous in Lemma 6.6.3.

Example 6.6.1. Consider the function $f : [1, +\infty[\to [1, +\infty[$ defined by $f(x) = x + \frac{1}{x}$. Clearly, f is a fixed point free mapping and, moreover, $(I - f)^{-1} : [-1, 0) \longrightarrow [1, +\infty)$ is given by $(I-f)^{-1}(x) = -\frac{1}{x}$, which is continuous but not uniformly continuous. Finally, f admits almost fixed point sequences.

Corollary 6.6.2. *Let K be a nonempty, closed, bounded subset of a Banach space X, and let $f : K \to K$ be a continuous mapping such that $I - f : K \to X$ is ϕ-expansive. If there exists an almost fixed point sequence $(x_n)_{n\in\mathbb{N}}$ of f in K, then f has a unique fixed point $x_0 \in K$. Furthermore, $x_n \to x_0$ as $n \to +\infty$.*

Proof. Since $I - f : K \to X$ is ϕ-expansive and continuous, it follows from Lemma 6.6.2 that $(I-f)^{-1} : \mathrm{R}(I-f) \to K$ is uniformly continuous. Now the result is a consequence of Lemma 6.6.3. ☐

Remark 6.6.2. In Corollary 6.6.2, it is assumed that $I - f$ is a ϕ-expansive mapping. If ϕ is nondecreasing, then it is easy to see that $(I-f)^{-1} : \mathrm{R}(f) \to K$ is uniformly continuous although K becomes unbounded (see the proof of Lemma 6.6.2). Nevertheless, if ϕ is continuous, the conclusion of Corollary 6.6.2 remains true if we remove the boundedness of K but we add the existence of a bounded almost fixed point sequence for f.

Definition 6.6.3. Let X be a metric space. We say that a mapping $f : D(f) \subset X \to X$ is weakly inward on $D(f)$ if

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda} d\big((1-\lambda)x + \lambda f(x), D(f)\big) = 0 \quad \text{for all } x \in D(f). \tag{6.14}$$

Note that condition (6.14) is always weaker than the assumption that f maps the boundary of $D(f)$ into $D(f)$. Recall that, if $f : D(f) \to X$ is a continuous accretive mapping, $D(f)$ is convex and closed and, if $I - f$ is weakly inward on $D(f)$, then f has the range condition (see [169]).

We now introduce the concept of a pseudocontractive map.

Definition 6.6.4. Let X be Banach space and $f : D(f) \subseteq X \to X$ a map. We say that f is pseudocontractive if, for every $x, y \in D(f)$ and for all $r > 0$, the inequality

$$\|x-y\| \le \big\|(1+r)(x-y) + r\big(f(y) - f(x)\big)\big\|$$

holds.

It is not difficult to see that pseudocontractive maps are more general than nonexpansive maps. The interest in these maps also stems from the fact that they are firmly connected to the class of accretive maps. Specifically, f is pseudocontractive if and only if $I - f$ is accretive where I denotes the identity mapping (see [52]).

Corollary 6.6.3. *Let K be a nonempty, bounded, closed convex subset of a Banach space X. Assume that the map $f : K \to X$ is weakly inward on K, continuous pseudocontractive mapping such that $(I-f)^{-1} : \mathrm{R}(I-f) \to K$ is uniformly continuous (in particular, if $I - f$ is ϕ-expansive). Then f has a unique fixed point in K.*

Proof. Since f is a continuous, pseudocontractive, and weakly inward on K, it is well known that $I{-}f : K \to X$ is an accretive operator with the range condition. Consequently, the resolvent $(2I{-}f)^{-1} : K \to K$ is a single-valued and nonexpansive mapping. Moreover, since K is bounded, closed, convex, there exists a sequence $(x_n)_{n\in N}$ in K such that $x_n - (2I - f)^{-1}(x_n) \to 0$. If we let $y_n = (2I - f)^{-1}(x_n)$, then $y_n + y_n - f(y_n) = x_n$ and therefore

$$y_n - f(y_n) = x_n - (2I - f)^{-1}(x_n),$$

which implies that $(y_n)_{n\in\mathbb{N}}$ is a bounded almost fixed point sequence in K for f. Now, invoking Lemma 6.6.3, we obtain the result. ☐

6.7 Zeros of *m*-accretive operators

A classic problem in convex analysis is Euler's variational problem. Let $\varphi : H \to]-\infty, +\infty]$ be a proper convex function. When does φ have a minimum? This problem can be written as follows: let $\partial\varphi : H \to 2^H$ be the subdifferential of φ, i. e.,

$$\partial\varphi(x_0) = \{y \in H : \langle y, z - x_0\rangle \le \varphi(z) - \varphi(x_0)\ \forall z \in H\},$$

then Euler's variational problem is reduced to studying if $0 \in \mathsf{R}(\partial\varphi)$. Indeed, if $0 \in \mathsf{R}(\partial\varphi)$, there exists $x_0 \in D(\partial\varphi)$ such that $0 \in \partial\varphi(x_0)$, and therefore

$$\langle 0, z - x_0\rangle \le \varphi(z) - \varphi(x_0) \quad \forall z \in H,$$

which implies that $\varphi(x_0) \le \varphi(z)\ \forall z \in H$, that is, x_0 is a minimum of φ.

From above we may say that Euler's variational problem will have a solution if the operator $\partial\varphi$ has some zero. In Example 3.3.3 we showed that $\partial\varphi$ is an accretive operator and, by Corollary 3.4.1, its resolvents are nonexpansive mappings. Let us see that the zeroes of an accretive operator coincide with the fixed points of its resolvents.

Proposition 6.7.1. *Let X be a Banach space and let* $A : D(A) \to 2^X$ *be an accretive operator. Then*

$$A^{-1}0 = \{x \in D_\lambda : J_\lambda x = x\} =: \mathsf{Fix}(J_\lambda).$$

Proof. Let $x \in A^{-1}0$. This means that $0 \in Ax$, hence $0 \in \lambda Ax$ for all $\lambda > 0$. Thus, for all $\lambda > 0$, we have $x \in x + \lambda Ax$. Consequently, $J_\lambda x = x$, $\forall\lambda > 0$.

Now, let $x \in \mathsf{Fix}(J_\lambda)$. In this case, we have that $J_\lambda x = x$, then $x \in x + \lambda Ax$, that is, $0 \in \lambda Ax$. This implies that $0 \in Ax$ and therefore $x \in A^{-1}0$. ☐

Next we are going to apply fixed point results for nonexpansive mappings in order to obtain conditions under which Euler's variational problem admits a solution.

Theorem 6.7.1. *Let H be a real Hilbert space. If $\varphi : H \to]-\infty, +\infty]$ is a proper lower semi-continuous convex function with its effective domain $D(\varphi)$ bounded. Then the problem*

$$\varphi(x) = \min_{z \in D(\varphi)} \varphi(z) \tag{6.15}$$

has a solution.

Proof. Proposition 3.6.1 shows that $\partial\varphi$ is an m-accretive operator. Then applying Theorem 3.8.1, we infer that $\overline{D(\varphi)}$ is convex. Therefore, we may assume that $J_1 : \overline{D(\varphi)} \to \overline{D(\varphi)}$. Since H has (FPP) (see Theorem 6.5.1), there exists $x_0 \in D(\varphi)$ such that $x_0 = J_1 x_0$. Next, using Proposition 6.7.1, we conclude that x_0 is a solution of (6.15). ☐

As we have just seen, the study of the existence of zeroes for an accretive operator is strongly related to the existence of a fixed point for nonexpansive mappings.

Remark 6.7.1. Let X be a Banach space and let $A \subseteq X \times X$ be an accretive operator. If we consider the differential inclusion

$$u' + Au \ni 0,$$

the existence of a zero for A guarantees the existence of a constant solution (equilibrium point) for the above differential inclusion.

Theorem 6.7.2. *Let X be a uniformly convex Banach space and let $A \subset X \times X$ be an m-accretive operator. The following assumptions are equivalent:*

(a) $0 \in \mathrm{R}(A)$,

(b) *there exist $x_0 \in D(A)$ and a bounded neighborhood $\mathcal{U}$ of x_0 such that for every $x \in \partial\mathcal{U} \cap D(A)$, $\langle y, x - x_0\rangle_+ \geq 0$ for all $y \in A(x)$,*

(c) *there exists $x_0 \in \overline{D(A)}$ such that $E := \{x \in D(A) : t(x - x_0) \in A(x), t < 0\}$ is bounded.*

Proof. (a) $\Longrightarrow$ (b) By hypotheses, $0 \in \mathrm{R}(A)$, thus there exists $x_0 \in D(A)$ such that $0 \in Ax_0$. Let $\mathcal{U}$ be a bounded open neighborhood of x_0. Given $x \in \partial\mathcal{U} \cap D(A)$, since A is an accretive operator and $0 \in Ax_0$, we have $\langle y, x - x_0\rangle_+ \geq 0$, for all $y \in A(x)$.

(b) $\Longrightarrow$ (c) Suppose, for a contradiction, that E is unbounded. Then, we may assume that $\{J_\lambda x_0 : \lambda \geq 0\}$ is not bounded. Therefore, there exists $\lambda_0 > 0$ such that $J_{\lambda_0} x_0 \notin \mathcal{U}$ and, by Proposition 3.4.1(d), we know that

$$\lim_{\lambda \to 0^+} J_\lambda x_0 = x_0.$$

Now, by connectedness arguments, there exists $\lambda_1 \in (0, \lambda_0)$ such that $J_{\lambda_1} x_0 \in \partial\mathcal{U} \cap D(A)$. Nevertheless, Proposition 3.4.1(a) says that $A_{\lambda_1} x_0 \in AJ_{\lambda_1} x_0$ and, as a consequence, we have $\langle A_{\lambda_1} x_0, J_{\lambda_1} x_0 - x_0\rangle_+ \geq 0$, which is a contradiction.

(c) $\Longrightarrow$ (a) Given $n \in \mathbb{N}$, consider $x_n \in E$ such that $\frac{-1}{n}(x_n - x_0) \in Ax_n$. By hypothesis, E is a bounded set, thus we may assume that $(x_n)_{n\in\mathbb{N}}$ is weakly convergent to $y_0 \in \overline{D(A)}$.

Denote

$$R := \limsup_{n\to+\infty} \|x_n - y_0\|$$

and take

$$K := \{y \in \overline{D(A)} \cap \mathbb{B}_{2R}(y_0) : \phi_{(x_n)}(y) \le R\},$$

where $\phi_{(x_n)}(y) := \limsup_{n\to+\infty} \|x_n - y\|$. Clearly, K is a closed, bounded, convex, and nonempty subset of X. Since X has (FPP) (see Theorem 6.5.3), it suffices to show that K is J_1-invariant. Indeed, let $y \in K$. Since

$$\phi_{(x_n)}(J_1 y) \le \limsup_{n\to+\infty} \|J_1(x_n) - J_1 y\| + \limsup_{n\to+\infty} \|J_1(x_n) - x_n\|,$$

it follows form the fact that J_1 in nonexpansive and the definition of the Yosida approximant that

$$\phi_{(x_n)}(J_1 y) \le \phi_{(x_n)}(y) + \lim_{n\to+\infty} \|A_1 x_n\|.$$

Moreover, the construction of the sequence $(x_n)_{n\in\mathbb{N}}$ yields $|Ax_n| \to 0$ as $n \to +\infty$ and, by Proposition 3.4.1(d), $\|A_1 x_n\| \le |Ax_n|$ which implies that

$$\phi_{(x_n)}(J_1 y) \le R.$$

Finally, because $\|J_1 y - y_0\| \le \phi_{(x_n)}(y) + \phi_{(x_n)}(y_0) \le 2R$, we easily conclude that K is J_1-invariant. ☐

Theorem 6.7.1 shows that, when φ is a proper lower semicontinuous convex functions, Euler's variational problem has a solution if $D(\varphi)$ is bounded. Theorem 6.7.2 allows us to give the following result (see [105]).

Corollary 6.7.1. *Let H be a real Hilbert space. Let $\varphi : H \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function with effective domain $D(\varphi)$. Suppose that, for some $z_0 \in D(\varphi)$, there exists $R > 0$ such that $\varphi(z_0) < \varphi(x)$ for every $x \in D(\varphi)$ with $\|x\| \ge R$. Then φ has a minimum in H.*

Proof. By Proposition 3.6.1, we know that $\partial\varphi$ is an m-accretive operator on H. Moreover, as we have mention in this subsection, φ will have a minimum if its subdifferential admits some zero. Hence, we only have to show that $0 \in \mathrm{R}(\partial\varphi)$.

First, we note that since $\partial\varphi$ is m-accretive on H, by Theorem 3.7.1, $\overline{D(\partial\varphi)}$ is a convex subset. Therefore, invoking Theorem 6.7.2, we obtain that $0 \in \mathrm{R}(\partial\varphi)$ if and only if $E = \{x \in D(\partial\varphi) : tx \in \partial\varphi(x); t < 0\}$ is a bounded set.

Suppose, in order to get a contradiction, that E is unbounded. This fact means that for every $n \in \mathbb{N}$ there exists $x_n \in E$ such that $\|x_n\| > n$. Since $x_n \in E$, there are $t_n < 0$ and $y_n \in \partial\varphi(x_n)$ with $t_n x_n = y_n$. By the definition of subdifferential, we have that

$$\langle y_n, z_0 - x_n\rangle \leq \varphi(z_0) - \varphi(x_n).$$

Now, considering $\|x_n\| > \max\{R, \|z_0\|\}$ and bearing in mind that $y_n = t_n x_n$, we obtain

$$0 < -t_n\big(\|x_n\| - \|z_0\|\big)\|x_n\| \leq \langle t_n x_n, z_0 - x_n\rangle \leq \varphi(z_0) - \varphi(x_n) \leq 0,$$

which is a contradiction. □

Proposition 6.7.2. *Let X be a uniformly convex Banach space and let $A \subset X \times X$ be an m-accretive operator on X. If there exists $\lambda > 0$ such that J_λ is bounded on $\overline{D(A)}$, then A has a zero.*

Proof. Suppose that there exists $\lambda > 0$ such that J_λ is bounded on $\overline{D(A)}$, then there exists $r > 0$ such that $J_\lambda x \in \mathbb{B}_r$ for every $x \in \overline{D(A)}$ which implies that

$$J_\lambda : \overline{D(A)} \cap \mathbb{B}_r \to \overline{D(A)} \cap \mathbb{B}_r.$$

Now, by Theorem 3.8.1, we know that $\overline{D(A)} \cap \mathbb{B}_r$ is a bounded, convex, closed, and J_λ-invariant subset. Since X has (FPP) (see Theorem 6.5.3), we conclude that J_λ has a fixed point and therefore $0 \in \mathrm{R}(A)$. This completes the proof. □

Lemma 6.7.1. *Let X be a Banach space and let $A : D(A) \subset X \to 2^X$ be an m-accretive operator on X. If $x_0 \in D(A)$ is such that there is a bounded neighborhood of x_0, $\mathcal{U}$, such that*

$$|Ax_0| < |Ax| \quad \forall x \in \partial\mathcal{U} \cap D(A),$$

then, for all $\lambda > 0$, $J_\lambda x_0 \in \mathcal{U}$.

Proof. Theorem 3.4.1(c) shows that $\|A_\lambda x_0\| \leq |Ax_0|$. Moreover, Proposition 3.4.1(a) implies that $A_\lambda x_0 \in AJ_\lambda x_0$ and therefore, $|AJ_\lambda x_0| \leq |Ax_0|$. Since by hypotheses $|Ax_0| < |Ax|$ $\forall x \in \partial\mathcal{U} \cap D(A)$, it is clear that $J_\lambda x_0 \notin \partial\mathcal{U}$. On the other hand, since $x_0 \in \mathcal{U}$, by Proposition 3.4.1(d), we have that $\lim_{\lambda\to 0^+} J_\lambda x_0 = x_0$. (This means that there exists $\delta > 0$ so that if $0 < \lambda < \delta$ then $J_\lambda x_0 \in \mathcal{U}$.) Let $k := \sup\{\lambda > 0 : J_\lambda x_0 \in \mathcal{U}\}$. We shall check that $k = +\infty$. Otherwise, if $k \in \mathbb{R}$, then $J_k x_0 \in \mathcal{U}$. Indeed, by the above argument,

$$J_k x_0 \notin \partial\mathcal{U}. \tag{6.16}$$

Nevertheless, since there exists λ_n such that $\lambda_n \to k$, by Proposition 3.4.1, we have

$$J_{\lambda_n} x_0 \to J_k x_0.$$

Hence

$$J_k x_0 \in \overline{\mathcal{U}}. \tag{6.17}$$

From (6.16) and (6.17), we obtain that $J_k x_0 \in \mathcal{U}$. Using again the continuity of the mapping $\lambda \to J_\lambda x$ and $J_k x_0 \in \mathcal{U}$, we obtain a contradiction because there will be $\lambda > k$ such that $J_\lambda x_0 \in \mathcal{U}$. □

As a consequence of Theorem 6.7.2 and Lemma 6.7.1, we have the following result.

Corollary 6.7.2. *Let X be a uniformly convex Banach space and let $A : D(A) \to 2^X$ be an m-accretive operator. Suppose that $\mathcal{U}$ is a bounded neighborhood of $x_0 \in D(A)$ such that*

$$|Ax_0| < |Ax| \quad \forall x \in \partial\mathcal{U} \cap D(A).$$

Then $0 \in \mathrm{R}(A)$.

Theorem 6.7.3. *Let X be a uniformly convex Banach space. Under the conditions of Lemma* 6.7.1, *there exists $y \in \mathcal{U} \cap D(A)$ such that* $0 \in Ay$.

Proof. According to Lemma 6.7.1 we have $J_\lambda x_0 \in \mathcal{U}$, $\forall \lambda > 0$. Since $|AJ_\lambda x_0| \leq \|A_\lambda x_0\| = \frac{1}{\lambda}\|x_0 - J_\lambda x_0\|$, using the fact that $\mathcal{U}$ is bounded, we obtain

$$\lim_{\lambda\to\infty} |AJ_\lambda x_0| = 0. \tag{6.18}$$

Set $d := \inf\{|Ax| : x \in \partial\mathcal{U} \cap D(A)\}$. There are two possibilities:
(a) $0 \in Ax_0$. In this case there is nothing to prove.
(b) $0 \notin Ax_0$.

Case (b) For all $x \in \partial\mathcal{U} \cap D(A)$, we have $|Ax_0| < |Ax|$ and therefore $d > 0$. This follows from the closedness of A (use Proposition 3.7.1). To see this, assume that $d = 0$, which yields that $|Ax_0| = 0$, meaning that there exists a sequence $(x_0, y_n) \in A$ such that $\|y_n\| \to 0$, hence A is closed, $(x_0, 0) \in A$, and this is a contradiction.

Now we may assume that $|Ax_0| < d$, otherwise, we would work with $J_\lambda x_0$ taking λ large enough so that $|AJ_\lambda x_0| < d$ (see (6.18)). Since $\mathcal{U}$ is bounded and $|Ax_0| < d$, there exist $R > 0$ and $r > 0$ such that

$$\mathcal{U} \subset \mathbb{B}_R(x_0),$$

where R and r satisfy $|Ax_0| + \frac{2}{r}R < d$.

Fix $y \in \mathbb{B}_R(x_0)$. Since A is m-accretive, $y \in D_\lambda$, $\forall \lambda > 0$. Hence $J_r y \in D(A)$. Let $x = J_r y$. We assert that $x \in \mathcal{U}$. To see this, define the following operator:

$$\tilde{A} : D(A) \to 2^X, \quad \tilde{A}x := Ax + \frac{1}{r}(x_0 - y).$$

It is not hard to show that $\tilde{A}$ is also an m-accretive operator. Denote by $\tilde{J}_\lambda$ its resolvent. Then for $z \in \partial\mathcal{U} \cap D(A)$, we have

$$\begin{aligned}
|\tilde{A}z| &= \left|Az + \frac{1}{r}(x_0 - y)\right| \geq |Az| - \frac{1}{r}\|x_0 - y\| \\
&\geq d - \frac{1}{r}\|x_0 - y\| > |Ax_0| + \frac{R}{r} \\
&\geq \left|Ax_0 + \frac{1}{r}(x_0 - y)\right| \\
&= |\tilde{A}x_0|.
\end{aligned}$$

Applying Lemma 6.7.1 to $\tilde{A}$, we conclude that $x = \tilde{J}_r x_0 \in \mathcal{U}$. Indeed, $x = J_r y$ if and only if $y \in x + rAx$, if and only if $x_0 \in x + rAx + x_0 - y$, if and only if $x_0 \in x + rAx + r\frac{1}{r}(x_0 - y)$, if and only if $x_0 \in (I + r\tilde{A})x$, if and only if $x = \tilde{J}_r x_0$.

The conclusion is now immediate. Indeed, consider the mapping

$$J_r : \mathbb{B}_R(x_0) \to \mathcal{U} \subset \mathbb{B}_R(x_0).$$

Since J_r is nonexpansive and X has (FPP) for nonexpansive mappings, there exists $y \in \mathbb{B}_R(x_0)$ such that $J_r y = y$. Now applying Proposition 6.7.1, we conclude that $0 \in Ay$. ☐

Next we shall discuss under which conditions an m-accretive operator is surjective in the sense that its range is the whole space.

Lemma 6.7.2. *Let X be a uniformly convex Banach space and let $A : D(A) \to 2^X$ be an m-accretive operator. If there exists $x_0 \in D(A)$ such that $|Ax_0| < r \leq \lim_{\|x\|\to+\infty} |Ax|$, then*

$$\mathbb{B}_{\frac{1}{2}(r-|Ax_0|)} \subset \mathrm{R}(A).$$

Proof. Let $y \in X$ be such that $\|y\| < \frac{1}{2}(r - |Ax_0|)$ and define the operator $\tilde{A} = A - y$. The operator $\tilde{A}$ is m-accretive. Indeed, since A is m-accretive, given $z + \lambda y \in X$ there is $v \in X$ such that $z + \lambda y \in v + \lambda Av$, which means $z \in v + \lambda Av - \lambda y$, that is, $z \in (I + \lambda\tilde{A})v$. Therefore, $X = \mathrm{R}(I + \lambda\tilde{A})$.

Now, we claim that there exists $R > 0$ such that

$$|\tilde{A}x_0| < |\tilde{A}x| \quad \forall x \in D(A) : \|x - x_0\| \geq R. \tag{6.19}$$

If no such R exists, then there is a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $\|x_n\| \to +\infty$ and $|\tilde{A}x_n| \leq |\tilde{A}x_0|$. It follows that $|Ax_n - y| \leq |Ax_0 - y|$, hence $|Ax_n| - \|y\| \leq \|y\| + |Ax_0|$. This implies that $|Ax_n| - |Ax_0| < 2\|y\|$, that is, $\frac{1}{2}(|Ax_n| - |Ax_0|) < \|y\|$. However, $r \leq \liminf_{n\to+\infty} |Ax_n|$, then $\frac{1}{2}(r - |Ax_0|) < \|y\|$ which is a contradiction because $y \in \mathbb{B}_{\frac{1}{2}(r-|Ax_0|)}$. This shows that $\tilde{A}$ satisfies the hypothesis of Theorem 6.7.3, and therefore $0 \in \mathrm{R}(\tilde{A})$. So, there exists $v \in X$ such that $0 \in \tilde{A}v$, therefore $y \in Av$. This proves the result. ☐

Lemma 6.7.3. *Let X be a uniformly convex Banach space and let $A : D(A) \to 2^X$ be an m-accretive operator. If there exists $x_0 \in D(A)$ such that $|Ax_0| < r \leq \lim_{\|x\|\to+\infty} |Ax|$, then* $\mathbb{B}_r \subset \mathrm{R}(A)$.

Proof. Fix $y \in X$ with $\|y\| \leq r$ and set

$$M := \{t \in [0,1] : ty \in \mathrm{R}(A)\}.$$

By Lemma 6.7.2, $M \neq \emptyset$ because there exists $t \in [0,1]$ such that $t\|y\| \in \mathbb{B}_{\frac{1}{2}(r-|Ax_0|)} \subset \mathrm{R}(A)$. We claim that $1 \in M$. To see this, let $t_0 := \sup M$ and choose a sequence $(t_n)_{n\in\mathbb{N}} \subset M$ such that $t_n \to t_0$ as $n \to \infty$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of $D(A)$ such that $t_n y \in Ax_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define the mapping

$$A_n x := A(x + x_n) - t_n y, \quad x \in D(A_n) = D(A) - x_n.$$

Clearly, $0 \in A_n 0$ and

$$\liminf_{\|x\|\to+\infty} |A_n x| \geq r - t_n\|y\|.$$

Applying Lemma 6.7.2 to A_n (replacing r by $r - t_n\|y\|$), we get

$$\mathbb{B}_{\frac{1}{2}(r-t_n\|y\|)} \subset \mathrm{R}(A_n).$$

But, if $t \geq t_0$ is sufficiently close to t_0, it is possible to choose $n \in \mathbb{N}$ such that

$$(t - t_n)\|y\| < \frac{1}{2}(r - t_n\|y\|).$$

For such n, there exists $z_n \in D(A) - x_n$ so that $(t - t_n)y \in A_n z_n$, i. e., $ty \in A(z_n + x_n)$, which proves that $1 \in M$. ☐

As an easy consequence of the last result, we have

Theorem 6.7.4. *Let X be a uniformly convex Banach space and let $A : D(A) \to 2^X$ be an m-accretive operator such that*

$$\lim_{\|x\|\to+\infty} |Ax| = +\infty.$$

Then $\mathrm{R}(A) = X$.

Remark 6.7.2. The above results hold for uniformly convex Banach spaces. Most of these results could be extended to spaces with (FPP) for nonexpansive mappings (see [115, 182, 201]). Nevertheless, it is well known that there are Banach spaces without (FPP) for nonexpansive mappings and then the above results do not apply in these spaces. The reader is also referred to [105, 112] where several results in this sense can be found.

6.8 Zeros of *m*-accretive and ϕ-expansive operators

In this section we shall work with a subclass of accretive operators (those which are also ϕ-expansive) for which the geometric properties of the Banach space framework are irrelevant. Next we give an example of such operators.

Example 6.8.1 ([97]). Consider the Banach space $(\ell_2, \|\cdot\|_2)$ and let $\{e_n\}$, $n = 1, 2, \ldots$ be the usual Schauder basis of such a space. Define the following operator:

$$f\left(\sum_{k=1}^{+\infty} x_k e_k\right) = \sum_{i=1}^{+\infty} z_i e_i$$

where

$$z_i = \begin{cases} x_{2k}, & i = 2k-1, \\ -x_{2k-1}, & i = 2k. \end{cases}$$

Let us see that f is accretive:

$$\begin{aligned} \langle f(x) - f(y), x - y\rangle &= \left\langle \sum_{k=1}^{+\infty}(x_{2k} - y_{2k})e_{2k-1} + \sum_{k=1}^{+\infty}(y_{2k-1} - x_{2k-1})e_{2k}, \sum_{k=1}^{+\infty}(x_k - y_k)e_k \right\rangle \\ &= \sum_{k=1}^{+\infty}(x_{2k} - y_{2k})(x_{2k-1} - y_{2k-1}) + \sum_{k=1}^{+\infty}(y_{2k-1} - x_{2k-1})(x_{2k} - y_{2k}) = 0. \end{aligned}$$

Moreover, if we take $\phi(t) = t$, then we obtain

$$\|f(x) - f(y)\| = \left\| \sum_{k=1}^{+\infty}(x_{2k} - y_{2k})e_{2k-1} + \sum_{k=1}^{+\infty}(y_{2k-1} - x_{2k-1})e_{2k} \right\| = \|x - y\| = \phi(\|x - y\|).$$

Proposition 6.8.1. *Let X be a Banach space, let $A : D(A) \subseteq X \to X$ be an m-accretive and ϕ-expansive mapping. Let U be a bounded neighborhood of $x_0 \in D(A)$ such that*

$$\|A(x_0)\| < \|A(x)\| \quad \textit{for all } x \in \partial U \cap D(A).$$

Then $0 \in \mathrm{R}(A)$.

Proof. Assume that $\|A(x_0)\| > 0$. Otherwise, the proposition holds. Suppose that there exists $\lambda > 0$ such that $J_\lambda x_0 \in \partial U$. It follows from Proposition 3.4.1 that $A_\lambda x_0 = AJ_\lambda x_0$. But Proposition 3.4.1 says that

$$\|AJ_\lambda x_0\| = \|A_\lambda x_0\| \leq \|Ax_0\|,$$

which is a contradiction. As a consequence of this, along with the fact that $\lambda \mapsto J_\lambda x_0$ is a continuous function on $[0, \infty)$ and $\lim_{\lambda \downarrow 0} J_\lambda x_0 = x_0$, yields that, for every $\lambda > 0$, we have $J_\lambda x_0 \in U$. Since U is a bounded set and $A_\lambda x_0 \in \mathrm{R}(A)$, we infer that $0 \in \overline{\mathrm{R}(A)}$. That is, $\inf\{\|A(x)\| : x \in D(A)\} = 0$. We may suppose (possibly by redefining x_0) that

$$\|A(x_0)\| < \rho = \inf\{\|A(x)\| : x \in \partial U \cap D(A)\}.$$

Select $R > 0$ so that $U \subset \mathbb{B}_R(x_0)$. Then there exists $r > 0$ such that

$$\|A(x_0)\| + 2r^{-1}R < \rho.$$

We claim that the resolvent J_r maps $K := \mathbb{B}_R(x_0)$ into itself. To see this, let $u \in K$ and $v = J_r(u)$. Then $v \in D(A)$. This means that we just need to show that $v \in \mathbb{B}_R(x_0)$. To see this, define $\tilde{A}$ by

$$\tilde{A}(x) = A(x) + r^{-1}(x_0 - u), \quad x \in D(A),$$

and let $\tilde{J}_r$ denote the resolvent of $\tilde{A}$. Consider $x \in \partial\mathbb{B}_R(x_0) \cap D(A)$, then

$$\begin{aligned}\|\tilde{A}(x_0)\| = \|A(x_0) + r^{-1}(x_0 - u)\| &\leq \|A(x_0)\| + r^{-1}R \\ &< \rho - r^{-1}R \leq \|A(x)\| - r^{-1}\|u - x_0\| \\ &\leq \|A(x) + r^{-1}(x_0 - u)\| = \|\tilde{A}(x)\|.\end{aligned}$$

Since $x_0 \in \mathrm{R}(I + \lambda\tilde{A})$ for all $\lambda \in (0, r]$, we may use the same argument as above to derive $v = \tilde{J}_r x_0 \in \mathbb{B}_R(x_0) \cap D(A)$ and therefore $v \in K$. On the other hand, since A is ϕ-expansive, by definition we have

$$\|Ax - Ay\| \geq \phi(\|x - y\|),$$

and, in this case, the use of Proposition 3.4.1 yields $A_r x \in AJ_r x$. Hence

$$\phi(\|J_r x - J_r y\|) \leq \|A_r x - A_r y\| \leq \frac{1}{r}\|x - J_r x - (y - J_r y)\|.$$

Following Lemma 6.6.1, this means that J_r has a fixed point in $\mathbb{B}_R(x_0)$, which implies that $0 \in A(\mathbb{B}_R(x_0) \cap D(A))$. ☐

Theorem 6.8.1. *Let X be a Banach space. Suppose $A : D(A) \subset X \to X$ is an m-accretive and ϕ-expansive mapping on $D(A)$. Then A is bijective.*

Proof. We shall first check that A is one-to-one. Let x, y be two elements $D(A)$. Since A is ϕ-expansive, we have

$$\|Ax - Ay\| \geq \phi(\|x - y\|).$$

So, if $x \neq y$, then $\|Ax - Ay\| \geq \phi(\|x - y\|) > 0$, that is, $Ax \neq Ay$.

Now, we are going to prove that $\mathrm{R}(A) = X$. To do this, we will show that $\mathrm{R}(A)$ is a nonempty set which is open and closed at the same time.

Let us see that $\mathrm{R}(A)$ is an open set. Indeed, let $y_0 \in \mathrm{R}(A)$ in this case we know that there exists $x_0 \in D(A)$ such that $y_0 = A(x_0)$. We claim that for every $r > 0$, $\partial\mathbb{B}_r(x_0) \cap D(A) \neq \emptyset$. Otherwise, there exists $r_0 > 0$ so that $\partial\mathbb{B}_{r_0}(x_0) \cap D(A) = \emptyset$. In this case, by Proposition 3.4.1, we have $\lambda \mapsto J_\lambda x_0$ is a continuous function on $[0, \infty)$ and $\lim_{\lambda\downarrow 0} J_\lambda x_0 = x_0$. Thus, for every $\lambda > 0$, we have $J_\lambda x_0 \in \mathbb{B}_{r_0}(x_0)$.

The resolvent operator J_1 maps $\mathbb{B}_{r_0}(x_0)$ into itself. To see this, let $y \in \mathbb{B}_{r_0}(x_0)$, $x = J_1 y$, and define the mapping

$$\tilde{A}(w) = A(w) + x_0 - y.$$

Then $x = J_1^{\tilde{A}}(x_0)$. Since $\tilde{A}$ is also m-accretive with domain $D(\tilde{A}) = D(A)$, by the same argument as above, we have that $J_\lambda^{\tilde{A}} x_0 \in \mathbb{B}_{r_0}(x_0)$. This means that $x \in \mathbb{B}_{r_0}(x_0)$. Therefore the nonexpansive mapping J_1 maps the closed ball $\mathbb{B}_{r_0}(x_0)$ into itself. Since A is accretive and ϕ-expansive, the use of Proposition 3.4.1, together with $A_1 x \in AJ_1 x$, implies

$$\phi(\|J_1 x - J_1 y\|) \leq \|A_1 x - A_1 y\| \leq \big\|x - J_1 x - (y - J_1 y)\big\|.$$

Hence, applying Lemma 6.6.1, we conclude that J_1 has a fixed point in $\mathbb{B}_{r_0}(x_0)$, which implies that $0 \in A(\mathbb{B}_{r_0}(x_0) \cap D(A))$.

Let $z \in X$ and define the mapping $B := A - z$. It is clear that B is m-accretive, $D(A) = D(B)$, ϕ-expansive, and $\partial\mathbb{B}_{r_0}(x_0) \cap D(B) = \emptyset$. Hence $0 \in \mathrm{R}(B)$. This implies that $z \in \mathrm{R}(A)$ and then $\mathrm{R}(A) = X$.

Consequently, we can assume our claim, i. e., $\partial\mathbb{B}_r(x_0) \cap D(A) \neq \emptyset$, for every $r > 0$.

Define

$$\tilde{A}(x) = A(x + x_0) - y_0 \quad \text{for } x \in D(\tilde{A}) := D(A) - x_0.$$

Let $y = A(x + x_0)$ and $x \in \partial\mathbb{B}_r \cap D(\tilde{A})$, then

$$\|y - y_0\| \geq \phi(\|x\|) = \phi(r) > \big|\tilde{A}(0)\big| = 0.$$

Therefore, $\|\tilde{A}(0)\| < \phi(r) \leq \|\tilde{A}(x)\|$. We first show that $\mathbb{B}_{\phi(r)/2} \subset \mathrm{R}(\tilde{A})$. To see this, let $z \in \mathbb{B}_{\phi(r)/2}$ and $\tilde{B}(x) = \tilde{A}(x) - z$. If $x \in \partial\mathbb{B}_r \cap D(\tilde{A})$, then

$$\begin{aligned}
\big\|\tilde{B}(0)\big\| &\leq \big\|\tilde{A}(0)\big\| + \|z\| \\
&< \phi(r) - \|z\| \\
&\leq \big\|\tilde{A}(x)\big\| - \|z\| \\
&\leq \big\|\tilde{A}(x) - z\big\| = \big\|\tilde{B}(x)\big\|.
\end{aligned}$$

Since $\tilde{B}$ satisfies the assumptions of Proposition 6.8.1, $\tilde{B}$ has a unique zero in $D(\tilde{A})$, and hence $z \in \mathrm{R}(\tilde{A})$. This implies that $\mathbb{B}_{\phi(r)/2}(y_0) \subseteq \mathrm{R}(A)$ and therefore $\mathrm{R}(A)$ is an open.

Suppose that $\mathrm{R}(A)$ is not a closed set. This means that there exists $z \in \partial\mathrm{R}(A)$ such that $z \notin \mathrm{R}(A)$, and then we may choose $z_0 \in \mathrm{R}(A)$ such that $\|z - z_0\| \le \phi(r)/3$. However, repeating the same argument once again, we obtain $\mathbb{B}_{\phi(r)/2}(z_0) \subset \mathrm{R}(A)$. This implies that $z \in \mathrm{R}(A)$, which is a contradiction. Therefore, $\mathrm{R}(A)$ is a closed set. Thus A is surjective because $\mathrm{R}(A)$ is an open and closed nonempty subset of X. ☐

Remark 6.8.1. In [107] it was proved that if X is a real Banach space and $A : D(A) \to 2^X$ is a ϕ-expansive, m-accretive operator, then $\mathrm{R}(A) = X$. We have just proved that every m-accretive and ϕ-expansive operator $A : D(A) \subseteq X \to X$, independently of the Banach space where it is defined, is bijective. This means that given $u \in X$ there exists a unique $x \in D(A)$ such that $u = A(x)$, and hence one can define a mapping $A^{-1} : X \to D(A)$ by $A^{-1}(u) = x$. Moreover, when ϕ is a continuous mapping with $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$, it is not difficult to see that A^{-1} is continuous. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $x_n \to x$, we have to prove that $A^{-1}x_n \to A^{-1}x$. This follows from the following estimate:

$$\|x_n - x\| \ge \phi(\|A^{-1}(x_n) - A^{-1}(x)\|).$$

Proposition 6.8.2. *Let $1 \le p < \infty$. If $A : D(A) \subset L^p(\Omega) \to L^p(\Omega)$ is an accretive operator and there exists $A^{-1} : L^p(\Omega) \to D(A)$ which is continuous, then A^{-1} is m-accretive.*

Proof. First case: $1 < p < \infty$. If A is accretive in $L^p(\Omega)$, we have that, for every $u, v \in D(A)$,

$$0 \le \langle Au - Av, u - v\rangle_+ = \|u - v\|_p^{2-p} \int_\Omega (Au(x) - Av(x))(u(x) - v(x))|u(x) - v(x)|^{p-2}\,dx. \tag{6.20}$$

Given $w, z \in L^p(\Omega)$, by hypothesis there exists $u, v \in D(A)$ such that $u = A^{-1}(w)$ and $v = A^{-1}(z)$. Therefore,

$$\langle A^{-1}(w) - A^{-1}(z), w - z\rangle_+ = \|w - z\|_p^{2-p} \int_\Omega (u(x) - v(x))(Au(x) - Av(x))|w - z|^{p-2}\,dx.$$

Notice that expression (6.20) allows us to derive that

$$\langle A^{-1}(w) - A^{-1}(z), w - z\rangle_+ \ge 0,$$

which means that A^{-1} is accretive in $L^p(\Omega)$.

Second case: $p = 1$. If A is accretive in $L^1(\Omega)$, we have that, for every $u, v \in D(A)$,

$$\begin{aligned}0 \le \langle Au - Av, u - v\rangle_+ = \|u - v\|_1 &\int_{\{x\in\Omega: u(x)\ne v(x)\}} (Au(x) - Av(x))\frac{u(x) - v(x)}{|u(x) - v(x)|}\,dx \\ &+ \int_{\{x\in\Omega: u(x)=v(x)\}} |Au(x) - Av(x)|\,dx.\end{aligned} \tag{6.21}$$

Since A is a mapping in $L^1(\Omega)$, we have that either $Au(x) = Av(x)$ a. e. in the set Ω_1 given by $\Omega_1 := \{x \in \Omega : u(x) = v(x)\}$ or Ω_1 is a null set; both cases imply that

$$0 \leq \langle Au - Av, u - v\rangle_+ = \|u - v\|_1 \int\limits_{\{x\in\Omega:u(x)\neq v(x)\}} (Au(x) - Av(x))\frac{u(x) - v(x)}{|u(x) - v(x)|}\,dx. \tag{6.22}$$

On the other hand, since A is one-to-one on $L^1(\Omega)$, given $w, z \in L^1(\Omega)$, and $u = A^{-1}w$, $v = A^{-1}z$, it is clear that $\{x \in \Omega : u(x) \neq v(x)\} = \{x \in \Omega : w(x) \neq z(x)\}$ a. e. in Ω. Therefore

$$\begin{aligned}\langle A^{-1}w - A^{-1}z, w - z\rangle_+ &= \|w - z\|_1 \int\limits_{\{x\in\Omega:w(x)\neq z(x)\}} (u(x) - v(x))\frac{Au(x) - Av(x)}{|w(x) - z(x)|}\,dx \\ &= \|w - z\|_1 \int\limits_{\{x\in\Omega:u(x)\neq v(x)\}} (u(x) - v(x))\frac{Au(x) - Av(x)}{|w(x) - z(x)|}\,dx.\end{aligned}$$

Notice that, by (6.22), we can conclude that

$$\langle A^{-1}(w) - A^{-1}(z), w - z\rangle_+ \geq 0,$$

which means that A^{-1} is accretive in $L^1(\Omega)$.

We have just proved that if A is accretive in $L^p(\Omega)$, then $A^{-1} : L^p(\Omega) \to D(A)$ is a continuous accretive operator. Hence, by Theorem 4.4.3, A^{-1} is m-accretive. ☐

6.9 Bibliographical remarks

The content of Section 6.2 is classic and can be found, for example, in the books [87, 115, 141, 152] or [244]. In the literature, there are many extensions of the Banach contraction principle. In Section 6.3 we presented some of them. In Section 6.3.1, Theorem 6.3.1 is due to D. W. Boyd and J. S. W. Wong [44]. For separate contraction mappings, Theorem 6.3.3 is due to Y. Liu and Z. Li [164]. In Section 6.3.3 we discuss the class of expansive operators. Theorem 6.3.4 may be found in [241].

Section 6.5 deals with nonexpansive mappings. In this section we only gave the classical results due to F. E. Browder [50], D. Göhde [117], and W. A. Kirk [142], and, as we have mentioned, in the books [115, 143] many results about this theory were collected. With respect to (FPP), the absence of normal structure allows us to quote the books by A. G. Aksoy and M. A. Khamsi [10], and B. Sim [218]. On the other hand, it was an open question if (FPP) is a sufficient condition for the reflexivity. This question was resolved negatively by P. K. Lin [163]. In this sense, we would like to remember that it is still an open question if reflexivity implies (FPP).

In Section 6.6 we gathered some fixed point theorems for ϕ-expansive mappings. The results of this section are recent and taken essentially from the papers [98] and [109].

The content of Sections 6.7 and 6.8 can be found, for example, in [107, 112, 115, 182].

7 Topological fixed point theorems

7.1 Introduction

Many mathematical and applied science problems can be formulated as equations of the type $f(x) = x$ or $f(x) + g(x) = x$, where x is an element of a certain set K and f, g are nonlinear operators. These equations are strongly related and are generally classified as fixed point equations. Unlike in Chapter 6, now we will focus on the topological fixed point theory for the weak topology (results using topological properties). With respect to the topological fixed point theory, the main two theorems are Brouwer fixed point theorem and its infinite-dimensional version, Schauder fixed point theorem [212]. In both theorems compactness plays an essential role. In 1955, G. Darbo [73] extended Schauder theorem to the setting of noncompact operators, introducing the notion of a k-set-contraction. In 1958, M. A. Krasnoselskii established that the sum of two operators $f + g$ has a fixed point in a nonempty closed convex subset K of a Banach space X, whenever f and g satisfy: (i) $f(x) + g(y)$ belongs to K for all x, y in K, (ii) f is continuous on K and $f(K)$ is contained in a compact subset of X, (iii) g is k-contractive on X with $0 \leq k < 1$. This result combines both the Banach contraction principle and Schauder fixed point theorem, linking metric and topological fixed point theories. Nevertheless, it is not hard to see that Krasnoselskii fixed point theorem is a particular case of G. Darbo fixed point theorem. Namely, it appears that $f + g$ is a k-set contractive map with respect to the K. Kuratowski measure of noncompactness. In 1967, V. N. Sadovskii [210] gave a more general fixed point result than Darbo theorem using the concept of a condensing mapping.

The examination of some particular examples of nonlinear functional equations involving weakly continuous or weakly sequentially continuous maps in an infinite-dimensional Banach space shows that the classical Schauder fixed point theorem and its corollaries (see, for example, [3, 87, 115, 141, 221]) are not adapted to this kind of problems and do not make it possible to establish existence results. The fixed point theorem of A. N. Tychonoff, which is an extension to Hausdorff locally convex topological vector spaces of Schauder fixed point theorem, ensures the existence of fixed points for continuous mappings from a compact convex subset K of X into itself. We note that, if X is a normed space, then X, endowed with its weak topology $\sigma(X, X^*)$, is a Hausdorff locally convex topological vector space. According to Tychonoff fixed point theorem, if X is a normed space, then in $(X, \sigma(X, X^*))$ every weakly continuous mapping from a weakly compact convex subset of X into itself admits a fixed point (see Corollary 7.5.2). The disadvantage of this result is that, in practice, it is often difficult to verify that a map is weakly continuous. In [21], O. Arino, S. Gautier, and J. P. Penot have shown that Tychonoff fixed point theorem is valid for weakly sequentially continuous mappings in metrizable locally convex topological vector spaces. Except for the metrizability of the space, this result does not require any additional restrictive hypothesis such as the reflexivity or the separability of

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the space. The essential argument in the proof of this result is the theorem of Eberlein–Šmulian (Theorem 1.7.3). In addition, the weak sequential continuity for the topology of a map is often easier to verify than the weak continuity. Knowing this result, in the 1990s, D. O'Regan undertook, in a series of papers, the study of the existence of fixed points for weakly continuous, as well as sequentially weakly continuous, mappings [185–193]. Motivated by the existence of solutions of a stationary neutron transport equation in [159], the authors introduced two classes of operators, ws-compact and ww-compact maps, and have derived new fixed point theorems of Schauder, Darbo, and Krasnosel'skii type for these classes of operators. The interest in ws-compact operators lies in the fact that they make it possible to obtain fixed point theorems of Schauder, Darbo, or Sadovski type for operators transforming a weakly compact subset into itself without assuming that these operators are weakly continuous. Obviously, for theorems of Darbo–Sadovskii type, we use the concept of a weak noncompactness measure. The ww-compact operators often intervene in fixed points theorems of Krasnosel'skii type when the perturbed operator is ws-compact. Since 2006, these two classes of operators have given rise to numerous developments (see, for example, [4, 55, 83, 98, 99, 101–103, 158, 159, 194, 235, 237]).

The purpose of this chapter is to present an introduction to the topological fixed point theory in Banach spaces, not necessarily reflexive, involving the weak topology. For completeness, in Section 7.2 we present Brouwer and Schauder fixed point theorems. The concept of a measure of noncompactness is introduced in Section 7.3, and some new results of Darbo and Sadovskii type are given. To discuss fixed point theorems involving the weak topology, our starting point is Tychonoff fixed point theorem. We give in Section 7.4 its formulation in a Banach space endowed with its weak topology and then its formulation for weakly sequentially continuous maps in metrizable locally convex topological vector spaces. We introduce in Section 7.5 the classes of ws-compact and ww-compact maps and give the analog of Schauder fixed point theorem for ws-compact maps. Section 7.6 deals with the abstract formulation of the concept of a measure of weak noncompactness, we discuss in detail the measure of weak compactness of F. De Blasi [75] and that of J. Banaś and Z. Knap [27] and show that they are regular measures of weak noncompactness in the sense of Definition 7.7.2. In Section 7.7 we present some theorems for weakly continuous and weakly sequentially continuous set contraction maps with respect to a measure of weak noncompactness. Theorems of Darbo and Sadovskii type for weakly continuous and weakly sequentially continuous maps are given. Some nonlinear alternatives of Leray–Schauder type for weakly sequentially continuous maps are the subject of Section 7.10. Section 7.11 is dedicated to fixed point theorems of Krasnosel'skii type. We give numerous fixed point theorem for maps involving continuity, weak continuity, weak sequential continuity, ws-compactness, ww-compactness, and many other kinds of perturbations. Nonlinear alternatives of Krasnosel'skii–Leray–Schauder and Krasnosel'skii–Schaefer type are also presented at the end of this section.

7.2 Brouwer and Schauder fixed point theorems

7.2.1 Brouwer fixed point theorem

In this section we shall recall Brouwer fixed point theorem and some of its consequences. It is a profound result which is at the origin of the topological fixed point theory.

We shall assume that $\mathbb{R}^n$ is endowed with its standard inner product, $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$ and norm $\|x\| = \langle x, x\rangle^{1/2}$. In this subsection $\mathbb{B}_1$ denotes the closed unit ball of $\mathbb{R}^n$ and $\mathbb{S}^{n-1} = \partial\mathbb{B}_1$ denotes the unit sphere in $\mathbb{R}^n$.

Theorem 7.2.1 (Brouwer). *Every continuous map $f : \mathbb{B}_1 \to \mathbb{B}_1$ has a fixed point, i. e.,* $\mathrm{Fix}(f) \neq \emptyset$.

Despite the simplicity of the statement of Brouwer theorem, his proof is not obvious even in dimension 2 and uses arguments from the algebraic topology. In [176], J. Milnor gave a proof of this result based on elementary multidimensional integral calculus. Milnor's proof was simplified by C. A. Rogers in [204]. In [126], we find an exposition of the Milnor–Rogers' proof. In Section 2 of his thesis [222], T. Stuckless gathered various proofs of Brouwer theorem.

Let us first give the relationship between Brouwer fixed point theorem and retractions.

Theorem 7.2.2. *The Brouwer fixed point theorem is equivalent to the following assertion: No infinitely differentiable retraction exists from $\mathbb{B}_1$ onto $\mathbb{S}^{n-1}$.*

Proof. Suppose that Brouwer fixed point theorem holds but there exists an infinitely differentiable retraction $f : \mathbb{B}_1 \to \mathbb{S}^{n-1}$. Let $g : \mathbb{B}_1 \to \mathbb{B}_1$ be given by $g(x) = -f(x)$. Therefore, g also maps $\mathbb{B}_1$ into $\mathbb{S}^{n-1}$. Hence, if $x = g(x)$, we have $x \in \mathbb{S}^{n-1}$. But for $x \in \mathbb{S}^{n-1}$, $g(x) = -f(x) = -x \neq x$. Thus g has no fixed point, contradicting Theorem 7.2.1.

Conversely, suppose that there is no such an infinitely differentiable retraction. We are going to see that if $f : \mathbb{B}_1 \to \mathbb{B}_1$ is any infinitely differentiable mapping, then it has a fixed point. Otherwise, we would have for each $x \in \mathbb{B}_1$ that $x \neq f(x)$ and thus we could consider the semiline starting at $f(x)$ and passing through x. This semiline intersects $\mathbb{S}^{n-1}$ at one point, say $g(x)$. Hence, we may define a mapping $g : \mathbb{B}_1 \to \mathbb{S}^{n-1}$ such that $g(x) \neq f(x)$ for all $x \in \mathbb{B}_1$ and $g(x) = x$ for all $x \in \mathbb{S}^{n-1}$. Let us show that g is infinitely differentiable. Indeed, for every $x \in \mathbb{B}_1$, we have

$$g(x) = \beta(x)x + (1 - \beta(x))f(x),$$

where $\beta(x)$ is chosen satisfying $\langle g(x), g(x)\rangle = 1$. This leads to the equation

$$\beta(x)^2\|x\|^2 + 2\beta(x)(1 - \beta(x))\langle x, f(x)\rangle + (1 - \beta(x))^2\|f(x)\|^2 = 1,$$

which gives us $\beta(x)$ as the solution of a second-degree equation with infinitely differentiable coefficients. It follows that the solution has the same property. Hence the function g is an infinitely differentiable retraction from $\mathbb{B}_1$ onto $\mathbb{S}^{n-1}$, which is a contradiction and, consequently, f must have a fixed point.

Finally, let $h : \mathbb{B}_1 \to \mathbb{B}_1$ be a continuous mapping. The Weierstrass approximation theorem says that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of infinitely differentiable functions which converges uniformly to h in $\mathbb{B}_1$. Since, for each $n \in \mathbb{N}$, f_n is infinitely differentiable, we obtain a sequence $(x_n)_{n\in N}$ in $\mathbb{B}_1$ such that $f_n(x_n) = x_n$. Because $(x_n)_{n\in\mathbb{N}}$ lies in the compact $\mathbb{B}_1$, there is a subsequence (also denoted by $(x_n)_{n\in N}$) which converges to $x_0 \in \mathbb{B}_1$. It follows that $f_n(x_n) \to h(x_0)$, and so x_0 is a fixed point for h. ☐

Proof of Theorem 7.2.1. By Theorem 7.2.2, we only need to see that no infinitely differentiable retraction of $\mathbb{B}_1$ onto $\mathbb{S}^{n-1}$ exists. In order to get a contradiction, suppose that such a retraction exists, say f. We may consider the differential form $\alpha := x_1\,dx_2 \wedge \cdots \wedge dx_n$. Then Stokes theorem (see, for example, [96]) gives

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \alpha &= \int_{f(\mathbb{S}^{n-1})} \alpha = \int_{\mathbb{S}^{n-1}} f^*\alpha = \int_{\mathbb{B}_1} df^*\alpha \\ &= \int_{\mathbb{B}_1} f^*\,d\alpha = \int_{f(\mathbb{B}_1)} d\alpha = \int_{\mathbb{S}^{n-1}} d\alpha = 0.\end{aligned}$$

Next, applying Stokes theorem, we obtain

$$\int_{\mathbb{S}^{n-1}} \alpha = \int_{\mathbb{B}_1} d\alpha = \text{volume}(\mathbb{B}_1) > 0,$$

which is a contradiction. ☐

Let us now introduce the following definition.

Definition 7.2.1. Let X be a topological space. Then X is said to have the (topological) fixed point property if each continuous mapping $f : X \to X$ has a fixed point.

Lemma 7.2.1. *Let X and Y be two topological spaces. If X and Y are homeomorphic and if X has the fixed point property, then Y also has the fixed point property.*

This lemma expresses that the fixed point property for continuous mappings is a topological invariant.

Proof. Let $h : X \to Y$ be a homeomorphism with $h(X) = Y$, and suppose that $f : Y \to Y$ is continuous. Then $g := h^{-1} \circ f \circ h$ is a continuous map from X into X. Hence there exists $x \in X$ such that $g(x) = x$, that is, $h^{-1} \circ f \circ h(x) = x$. This implies that $f(h(x)) = h(x)$. ☐

Note that Brouwer fixed point theorem expresses the fact that the closed unit ball $\mathbb{B}_1$ of $\mathbb{R}^n$ has the fixed point property. But $\mathbb{B}_1$ is not the only subset of $\mathbb{R}^n$ having the fixed point property. In fact, as the following two results show, Brouwer fixed point theorem extends to compact convex subsets of $\mathbb{R}^n$.

Corollary 7.2.1. *Let* $R > 0$. *If* $f : \mathbb{B}_R \to \mathbb{B}_R$ *is continuous, then* $\operatorname{Fix}(f) \neq \emptyset$.

Proof. This follows from the fact that $\mathbb{B}_1$ and $\mathbb{B}_R$ are homeomorphic and the use of Theorem 7.2.1 and Lemma 7.2.1. □

Corollary 7.2.2 (Brouwer, general formulation). *Let* K *be a convex, compact subset of* $\mathbb{R}^n$ *and* $f : K \to K$ *a continuous mapping. Then* $\operatorname{Fix}(f) \neq \emptyset$.

Proof. As all the norms on $\mathbb{R}^n$ are equivalent, by Lemma 7.2.1, we may assume that we are working with the norm that comes from the inner product. Let $P_K : \mathbb{R}^n \to K$ be the projection onto K (P_K is well defined because $\mathbb{R}^n$ is a Hilbert space). Since K is compact, there exists $R > 0$ such that $K \subseteq \mathbb{B}_R$. Define the mapping $\tilde{f} : \mathbb{B}_R \to \mathbb{B}_R$ by

$$\tilde{f}(x) = f(P_K(x)).$$

According to Corollary 7.2.1, $\tilde{f}$ has a fixed point, say $x_0 \in \mathbb{B}_R$ ($x_0 = \tilde{f}(x_0) = f(P_K(x_0))$). Since the range of $\tilde{f}$ is contained in K and $P_K(x_0) = x_0$, we obtain $x_0 = f(x_0)$. □

Note that, in general, the fixed point derived via Brouwer fixed point theorem is not unique.

7.2.2 Schauder fixed point theorem

We now discuss the possibility of extending Brouwer theorem to infinite-dimensional normed spaces. The following example, due to S. Kakutani [132], shows that, in the form of Theorem 7.2.1, this is not possible.

Example 7.2.1. Let $X = \ell_2$ be the space of square summable sequences. Let $\mathbb{B}_1$ be the unit ball of X and $\partial\mathbb{B}_1$ its boundary. Define the function f on $\mathbb{B}_1$ by

$$x = (x_n)_{n\in\mathbb{N}^*} \mapsto f(x) = \left(\sqrt{1 - \|x\|^2}, x_1, x_2, \dots\right) \quad \text{where } \|x\|^2 = \sum_{n=1}^{+\infty} |x_n|^2.$$

It is clear that f is continuous and $f(\mathbb{B}_1) \subseteq \partial\mathbb{B}_1 \subseteq \mathbb{B}_1$. Note, however, if x is fixed point of f, $f(x) = x$, then $\|x\| = 1$ and

$$x = (x_1, x_2, \dots) = (0, x_1, x_2, \dots) = f(x).$$

This implies that $x_1 = 0$ and successively $x_2 = x_1 = 0$, and so on. Thus we get $x = 0$ and $\|x\| = 1$ which is a contradiction. Hence f has no fixed point in $\mathbb{B}_1$.

In the preceding example, the reason why f fails to have a fixed point is that $\mathbb{B}_1$ is not compact since the space ℓ_2 has infinite dimension.

Definition 7.2.2. Let X and Y be two normed spaces and K a subset of X.
(a) A map $f : K \to Y$ is said to be compact if f is continuous and $f(K)$ is relatively compact.
(b) A map $f : X \to Y$ is said to be completely continuous if f is continuous and, for all bounded subset M of X, $f(M)$ is relatively compact.

Compact or completely continuous operators play a central role in nonlinear analysis. Their importance is due to the fact that some results of continuous functions on $\mathbb{R}^n$ extend to Banach spaces when continuity is replaced by compactness.

The following result shows that compact maps can be approximated by sequences of finite rank operators.

Proposition 7.2.1. *Let X and Y be two Banach spaces, K a bounded subset of X, and $f : K \to Y$ a map. Then the following two assertions are equivalent:*
(i) *f is compact.*
(ii) *For all $n \in \mathbb{N}$, there exists a compact operator $P_n : K \longrightarrow Y$ such that*

$$\sup_{x\in K}\|f(x) - P_n(x)\| \le 1/n \quad \textit{and} \quad \dim(\mathrm{vect}(P_n(K))) < \infty. \tag{7.1}$$

The operators P_n are called Schauder projections.

Proof. If f is compact, then $f(K)$ is relatively compact and so, for all $n \in \mathbb{N}$, there exist elements $y_i \in f(K)$, $i = 1, 2, \dots, N = N(n)$, such that

$$f(K) \subset \bigcup_{i=1}^{N} \mathbb{B}_{1/n}(y_i)$$

and then, $\forall x \in K, \exists y_i \in \{y_1, \dots, y_N\}$ such that $\|f(x) - y_i\|_Y < 1/n$. We construct a partition of the unit of K as follows: we define the functions $\alpha_i : K \to \mathbb{R}$ by

$$\alpha_i(x) := \begin{cases} 0, & \text{if } \|f(x) - y_i\|_Y \ge \frac{1}{n}, \\ \frac{1}{n} - \|f(x) - y_i\|_Y & \text{if } \|f(x) - y_i\|_Y < \frac{1}{n}, \end{cases} \quad \text{for } i = 1, \dots, N.$$

We have the following properties:
(a) the functions α_i, $i = 1, \dots, N$, are continuous because f is continuous and the maximum of two continuous functions is continuous;
(b) $\alpha_i(x) \ge 0$ for all $x \in K$;
(c) $\sum_{i=1}^{N} \alpha_i(x) > 0$ for all $x \in K$;
(d) if $x \in K$ such that $\alpha_i(x) > 0$, then $\|f(x) - y_i\|_Y < 1/n$.

Using property (c), we can define the functions $\lambda_i : K \to \mathbb{R}$ by

$$\lambda_i(x) := a_i(x)\left(\sum_{j=1}^{N} a_j(x)\right)^{-1} \quad \text{for } i = 1, \dots, N.$$

The functions λ_i, $i = 1, \dots, N$, form the desired partition. Indeed,

- for all $i = 1, \dots, N$, λ_i is continuous;
- for all $x \in K$, we have $\sum_{i=1}^{n} \lambda_i(x) = 1$;
- if $x \in K$ is such that $\lambda_i(x) > 0$, then $\|f(x) - y_i\|_Y < 1/n$.

Set

$$K_n = \operatorname{co}(y_1, \dots, y_N). \tag{7.2}$$

It is clear that $K_n \subset \operatorname{vect}\{y_1, \dots, y_N\}$. We define Schauder projections $P_n : K \to K_n$, $n \in \mathbb{N}$, by

$$P_n(x) = \sum_{i=1}^{N} \lambda_i(x) y_i. \tag{7.3}$$

Let $x \in K$. Using the properties of the functions λ_i, $i = 1, \dots, N$, we get

$$\|f(x) - P_n(x)\|_Y = \left\| f(x) - \sum_{i=1}^{n} \lambda_i(x) y_i \right\|_Y \le \sum_{i=1}^{n} \lambda_i(x) \|f(x) - y_i\|_Y \le \frac{1}{n}.$$

By definition, $P_n(K) \subset \operatorname{vect}\{y_1, \dots, y_N\}$, hence the image of P_n is contained in a finite dimensional vector subspace of Y. It remains to show that P_n is a compact operator. The continuity of P_n is a consequence of the continuity of the functions λ_i. Let S be a bounded subset of K. Since f is compact, there exists $c \ge 0$ such that, for all $x \in S$, we have $\|f(x)\| \le c$, and consequently,

$$\|P_n(x)\|_Y \le \|P_n(x) - f(x)\|_Y + \|f(x)\|_Y \le \frac{1}{n} + c, \quad \forall x \in S.$$

This implies that $P_n(S)$ is bounded and therefore relatively compact.

Conversely, let n be an arbitrary (but fixed) nonnegative integer. There is a real δ such that for all $x, y \in K$ satisfying $\|x - y\|_X < \delta$, we have

$$\begin{aligned} \|f(x) - f(y)\|_Y &\le \|f(x) - P_n(x)\|_Y + \|P_n(x) - P_n(y)\|_Y + \|P_n(y) - f(y)\|_Y \\ &\le 1/n + \|P_n(x) - P_n(y)\|_Y + 1/n \le \frac{3}{n}. \end{aligned} \tag{7.4}$$

This proves the continuity of f. To show that $f(K)$ is relatively compact, we shall establish that from each open cover of $f(K)$, one can extract a finite open cover of $f(K)$.

By the compactness of P_n and the fact that K is bounded, we conclude that there exist $x_1, \dots, x_N \in K$ such that

$$P_n(K) \subset \bigcup_{i=1}^{N} \mathring{\mathbb{B}}_{1/n}(P_n(x_i)).$$

So, for all $y \in K$, there exists $i \in \{1, \dots, N\}$ such that $\|P_n(x_i) - P_n(y)\|_Y < 1/n$. This, combined with (7.4), shows that for each $y \in K$ there exists $i \in \{1, \dots, N\}$ such that $\|f(x_i) - f(y)\|_Y < \frac{3}{n}$. This shows that $f(K)$ admits a finite open cover, which completes the proof. ☐

Theorem 7.2.3 (Schauder). *Let X be a Banach space and K a nonempty subset of X. Suppose*

(a) *K is convex and compact,*
(b) *$f : K \to K$ is a continuous map.*

Then $\mathsf{Fix}(f) \neq \emptyset$.

Proof. Note that the operator f is compact since it is continuous and $f(K)$ is relatively compact. It follows from Proposition 7.2.1 that there exist operators $P_n : K \to K_n$ (K_n is given by (7.2)) such that

$$\|f(x) - P_n(x)\|_X \leq 1/n. \tag{7.5}$$

The convexity of K implies that $K_n \subseteq \mathrm{co}(f(K)) \subseteq K$. Accordingly, the operator

$$\mathbb{P}_n := P_{n|K_n} : K_n \to K_n$$

is continuous. The set K_n is closed and homomorphic to the closed unit ball $\mathbb{B}_1$ of $\mathbb{R}^N$. It follows from Corollary 7.2.2 that for every $n \in \mathbb{N}$, there exists $x_n \in K_n \subset K$ such that

$$\mathbb{P}_n(x_n) = x_n. \tag{7.6}$$

Because K is compact, there exist a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to +\infty$.

The point x is a fixed point of f. Indeed, using (7.6), the continuity of f, and (7.5), we obtain

$$\|f(x) - x_n\|_X \leq \|f(x) - f(x_n)\|_X + \|f(x_n) - \mathbb{P}_n(x_n)\|_X,$$

and then $\lim_{n\to+\infty} \|f(x) - x_n\|_X = 0$. Hence, $f(x) = x$, and so $\mathsf{Fix}(f) \neq \emptyset$. ☐

The following corollary (due to J. Schauder in 1930) is a version of Theorem 7.2.3 often used in applications. In general, it is easier to show that an operator is compact than to show that a subset of a normed space is compact. So, the compactness assumption of K will be replaced by the compactness of the operator f.

Corollary 7.2.3. *Let K be a nonempty, bounded, closed, convex subset of a Banach space X and $f : K \to K$ a mapping. If f is compact, then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof. Since K is bounded and $\overline{f(K)}$ is compact, By Theorem 1.4.1, $M = \overline{\mathrm{co}}(\overline{f(K)})$ is also a convex and compact subset of X. Because K is closed and $f(K) \subseteq K$, it is clear that $\overline{f(K)} \subseteq K$ and then $M \subseteq \overline{\mathrm{co}}(K) = K$. Moreover, we have $f(M) \subseteq M$ because

$$M \subseteq K \implies f(M) \subseteq f(K) \subseteq \overline{f(K)} \subseteq \overline{\mathrm{co}}(\overline{f(K)}) = M.$$

Now Theorem 7.2.3 ensures that f has a fixed point in $M \subseteq K$. ☐

7.3 Set contractive mappings

As we have seen in the previous section, compactness plays an essential role in the proof of Schauder fixed point theorem. However, there are some important problems where the operators are not compact. The first step to extend Schauder theorem to noncompact operators was done by G. Darbo in 1955 [73]. The main idea is to define a new class of operators which are contractive with respect to a measure of noncompactness. The first example of the notion of a measure of noncompactness (α-measure or set-measure) was introduced by Kuratowski in [148], and the associated notion of α-contractiveness has proved useful in several areas of functional analysis, operator theory, differential and integral equations.

7.3.1 Measure of noncompactness

Let X be a Banach space. We denote by

- $\mathbf{B}(X)$ the collection of bounded and nonempty subsets of X,
- $\mathbf{BF}(X)$ the collection of bounded and nonempty closed subsets of X,
- $\mathbf{C}(X)$ the collection of nonempty and compact subsets of X,
- $\mathbf{RC}(X)$ the collection of nonempty and relatively compact subsets of X.

In order to generalize the notion of compact operator in a reasonable way, we need the concept of a measure of noncompactness.

Definition 7.3.1. A map $\mu : \mathbf{B}(X) \to [0, +\infty)$ is called a measure of noncompactness on the Banach space X provided it satisfies the following conditions:

(1) the family $\ker \mu := \{M \in \mathbf{B}(X) : \mu(M) = 0\}$ is nonempty and $\ker \mu \subset \mathbf{RC}(X)$,
(2) $M_1 \subset M_2 \implies \mu(M_1) \leq \mu(M_2)$,
(3) $\mu(\overline{M}) = \mu(M)$,

(4) $\mu(\mathrm{co}(M)) = \mu(M)$,

(5) (Generalized Cantor's intersection theorem) if $(M_n)_{n\in\mathbb{N}}$ is a decreasing sequence (in the sense of inclusion) of nonempty, closed, and bounded subsets of X and if $\lim_{n\to+\infty}\mu(M_n) = 0$, then the set $M_\infty = \bigcap_{n=1}^{+\infty} M_n$ is nonempty and belongs to $\mathbf{C}(X)$.

The family $\ker\mu$ is called the kernel of the measure of noncompactness μ; the kernel of μ is contained in $\mathbf{RC}(X)$.

The measure of noncompactness μ is called homogeneous if, for all $M \in \mathbf{B}(X)$, we have

(6) $\mu(\lambda M) = |\lambda|\mu(M)\ \forall\lambda \in \mathbb{R}$.

It is called subadditive if

(7) $\mu(M + N) \le \mu(M) + \mu(N)\ \forall M, N \in \mathbf{B}(X)$.

We say that μ has the maximum property if

(8) $\mu(M \cup N) = \max(\mu(M), \mu(N)), \forall M, N \in \mathbf{B}(X)$.

We say that μ is nonsingular if

(9) $\mu(M \cup \{x_0\}) = \mu(M)$ for all $x_0 \in X$ and $N \in \mathbf{B}(X)$.

Definition 7.3.2. A subadditive measure of noncompactness $\mu(\cdot)$ which has the maximum property and is such that $\ker(\mu) = \mathbf{RC}(X)$ is called a regular measure.

Example 7.3.1. Let (X, d) be a complete metric space and M a subset of X. We recall that the diameter of M is defined by

$$\mathrm{diam}(M) = \sup\{(d(x,y) : x \in M, y \in M\}$$

with $\mathrm{diam}(\emptyset) = 0$. It is clear that for every bounded subset M, we have $\mathrm{diam}(M) < +\infty$, and $\mathrm{diam}(M) = 0$ if and only if M is empty or has only one element (then compact). The map

$$\mu_d : \mathbf{B}(X) \to [0, +\infty), \quad A \mapsto \mu_d(A) = \mathrm{diam}(A)$$

satisfies the following properties:

(a) if $M_1 \subseteq M_2$, then $\mu_d(M_1) \le \mu_d(M_2)$,

(b) $\mu_d(M) = \mu_d(\overline{M})$,

(c) *(Cantor's intersection theorem)* if $(M_n)_{n\in\mathbb{N}^*}$ is a decreasing sequence of nonempty, closed, bounded subsets of X and if $\lim_{n\to+\infty}\mu_d(M_n) = 0$, then the intersection $M_\infty = \bigcap_{n=1}^{+\infty} M_n$ is nonempty and consists of one point.

Moreover, if X is a Banach space, then

(e) $\mu_d(\lambda M) = |\lambda|\mu_d(M)$, for all real $\lambda \in \mathbb{R}$,

(f) $\mu_d(x + M) = \mu_d(M)$, for all $x \in X$,
(g) $\mu_d(M_1 + M_2) \le \mu_d(M_1) + \mu_d(M_2)$,
(h) $\mu_d(\mathrm{co}(M)) = \mu_d(M)$.

Properties (a)–(g) are immediate. We prove (h). If x and y are two points of $\mathrm{co}(M)$, then

$$x = \sum_{i=1}^{n} t_i x_i \quad \text{and} \quad y = \sum_{j=1}^{m} s_j y_j, \quad \text{with } x_i, y_j \in M,\ \sum_{i=1}^{n} t_i = 1, \text{ and } \sum_{j=1}^{m} s_j = 1.$$

Thus

$$\begin{aligned}\|x - y\| &= \left\| \sum_{i=1}^{n} t_i x_i - \sum_{j=1}^{m} s_j y_j \right\| = \left\| \sum_{j=1}^{m} \sum_{i=1}^{n} t_i s_j x_i - \sum_{i=1}^{n} \sum_{j=1}^{m} t_i s_j y_j \right\| \\ &\le \sum_{j=1}^{m} \sum_{i=1}^{n} t_i s_j \|x_i - y_j\| \le \mathrm{diam}(M) \sum_{j=1}^{m} \sum_{i=1}^{n} s_j t_i \\ &= \mathrm{diam}(M),\end{aligned}$$

and then $\mu_d(\mathrm{co}(M)) \le \mu_d(M)$. The opposite inequality is obvious.

We note that $\mu_d(\cdot)$ is a measure of noncompactness in the sense of Definition 7.3.1 but it is not regular because, among other things, it does not satisfy assumption (9).

7.3.1.1 Kuratowski measure of noncompactness

Recall that, if $M \in \mathbf{B}(X)$ is not a precompact set, there is a real number $\varepsilon > 0$ such that M cannot be covered by finitely many sets (or balls) with diameter $\le \varepsilon$. Hence, we can give the following definition.

Definition 7.3.3. Let (X, d) be a complete metric space. The Kuratowski measure of noncompactness $\alpha : \mathbf{B}(X) \to [0, +\infty)$ of a nonempty, bounded subset M of X, $\alpha(M)$, is the infimum of all numbers $\varepsilon > 0$ such that M can be covered by a finite number of sets with diameters $< \varepsilon$, that is,

$$\alpha(M) = \inf\left\{ \varepsilon > 0 : M \subset \bigcup_{i=1}^{n} S_i, S_i \subset X, \mathrm{diam}(S_i) < \varepsilon, i = 1, 2, \ldots, n, n \in \mathbb{N} \right\}.$$

It is clear that $\alpha(\emptyset) = 0$ and, for all $M \in \mathbf{B}(X)$, we have $0 \le \alpha(M) \le \mathrm{diam}(M)$.

The notion of a measure of noncompactness (α-measure or set-measure) was introduced by K. Kuratowski [148]. The following properties follow easily from Definition 7.3.3.

Proposition 7.3.1. *For all $A, B \in \mathbf{B}(X)$, we have*
(a) $\alpha(A) = 0 \Longleftrightarrow \bar{A}$ *is compact,*
(b) $A \subset B \Longrightarrow \alpha(A) \le \alpha(B)$,

(c) $\alpha(A) = \alpha(\bar{A})$,
(d) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,
(e) $\alpha(A \cap B) \leq \min\{\alpha(A), \alpha(B)\}$.

The next result, due to K. Kuratowski [148], is a generalization of the well-known Cantor intersection theorem.

Proposition 7.3.2. *Let* (X, d) *be a complete metric space. If* $(M_n)_{n\geq 1}$ *is a decreasing sequence of nonempty, closed, and bounded subsets of* X *such that* $\lim_{n\to+\infty} \alpha(M_n) = 0$, *then the intersection* $M_\infty = \bigcap_{n\geq 1} M_n$ *is a nonempty compact subset of* X.

If X is a Banach space, the following properties hold.

Proposition 7.3.3. *Let* X *be a Banach space. For all* $A, B \in \mathbf{B}(X)$, *we have*
(f) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$,
(g) $\alpha(A + x) = \alpha(A)$, *for all* $x \in X$,
(h) $\alpha(\lambda A) = |\lambda|\alpha(A)$, *for all* $\lambda \in \mathbb{R}$,
(i) $\alpha(\mathrm{co}(A)) = \alpha(A)$.

We refer to [9, 24] or [25] for a proof of Proposition 7.3.3.

We close this subsection by establishing the following elementary result.

Proposition 7.3.4. *Let* K *be a nonempty subset of a Banach space* X *and* $f : K \to X$ *a mapping. If* f *is* k*-Lipschitz, then for all bounded subsets* S *of* K, *we have* $\alpha(f(S)) \leq k\alpha(S)$.

Proof. Let $\varepsilon > 0$ be an arbitrary real number and suppose that $S \subseteq \bigcup_{i=1}^n S_i$ with $\mathrm{diam}(S_i) \leq \alpha(S) + \varepsilon$. Then we have $f(S) \subseteq \bigcup_{i=1}^n f(S_i) = \bigcup_{i=1}^n F_i$ where $F_i := f(S_i)$. Letting $y_1, y_2 \in F_i$ where $i \in \{1, 2, \ldots, n\}$, there exist $x_1, x_2 \in S_i$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since f is a k-Lipschitz map, we can write

$$\|f(x_2) - f(x_1)\| \leq k\|x_2 - x_1\| \leq k(\alpha(S) + \varepsilon).$$

This yields that $\mathrm{diam}(F_i) \leq k(\alpha(S) + \varepsilon)$ and therefore $\alpha(f(S)) \leq k(\alpha(S) + \varepsilon)$. Since ε is arbitrary, we get $\alpha(f(S)) \leq k\alpha(S)$. □

7.3.1.2 Hausdorff measure of noncompactness

To find the exact value of $\alpha(K)$ for $K \in \mathbf{B}(X)$ is not always an easy task. For this and other reasons, it is sometimes more useful to use the ball or Hausdorff measure of noncompactness, which was introduced by L. S. Goldenstein, I. T. Gohberg, and A. S. Markus in 1957 [118].

Definition 7.3.4. Let (X, d) be a complete metric space. The Hausdorff measure of noncompactness $\chi : \mathbf{B}(X) \to [0, +\infty)$ of a nonempty, bounded subset M of X, $\chi(M)$, is the infimum of all numbers $\varepsilon > 0$ such that M can be covered by a finite number of balls with radii $< \varepsilon$, that is,

$$\chi(M) = \inf\left\{\varepsilon > 0 : M \subset \bigcup_{i=1}^{n} \mathbb{B}_{r_i}(x_i), x_i \in X,\ r_i < \varepsilon, i = 1, 2, \dots, n, n \in \mathbb{N}\right\}.$$

If X is a Banach space, we have the following equivalent definition.

Definition 7.3.5. Let X be a Banach space. The Hausdorff measure of noncompactness χ of a nonempty bounded subset M of X, $\chi(M)$, is the infimum of all numbers $\varepsilon > 0$ such that M has a finite ε-net in X, that is,

$$\chi(M) = \inf\{\varepsilon > 0 : M \subset S + \varepsilon\mathbb{B}_1, S \subset X, S \text{ is finite}\}.$$

The following properties follow from Definition 7.3.5.

Proposition 7.3.5. *For all $A, B \in \mathbf{B}(X)$, we have*

(a) $0 \leq \alpha(A) \leq \chi(A) \leq diam(A)$,
(b) $\chi(A) = 0 \Longleftrightarrow \bar{A}$ *is compact,*
(c) $A \subset B \Longrightarrow \chi(A) \leq \chi(B)$,
(d) $\chi(A) = \chi(\bar{A})$,
(e) $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$,
(f) $\chi(A \cap B) \leq \min\{\chi(A), \chi(B)\}$.

If X is a Banach space, then, for all $A, B \in \mathbf{B}(X)$, we have

(g) $\chi(A + B) \leq \chi(A) + \chi(B)$,
(h) $\chi(A + x) = \chi(A)$, *for all* $x \in X$,
(i) $\chi(\lambda A) = |\lambda|\chi(A)$, *for all* $\lambda \in \mathbb{R}$,
(j) $\chi(\mathrm{co}(A)) = \chi(A)$.

The next result shows the equivalence of Kuratowski and Hausdorff measures of noncompactness.

Proposition 7.3.6. *Let (X, d) be a complete metric space and let M be a nonempty, bounded subset of X. Then*

$$\chi(M) \leq \alpha(M) \leq 2\chi(M).$$

For a proof of Proposition 7.3.6, we refer to [24] or [25, 26]. For deep discussions and more information on measures of noncompactness, we refer, for example, to the books [9, 24–26].

Throughout the two next subsections, $\mu(\cdot)$ will denote a measure of noncompactness in the sense of Definition 7.3.1.

7.3.2 Darbo-type fixed point theorems

Definition 7.3.6. Let X be a Banach space, K a nonempty subset of X, $\mu(\cdot)$ a measure of noncompactness on X, and $f : K \to X$ a continuous map such that, for each $B \in \mathbf{B}(X) \cap 2^K$, $f(B) \in \mathbf{B}(X)$.

(a) The map f is said to satisfy the Darbo condition for $\mu(\cdot)$ if there is a constant $k > 0$ such that

$$\mu(f(A)) \leq k\mu(A) \quad \forall A \in \mathbf{B}(X) \cap 2^K.$$

(b) The map f is said to be μ-k-contractive if it satisfies the Darbo condition for some $k \in [0,1)$.

(c) The map f is said to be μ-nonexpansive if it satisfies the Darbo condition with $k = 1$.

We note that the validity of Schauder fixed point theorem requires the compactness of the convex set K or the compactness of the map f. The following result, due to G. Darbo [73], is a generalization of Schauder fixed point theorem in the sense that neither the compactness of K nor the compactness of f is required.

Theorem 7.3.1 (Darbo). *Let X be a Banach space, K a nonempty, closed, convex element of $\mathbf{B}(X)$ and $f : K \to K$ a continuous mapping. If f is μ-k-contractive, then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof. We define a sequence of subsets $(K_n)_{n\in\mathbb{N}}$ by $K_0 = K$ and, for each integer $n \geq 1$, $K_{n+1} = \overline{\mathrm{co}}(f(K_n))$. We have $f(K_0) = f(K) \subset K = K_0$, $K_1 = \overline{\mathrm{co}}(f(K_0)) \subset K = K_0$. Continuing this process, we get $K_0 \supset K_1 \supset K_2 \supset \cdots$. It is clear that $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty, closed, convex, f-invariant subsets. Obviously, $\mu(K_{n+1}) = \mu(\overline{\mathrm{co}}(f(K_n))) = \mu(f(K_n)) \leq k\mu(K_n)$ and so, for every $n \in \mathbb{N}$, we have $\mu(K_{n+1}) \leq k^{n+1}\mu(K_0)$. Since $k \in [0,1)$, we get $\lim_{n\to+\infty} \mu(K_n) = 0$. According to axiom (5) of Definition 7.3.1, we deduce that $K_\infty = \bigcap_{n=0}^{+\infty} K_n$ is a nonempty, convex, compact subset of X. Further, for all $p \in \mathbb{N}$, we have $K_\infty = \bigcap_{n=1}^{+\infty} K_n \subset K_p$, which proves that $f(K_\infty) \subseteq K_\infty$. Finally, applying Theorem 7.2.3 to f as a map from K_∞ into K_∞ ensures that f has a fixed point in $K_\infty \subset K$. ☐

If one checks carefully the proof of any recent generalization of Darbo fixed point theorem (see, for instance, [6, 7, 20, 102] and the references therein), one notices that the essential ingredient of the proof is the following result which was explicitly proved by R. D. Nussbaum [183, Proposition 10] for the Kuratowski measure of noncompactness.

Proposition 7.3.7. *Let K be a closed, bounded, convex subset of a Banach space X and let $f : K \to K$ be a continuous mapping. Let $K_1 := \overline{\mathrm{co}} f(K)$ and $K_n := \overline{\mathrm{co}} f(K_{n-1})$ for $n > 1$. Assume that $\alpha(K_n) \to 0$, as $n \to \infty$. Then* $\mathrm{Fix}(f) \neq \emptyset$.

We note that the previous result holds true for any measure of noncompactness satisfying axiom (5) of Definition 7.3.1. However, the converse of this implication is in general false. Indeed, it is enough to consider the identity mapping defined on the unit ball of any infinite-dimensional Banach space.

We next prove an extension of Nussbaum's result considering a more general condition which will be used in order to characterize the existence of fixed points for continuous mappings.

Proposition 7.3.8. *Let K be a closed, bounded, convex subset of a Banach space X. Let $\mu(\cdot)$ be a measure of nonconpactness on X and $f : K \to K$ a continuous mapping. If there exists a decreasing sequence $(K_n)_{n\in\mathbb{N}}$ of closed, convex, f-invariant subsets of K such that $\mu(K_n) \to 0$, as $n \to +\infty$, then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof. By axiom (5) of Definition 7.3.1, we know that $K_\infty := \bigcap_{n\in\mathbb{N}} K_n$ is nonempty. Since $\mu(K_\infty) = \mu(\bigcap_{n\in\mathbb{N}} K_n) \leq \mu(K_n)$ for all $n \in \mathbb{N}$, by taking the limit as $n \to \infty$, we get $\mu(K_\infty) = 0$, that is, $K_\infty \in \ker(\mu)$. Moreover, K_∞ is convex and f-invariant because each K_n is convex and f-invariant. To end the proof, it suffices to apply Schauder fixed point theorem to the mapping $f : K_\infty \to K_\infty$. □

Nevertheless, if we assume that the kernel of the measure of noncompactness contains singleton sets, then we obtain the following result.

Theorem 7.3.2. *Let K be a closed, bounded, convex subset of a Banach space X and let $\mu(\cdot)$ be a measure of noncompactness on X such that any singleton subset of K belongs to* $\ker \mu$. *If $f : K \to K$ is a continuous map, then the following assertions are equivalent:*
(a) *there exists a decreasing sequence $(K_n)_{n\in\mathbb{N}}$ of convex, closed, f-invariant subsets of K such that $\mu(K_n) \to 0$, as $n \to +\infty$,*
(b) $\mathrm{Fix}(f) \neq \emptyset$.

Proof. The necessity follows from property (5) of Definition 7.3.1. Let us prove the sufficiency. Assume that f has at least a fixed point z in K. The sequence of sets defined by $K_n := \{z\}$ satisfies (a) because z is a fixed point of f and any singleton set belongs to $\ker(\mu)$. □

Next, we will give two generalizations of Darbo fixed point theorem which can be obtained by applying Proposition 7.3.8.

Definition 7.3.7. Let $\mathcal{G}$ denote the set of all functions $g : \mathbb{R}^+ \to [0,1)$ such that, for all monotone decreasing sequences $(t_n)_{n\in\mathbb{N}}$, we have that

$$\lim_{n\to+\infty} g(t_n) = 1 \quad \text{implies} \quad \lim_{n\to+\infty} t_n = 0.$$

We note that this class of functions was introduced by M. A. Geraghty [113], who did not make any continuity assumption on g in order to get an extension of the Banach contraction principle.

Definition 7.3.8. We define $\mathcal{F}$ to be the family of functions $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that $F(t) = 0$ if and only if $t = 0$, and such that either F is continuous on $\mathbb{R}^+$ or $F(t) \geq t$ for all $t \geq 0$.

Theorem 7.3.3. *Let X be a Banach space and let $\mu(\cdot)$ be a measure of noncompactness on X. Let K be a nonempty, closed, bounded, convex subset of X and $f : K \to K$ a continuous mapping. If there exist two functions $g \in \mathcal{G}$ and $F \in \mathcal{F}$ such that, for every f-invariant subset S of K,*

$$F(\mu(S)) \leq g(\mu(S))F(\mu(S)), \tag{7.7}$$

then $\mathrm{Fix}(f) \neq \emptyset$.

We recall that the class of functions $\mathcal{G}$ was given in Definition 7.3.7.

Proof. By property (5) of Definition 7.3.1, it will be enough to find a decreasing sequence $(K_n)_{n\in\mathbb{N}^*}$ of closed, f-invariant subsets of K such that $\mu(K_n) \to 0$, as $n \to +\infty$. Let $K_1 := K$ and define the set sequence $(K_n)_{n\in\mathbb{N}^*}$ by $K_{n+1} := \overline{\mathrm{co}} f(K_n)$ for each $n \geq 2$. Note that $f(K_1) = f(K) \subseteq K = K_1$, $K_2 = \overline{\mathrm{co}} f(K_1) \subseteq K = K_1$, and by induction we get that $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of f-invariant subsets of X, because for each $n \in \mathbb{N}$, we have $f(K_n) \subseteq \overline{\mathrm{co}} f(K_n) = K_{n+1} \subseteq K_n$.

If there exists $n_0 \in \mathbb{N}^*$ such that $F(\mu(K_{n_0})) = 0$, then $\mu(K_{n_0}) = 0$ since $F \in \mathcal{F}$. By property (b) of Definition 7.3.7, $\mu(K_n) = 0$ for every $n \geq n_0$ and, thus, $\mu(K_n) \to 0$, as $n \to +\infty$. Otherwise, we can assume that $F(\mu(K_n)) > 0$ for all $n \in \mathbb{N}$. Using (7.7) and the properties of the measure of noncompactness, we have

$$F(\mu(K_{n+1})) = F(\mu(\overline{\mathrm{co}} f(K_n))) = F(\mu(f(K_n))) \leq g(\mu(K_n))F(\mu(K_n)),$$

for each $n \in \mathbb{N}$. This implies that the sequence $\{F(\mu(K_n))\}_{n\in\mathbb{N}}$ is nonincreasing and nonnegative, since $g \in \mathcal{G}$. Thus, we infer that there exists $\ell \geq 0$ such that $\lim_{n\to+\infty} F(\mu(K_n)) = \ell$. We now distinguish two cases:

CASE 1. Assume that $\ell > 0$. Since $\frac{F(\mu(K_{n+1}))}{F(\mu(K_n))} \leq g(\mu(K_n)) < 1$ for all $n \in \mathbb{N}$, by the squeeze theorem, $g(\mu(K_n)) \to 1$ as $n \to \infty$. Since $g \in \mathcal{G}$, we conclude that $\lim_{n\to+\infty} \mu(K_n) = 0$.

CASE 2. Assume that $\ell = 0$. On the one hand, if F is continuous then $\mu(K_n) \to 0$ as $n \to \infty$, because $F(t) = 0 \iff t = 0$. On the other hand, if F satisfies $F(t) \geq t$ for all $t \geq 0$, then $F(\mu(K_n)) \geq \mu(K_n) \geq 0$ for all $n \in \mathbb{N}$. Thus, $\lim_{n\to+\infty} \mu(K_n) = 0$. ☐

Definition 7.3.9. A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison if ψ is nondecreasing and $\psi^n(t) \to 0$, as $n \to +\infty$, for all $t \geq 0$. We denote by Ψ the set of all comparison functions.

For properties and applications of comparison functions, we refer the reader to [208, Chapter 4] and [127, Lemma 3.1].

Theorem 7.3.4. *Let X be a Banach space and $\mu(\cdot)$ a measure of noncompactness on X. Let K be a nonempty, closed, bounded, convex subset of a Banach space X, and $f : K \to K$ a continuous mapping. If there exist two functions $\psi \in \Psi$ and $F \in \mathcal{F}$ such that, for every f-invariant subset S of K,*

$$F(\mu(f(S))) \leq \psi(F(\mu(S))), \tag{7.8}$$

then Fix$(f) \neq \emptyset$.

Proof. As in the proof of Theorem 7.3.3, we define, by induction, a sequence of subsets of X, $(K_n)_{n\in\mathbb{N}^*}$, where $K_1 := K$ and $K_{n+1} := \overline{\mathrm{co}}f(K_n)$ with $\mu(K_n) > 0$ for all $n \in \mathbb{N}^*$. From (7.8), for each $n \in \mathbb{N}^*$, we have

$$\begin{aligned} F(\mu(K_{n+1})) = F(\mu(\overline{\mathrm{co}}f(K_n))) = F(\mu(f(K_n))) \leq \psi(F(\mu(K_n))) \\ \leq \psi^2(F(\mu(K_{n-1}))) \leq \cdots \leq \psi^n(F(\mu(K))). \end{aligned}$$

Since $\psi \in \Psi$, we have $\lim_{n\to+\infty} F(\mu(K_n)) = 0$.

If F is continuous, then $\mu(K_n) \to 0$ as $n \to +\infty$ because $F(t) = 0$ if and only if $t = 0$. On the other hand, if $F(t) \geq t$ for all $t \in \mathbb{R}_+$, it is clear that $\mu(K_n) \to 0$ as $n \to +\infty$. ☐

7.3.3 Sadovskii-type fixed point theorems

Definition 7.3.10. Let X be a Banach space, K a nonempty subset of X, $\mu(\cdot)$ a measure of noncompactness on X, and $f : K \to X$ a continuous mapping such that, for all $B \in \mathbf{B}(X) \cap 2^K$, $f(B) \in \mathbf{B}(X)$. We say that f is μ-condensing if

$$\mu(f(A)) < \mu(A) \quad \forall A \in (\mathbf{B}(X) \cap 2^K) \setminus \ker \mu.$$

It is clear that any μ-k-contractive map is μ-condensing. In addition, if μ is a measure of noncompactness such that $\ker \mu = \mathbf{RC}(X)$, each compact map is μ-k-contractive and therefore μ-condensing.

In [210], B. N. Sadovskii showed the following fixed point theorem. It is an extension of Darbo fixed point theorem to μ-condensing mappings.

Theorem 7.3.5 (Sadovskii). *Let X be a Banach space, K a nonempty, closed, convex element of $\mathbf{B}(X)$, and $\mu(\cdot)$ a nonsingular measure of noncompactness on X (satisfying axiom (9)). If $f : K \to K$ is a μ-condensing map, then* Fix$(f) \neq \emptyset$.

Proof. Let $y \in K$ and define the set Σ by

$$\Sigma := \{D \subseteq K : D \text{ is closed, convex, } y \in D \text{ and } f : D \to D\}$$

and set

$$A = \bigcap_{D\in\Sigma} D \quad \text{and} \quad B = \overline{\mathrm{co}}(f(A) \cup \{y\}).$$

It is obvious that A is a closed, convex subset of X containing y. Moreover, by the definition of A, for each $D \in \Sigma$, we have $f(A) \subseteq f(D) \subseteq D$ and then $f(A) \subseteq \bigcap_{D\in\Sigma} D = A$. Since

$y \in A$, we see that $f(A) \cup \{y\} \subseteq A$ and thus, because A is closed, convex, we deduce that $f(B) \subseteq f(A) \subseteq B$. This proves that $B \in \Sigma$. This implies that $A \subseteq B$ and therefore $A = B$. Moreover, the properties of μ allow us to write

$$\mu(A) = \mu(B) = \mu(\overline{\text{co}}(f(A) \cup \{y\})) = \mu(f(A)) < \mu(A)$$

(because f is μ-condensing), which is a contradiction. Accordingly, $\mu(A) = 0$ and so A is compact. Applying Theorem 7.2.3 to the map $f_{|A} : A \to A$, we conclude that f admits a fixed point in $A \subseteq K$. □

The following example, due to M. Baronti, E. Casini, and P. L. Papini [33], shows that Sadovskii fixed point theorem fails if the measure of noncompactness μ is not regular, in particular if μ does not fulfill condition (9).

Example 7.3.2. Let X be the nonreflexive Banach space of all continuous real functions on the closed unit interval, with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \max_{0 \le t \le 1} |f(t)| + \int_0^1 |f(t)|\, dt.$$

Consider the following closed, convex, bounded subset:

$$K = \{f \in \mathbb{B}_1 : f(0) = 0, f(1) = 1, 0 \le f(t) \le t, f \text{ is monotone nondecreasing}\}.$$

The mapping $g : K \to K$ defined by

$$g(f)(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{2}, \\ (2t-1)f(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

The map g is fixed point free. Indeed, suppose that $f \in K$ is such that $g(f) = f$. It is clear that $f(x) = 0$ for every $x \in [0, \frac{1}{2}]$. If $x \in [\frac{1}{2}, 1]$, then $(2x-1)f(2x-1) = f(x)$ implies that $f(x) = 0$ for every $x \in [0, \frac{3}{4}]$. By iterating the reasoning, we can easily prove that $f(x) = 0$ for all $x \in [0, 1 - \frac{1}{2^n}]$ and all $n \in \mathbb{N}$. Since f is continuous and $f(1) = 1$, this is a contradiction proving that g is fixed point free.

Let μ_d be the set-function diameter defined in Example 7.3.1. The mapping g is μ_d-condensing. Indeed, let S be a closed subset of K such that $\text{diam}(S) > 0$. Let us first observe that, for any $f, h \in K$ with $f \ne h$, we have

$$\begin{aligned} \|g(f) - g(h)\| &= \max_{0 \le x \le 1} |g(f)(x) - g(h)(x)| + \frac{1}{2} \int_0^1 |g(f)(x) - g(h)(x)|\, dx \\ &= \max_{1/2 \le x \le 1} (2x-1) |(f(2x-1) - h(2x-1))| \end{aligned}$$

$$
\begin{aligned}
&+\int_{1/2}^{1}(2x-1)\big|\big(f(2x-1)-h(2x-1)\big)\big|\,dx \\
&= \max_{0\le x\le 1}\big|x\big(f(x)-h(x)\big)\big| + \frac{1}{2}\int_{0}^{1} x\big|f(x)-h(x)\big|\,dx \\
&< \|f-h\|_\infty + \frac{1}{2}\|f-h\|_1 \\
&= \|f-h\|.
\end{aligned}
$$

Hence, with the help of the calculation above, for two suitable sequences $(f_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ in S we get

$$
\begin{aligned}
\operatorname{diam}\big(g(S)\big) &= \lim_{n\to+\infty}\big\|g(f_n)-g(h_n)\big\| = \lim_{n\to+\infty}\big(\big\|g(f_n)-g(h_n)\big\|_\infty + \big\|g(f_n)-g(h_n)\big\|_1\big) \\
&\le \lim_{n\to+\infty}\Big(\big\|g(f_n)-g(h_n)\big\|_\infty + \frac{1}{2}\big\|g(f_n)-g(h_n)\big\|_1\Big) \\
&\le \lim_{n\to+\infty}\|f_n-h_n\| \le \operatorname{diam}(S).
\end{aligned}
$$

So, if we assume that $\operatorname{diam}(g(S)) = \operatorname{diam}(S)$, then (by passing again if necessary to a subsequence) we have

$$
\begin{aligned}
\lim_{n\to+\infty}\|f_n-h_n\|_1 &= \lim_{n\to+\infty}\big\|g(f_n)-g(h_n)\big\|_1 = 0, \\
\lim_{n\to+\infty}\|f_n-h_n\|_\infty &= \lim_{n\to+\infty}\big\|g(f_n)-g(h_n)\big\|_\infty = \operatorname{diam}\big(g(S)\big) = \operatorname{diam}(S).
\end{aligned}
$$

But we can choose a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\|g(f_n)-g(h_n)\|_\infty = x_n|f_n(x_n)-h_n(x_n)|$. By considering, if necessary, a subsequence, we may assume that $x_n \to x_0 \in [0,1]$. Then

$$
\operatorname{diam}(S) = \lim_{n\to+\infty} x_n\big|f_n(x_n)-h_n(x_n)\big| \le \lim_{n\to+\infty} x_n\big\|f_n(x_n)-h_n(x_n)\big\| = x_0\operatorname{diam}(S), \quad (7.9)
$$

thus $x_0 = 1$. By considering subsequences, and by eventually exchanging the sequences, we may assume that

$$
f_n(x_n)\to l \quad\text{and}\quad h_n(x_n)\to L \quad \text{with } L\le l\le 1.
$$

Therefore (7.9) implies that $l - L = \operatorname{diam}(S)$, so

$$
f_n(x_n)\to l, \quad h_n(x_n)\to l-\operatorname{diam}(S).
$$

Now take $f \in S$. Because $\lim_{n\to+\infty} x_n = 1$, we have

$$
\operatorname{diam}(S) \ge \big|f(x_n)-h_n(x_n)\big| \to \big|1-l+\operatorname{diam}(S)\big| \ge \operatorname{diam}(S) \quad \text{as } n\to+\infty.
$$

Thus we have $l = 1$; $\lim_{n\to+\infty}|f(x_n)-h_n(x_n)| = \operatorname{diam}(S)$ for every $f \in S$, and then

$$\lim_{n\to+\infty} \|f - h_n\|_\infty = \operatorname{diam}(S).$$

Now take $\varepsilon \in (0, \operatorname{diam}(S))$, then there exists $\eta > 0$ such that, for every $x \in [1-\eta, 1]$, we have $1-\varepsilon \le f(x) \le 1$. For n large enough, we get $x_n > 1-\eta$ and therefore the monotonicity assumption of the functions gives (for suitable points α_n)

$$\int_0^1 |f(x) - g_n(x)|\,dx \ge \int_{1-\eta}^{x_n} |f(x) - h_n(x)|\,dx = (x_n - 1 + \eta)|f(\alpha_n) - h_(\alpha_n)|$$
$$\ge (x_n - 1 + \eta)(1 - \varepsilon - h_n(x_n));$$

also, since $\lim_{n\to+\infty} h_n(x_n) = 1 - \operatorname{diam}(S)$, we have

$$\lim_{n\to+\infty} (x_n - 1 + \eta)(1 - \varepsilon - h_n(x_n)) = \eta\big(\operatorname{diam}(S) - \varepsilon\big),$$

and this implies that

$$\liminf_{n\to+\infty} \|f - h_n\| \ge \lim_{n\to+\infty} \|f - h_n\|_\infty + \liminf_{n\to+\infty} \|f - h_n\|_1 \ge \operatorname{diam}(S) + \eta\big(\operatorname{diam}(S) - \varepsilon\big).$$

This is a contradiction. Accordingly, $\mu_d(g(S)) < \mu_d(S)$; hence g is μ_d-condensing.

This example shows that Sadovskii fixed point theorem fails for the set-function $\mu_d(\cdot)$ because it is not nonsingular (cf. (9)).

Checking carefully the proof of Theorem 7.3.5, one can note that it is only required that f is a μ-condensing mapping for f-invariant, closed, convex subsets, that is, $\mu(f(S)) < \mu(S)$ for any closed and convex subset S of K with $\mu(S) > 0$ and $f(S) \subseteq S$.

The following example shows that the above condition is weaker than to be condensing.

Example 7.3.3. Let $\mathbb{B}_1^+$ be the first quadrant of the unit ball in the Euclidean plane with its usual norm, i. e., $\mathbb{B}_1^+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \ge 0, x_2 \ge 0, \|x\|_2 \le 1\}$. Consider the mapping $f : \mathbb{B}_1^+ \to \mathbb{B}_1^+$ defined by

$$f(x_1, x_2) = P_{B_1^+}\left(x_1, \frac{x_2 + 1}{2}\right),$$

where $P_{\mathbb{B}_1^+} : \mathbb{R}^2 \to \mathbb{B}_1^+$ denotes to the metric projection onto $\mathbb{B}_1^+$.

The mapping f is not μ_d-condensing since if we take the subset $S = [0, \frac{1}{2}] \times \{0\}$, clearly $f(S) = [0, \frac{1}{2}] \times \{\frac{1}{2}\}$, and consequently $\mu_d(f(S)) = \frac{1}{2} = \mu_d(S)$.

However, we shall prove that f verifies $\mu_d(f(S)) < \mu_d(S)$, for any f-invariant, convex, and closed subset S of $\mathbb{B}_1^+$.

Let S be an f-invariant, convex, and closed subset of $\mathbb{B}_1^+$. Since f is continuous and its unique fixed point is $(0, 1)$, by Brouwer theorem, we have $(0, 1) \in S$. Without loss of generality, we can suppose that $0 < \operatorname{diam}(S) = \|x^0 - y^0\|_2$, which implies that $\|u -$

$v\|_2 \leq \|x^0 - y^0\|_2$ for all $u, v \in S$. Then, bearing in mind that the metric projection $P_{B_1^+}$ is nonexpansive, we have

$$\begin{aligned}\|f(u) - f(v)\|_2 &= \left\| P_{B_1^+}\left(u_1, \frac{u_2+1}{2}\right) - P_{B_1^+}\left(v_1, \frac{v_2+1}{2}\right)\right\|_2 \\ &\leq \left\|\left(u_1, \frac{u_2+1}{2}\right) - \left(v_1, \frac{v_2+1}{2}\right)\right\|_2 \\ &= \left\|\left(u_1 - v_1, \frac{u_2 - v_2}{2}\right)\right\|_2 \\ &\leq \|(u_1 - v_1, u_2 - v_2)\|_2 = \|u - v\|_2.\end{aligned}$$

Thus, $\operatorname{diam}(f(S)) = \operatorname{diam}(S) \iff {x_2}^0 = {y_2}^0 < 1$ (otherwise, we would obtain that $x^0 = (0,1) = y^0$, which is a contradiction) but in this case we would have that $\overline{\mathrm{co}}\{(0,1), x^0, y^0\} \subseteq S$ and

$$\operatorname{diam}(\overline{\mathrm{co}}\{(0,1), x^0, y^0\}) = \|x^0 - y^0\|_2 = |x_1^0 - y_1^0| \leq \max\{x_1^0, y_1^0\},$$

which is a contradiction because, by Pythagorean theorem,

$$\|(0,1) - x^0\|_2 = \sqrt{(x_1^0)^2 + (1 - x_2^0)^2} > x_1^0$$

and

$$\|(0,1) - y^0\|_2 = \sqrt{(y_1^0)^2 + (1 - y_2^0)^2} > y_1^0.$$

Therefore, $\operatorname{diam}(f(S)) < \operatorname{diam}(S)$.

The previous example motivates the following concept which is more general than that of a condensing mapping.

Definition 7.3.11. Let $\mu(\cdot)$ be a measure of noncompactness on a Banach space X and let K be a nonempty, bounded subset of X. A mapping $f : K \to X$ is called μ-quasicondensing if $\mu(f(S)) < \mu(S)$ for all f-invariant, closed, convex subsets S of K with $\mu(S) > 0$.

Next we will prove a new result which indicates that, in order to guarantee the existence of a fixed point for a mapping, we can replace the concept of a μ-condensing mapping by that of a μ-quasicondensing mapping.

Theorem 7.3.6. *Let K be a nonempty, closed, bounded, and convex subset of a Banach space X, $\mu(\cdot)$ a nonsingular (cf. axiom (9)) measure of noncompactness on X, and $f : K \to K$ a continuous map. Suppose that*

$$\begin{cases} \text{there exists a function } h : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } h(\mu(f(S))) \neq h(\mu(S)), \\ \text{for every } f\text{-invariant closed, convex subset } S \text{ of } K \text{ with } \mu(S) > 0. \end{cases} \tag{7.10}$$

Then, $\operatorname{Fix}(f) \neq \emptyset$.

Proof. Fix a point $x_0 \in K$ and let Δ be the family of all closed, convex subsets S of K for which $x_0 \in S$ and $f(S) \subseteq S$. Putting $S_\infty := \bigcap_{S\in\Delta} S$. It is clear that $S_\infty \neq \emptyset$ because $x_0 \in S_\infty$.

We claim that $S_\infty = \overline{\text{co}}(f(S_\infty) \cup \{x_0\})$. Indeed, since $f(S_\infty) \subseteq S_\infty$, $x_0 \in S_\infty$, and S_∞ is closed and convex, we have that $\overline{\text{co}}(f(S_\infty) \cup \{x_0\}) \subseteq S_\infty$, and therefore, $f(\overline{\text{co}}(f(S_\infty) \cup \{x_0\})) \subseteq f(S_\infty) \subseteq \overline{\text{co}}(f(S_\infty) \cup \{x_0\})$. Bearing in mind that $x_0 \in \overline{\text{co}}(f(S_\infty) \cup \{x_0\})$ and $\overline{\text{co}}(f(S_\infty) \cup \{x_0\})$ is closed and convex, from the definition of S_∞, we have that $S_\infty \subseteq \overline{\text{co}}(f(S_\infty) \cup \{x_0\})$. So, $S_\infty = \overline{\text{co}}(f(S_\infty) \cup \{x_0\})$. Thus, using properties (2), (3), and (4) of Definition 7.3.1, and the fact that $\mu(\cdot)$ is nonsingular, we have

$$\begin{aligned} \mu(S_\infty) &= \mu(\overline{\text{co}}(f(S_\infty) \cup \{x_0\})) = \mu(\text{co}(f(S_\infty) \cup \{x_0\})) \\ &= \mu(f(S_\infty) \cup \{x_0\}) \\ &= \mu(f(S_\infty)). \end{aligned}$$

If $\mu(S_\infty) > 0$, then $h(\mu(S_\infty)) = h(\mu(f(S_\infty))) \neq h(\mu(S_\infty))$, which is a contradiction. Hence, $\mu(S_\infty) = 0$. If we define $K_n = S_\infty$ for every $n \in \mathbb{N}$, then, by Proposition 7.3.8, we conclude that $\operatorname{Fix}(f) \neq \emptyset$. ☐

Note that condition (7.10) in Theorem 7.3.6 seems to be more general than that for the map to be quasicondensing with respect to a measure of noncompactness, but, as we shall show in the following result, both conditions are equivalent.

Proposition 7.3.9. *Let K be a bounded subset of a Banach space X, $\mu(\cdot)$ a measure of noncompactness on X. A mapping $f : K \to K$ is μ-quasicondensing if and only if f satisfies condition* (7.10).

Proof. Suppose f is μ-quasicondensing. In this case, it is enough to consider the map $h : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $h(x) = x$. On the other hand, if there exists a function h satisfying condition (7.10), then, by the monotonicity property of μ, we have $\mu(f(S)) < \mu(S)$ whenever $f(S) \subseteq S$ and $\mu(S) > 0$. ☐

Since Example 7.3.3 shows the existence of quasicondensing maps which are not condensing, as a consequence of the above two results, we obtain the following generalization of Sadovskii fixed point theorem.

Corollary 7.3.1. *Let K be a bounded subset of a Banach space X, $\mu(\cdot)$ a nonsingular measure of noncompactness on X. If $f : K \to K$ is a continuous μ-quasicondensing map, then* $\operatorname{Fix}(f) \neq \emptyset$.

At this point, one may be interested in characterizing the existence of fixed points for μ-quasicondensing maps with respect to a general measure of noncompactness μ.

In this direction, we need the concept of a minimal f-invariant subset with respect to a general measure of noncompactness.

Definition 7.3.12. Let K be a nonempty, convex, closed, bounded subset of a Banach space X and $f : K \to K$ a map. A set $S \subseteq K$ is said to be minimal f-invariant if S is nonempty closed, convex, $f(S) \subseteq S$, and whenever Y is a nonempty, closed, convex subset of S with $f(Y) \subseteq Y$, it follows that $Y = S$.

Obviously, any single-point f-invariant set is necessarily a fixed point of the map f, and is minimal. In general, the existence of a minimal invariant subset for a mapping is not assured (cf. see Remark 7.3.1 below). Furthermore, identifying the minimal f-invariant sets is not an easy task, as is cited by K. Goebel and B. Sims in [116]. Nevertheless, we have the following result.

Theorem 7.3.7. *Let K be a nonempty, convex, closed, bounded subset of a Banach space X and $\mu(\cdot)$ be a measure of noncompactness on X. Assume that $f : K \to K$ is a μ-quasicondensing and continuous map. Then, the following assertions are equivalent:*

(a) *f has at least one fixed point in K,*
(b) *there exists a minimal f-invariant subset of K,*
(c) *if S is minimal f-invariant, then S is a singleton.*

Proof. $(a) \Longrightarrow (b)$ This is obvious because any fixed point of f is minimal f-invariant. On the other hand, $(c) \Longrightarrow (a)$ is clear. We next prove $(b) \Longrightarrow (c)$. Let S be a minimal f-invariant set. Then, $\overline{\text{co}}f(S) \subseteq S$ because S is convex and closed. Thus, $f(\overline{\text{co}}f(S)) \subseteq f(S) \subseteq \overline{\text{co}}f(S)$. Since S is minimal f-invariant, we have $S = \overline{\text{co}}f(S)$. Bearing in mind the properties of $\mu(\cdot)$ and the fact that f is μ-quasicondensing, we deduce that S is compact, and, by Schauder fixed point theorem, there exists $z \in S$ such that $f(z) = z$. Then, $\{z\}$ is a subset of S which is minimal f-invariant. Hence, $S = \{z\}$. ☐

Remark 7.3.1. We can use the previous result in order to prove that, for the map $f : K \to K$ given in Example 7.3.2, there is no minimal f-invariant subset, because otherwise f would have at least one fixed point in K, which is a contradiction.

It is interesting to note that when K is weakly compact, the existence of a minimal f-invariant set is obtained by using Zorn lemma. As a consequence of this fact and Theorem 7.3.7, we have

Corollary 7.3.2. *Let X be a Banach space and $\mu(\cdot)$ a measure of noncompactness on X. If K is a closed convex subset of X and $f : K \to K$ is a continuous, μ-condensing map, then f has at least one fixed point whenever $\overline{f(K)}$ is weakly compact.*

7.4 On maps f such that $I - f$ is ϕ-expansive

Let us recall the definition of separate contraction mappings in the context of Banach spaces (see Definition 6.3.2).

Definition 7.4.1. Let K be a subset of a Banach space X. We say that a map $f : K \longrightarrow X$ is a separate contraction if there exist two functions $\varrho, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:
(1) $\psi(0) = 0$, ψ is nondecreasing,
(2) $\|f(x) - f(y)\| \leq \varrho(\|x - y\|)$,
(3) $0 < \psi(r) \leq r - \varrho(r)$ for $r > 0$.

In the original definition of a separate contraction mapping, the function ψ is assumed to be strictly increasing which is a particular case of the above definition. Anyway, if f is a separate contraction, then f is a nonexpansive map and $I - f$ is ϕ-expansive with $\phi = \psi$. Consequently:

- If K is a nonempty, bounded, closed, convex subset of a Banach space and $f : K \to X$ is a weakly inward separate contraction on K, then, according to Corollary 6.6.3, f has a unique fixed point in K.
- If K is a nonempty, closed subset of X and $f : K \to K$ is a separate contraction, we recapture Theorem 6.3.3 since, on the one hand, ϕ is nondecreasing and, on the other hand, if $x_0 \in K$ then it is not hard to see that $(f^n x_0)_{n\in\mathbb{N}}$ is an almost fixed point sequence. Therefore, we may apply Corollary 6.6.2 and Remark 6.6.2.
- In [209] S. Sadiq Basha introduces the concept of a weak contraction of the first kind. Clearly, this type of mapping falls into the class of separate contractions and therefore Corollary 3.3 in [209] is an easy consequence of the above comment.

Let X be a Banach space and K a nonempty subset of X. A mapping $f : K \longrightarrow X$ is said to be expansive (see Section 6.3.3) if there exists $h > 1$ such that $\|f(x) - f(y)\| \geq h\|x - y\|$, for all $x, y \in K$. In this case, clearly, $(I - f)$ is ϕ-expansive with $\phi(t) = (h - 1)t$. Moreover, f is invertible and $f^{-1} : \mathsf{R}(f) \longrightarrow K$ is a $\frac{1}{h}$-contraction. This implies that $I - f^{-1} : \mathsf{R}(f) \longrightarrow X$ is ϕ-expansive with $\phi(r) = (1 - \frac{1}{h})r$. Thus, if $K \subseteq \mathsf{R}(f)$, then we conclude that f^{-1} has a unique fixed point.

Definition 7.4.2. Let K be a nonempty, closed, bounded, convex subset of a Banach space X and $f : K \to K$ a mapping. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in K is said to be an approximate fixed point sequence for f if $\lim_{n\to+\infty} \|x_n - f(x_n)\| = 0$.

Next we have the following result for μ-nonexpansive mappings.

Theorem 7.4.1. *Let K be a nonempty, closed, convex subset of a Banach space X, $\mu(\cdot)$ a subadditive, nonsingular measure of noncompactness on X, and $f : K \longrightarrow K$ a map. Suppose that the following two conditions hold true:*
(a) *f is a μ-nonexpansive mapping,*
(b) *there exist $R > 0$ and $x_0 \in K$ such that for all $x \in K \cap \partial\mathbb{B}_R(x_0)$ and, for all $\lambda > 1$, we have that $f(x) - x_0 \neq \lambda(x - x_0)$.*

Then there exists a bounded almost fixed point sequence $(x_n)_{n\in\mathbb{N}}$ of f. Moreover, if $I - f : K \to \mathsf{R}(I - f)$ is ϕ-expansive, then f has a unique fixed point $x \in K$ and $x_n \to x$.

Proof. We define $\mathbb{B}_R^K(x_0) = \{x \in K : \|x - x_0\| \le R\}$ and let $\rho : K \longrightarrow \mathbb{B}_R^K(x_0)$ be the map given by

$$\rho(x) = \begin{cases} x & \text{if } \|x - x_0\| \le R, \\ \frac{R}{\|x-x_0\|}x + (1 - \frac{R}{\|x-x_0\|})x_0 & \text{if } \|x - x_0\| > R. \end{cases}$$

Here $\mathbb{B}_R^K(x_0)$ is a nonempty, bounded, closed, and convex subset of K, and the mapping ρ is a continuous retraction of K on $\mathbb{B}_R^K(x_0)$. For each integer $n \ge 2$, we define the map $f_n : K \to K$ by

$$f_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)f(x).$$

It is clear that, for each $n \ge 2$, f_n is continuous μ-$(1 - \frac{1}{n})$-contractive, and then μ-condensing. Now we define the map $f_{n,\rho} : \mathbb{B}_R^K(x_0) \to \mathbb{B}_R^K(x_0)$ by $f_{n,\rho}(x) = \rho(f_n(x))$. The map $f_{n,\rho}$ is continuous and μ-$(1 - \frac{1}{n})$- contractive because f_n is μ-$(1 - \frac{1}{n})$-contractive and ρ is μ-nonexpansive. Therefore f_n is μ-condensing. Next, using Theorem 7.3.5, we conclude that $f_{n,\rho}$ has a fixed point, say $f_{n,\rho}(x_n) = x_n$. We shall check that $f_n(x_n) = x_n$. In order to prove this, we will prove that $\|f_n(x_n) - x_0\| \le R$. Assume for a contradiction that $\|f_n(x_n) - x_0\| > R$. Hence we can write

$$x_n = \rho(f_n(x_n)) = \frac{R}{\|f_n(x_n) - x_0\|}f_n(x_n) + \left(1 - \frac{R}{\|f_n(x_n) - x_0\|}\right)x_0,$$

and thus

$$\frac{R}{\|f_n(x_n) - x_0\|}\big(f_n(x_n) - x_0\big) = x_n - x_0.$$

Consequently, $x_n \in K \cap \partial\mathbb{B}_R(x_0)$.

We also have

$$f(x_n) - x_0 = \frac{n}{n-1}\big(f_n(x_n) - x_0\big) = \lambda_n(x_n - x_0)$$

with $\lambda_n = \frac{n}{n-1}\frac{\|f_n(x_n)-x_0\|}{R} > 1$, which contradicts condition (b). This shows that

$$\big\|f_n(x_n) - x_0\big\| \le R,$$

and therefore $x_n = \rho(f_n(x_n)) = f_n(x_n)$.

Next we shall prove that $(x_n)_{n\in\mathbb{N}}$ is an almost fixed point sequence for f. First we note that

$$\big\|x_0 - f(x_n)\big\| \le \big\|x_0 - f_n(x_n)\big\| + \big\|f_n(x_n) - f(x_n)\big\| \le R + \frac{1}{n}\big\|x_0 - f(x_n)\big\|$$

and thus $\|x_0 - f(x_n)\| \le \frac{n}{n-1}R$. Next, using the following inequality:

$$\|x_n - f(x_n)\| = \frac{1}{n}\|x_0 - f(x_n)\| \leq \frac{R}{n-1},$$

we conclude that $(x_n)_{n\in\mathbb{N}}$ is a bounded almost fixed point sequence for f. Finally, if $I-f : K \to \mathrm{R}(I-f)$ is ϕ-expansive, then the use of Corollary 6.6.2 ends the proof. □

Example 7.4.1. Let X be a infinite-dimensional Banach space and let $f : X \to X$ be the mapping defined as

$$f(x) = \begin{cases} -x & \text{if } \|x\| \leq 1, \\ -\frac{x}{\|x\|} & \text{if } \|x\| > 1. \end{cases}$$

We note that f is μ-nonexpansive. We also have that $I - f$ is ϕ-expansive. To see that f is μ-nonexpansive for any nonsingular measure of noncompactness $\mu(\cdot)$, we will verify that, for every subset K of X, we have $f(K) = \mathrm{co}(-K \cup \{0\})$. Indeed, let $x \in K$. If $\|x\| \leq 1$, then $f(x) = -x \in -K$. If $\|x\| > 1$, then

$$f(x) = \frac{1}{\|x\|}(-x) + \left(1 - \frac{1}{\|x\|}\right)0 \in \mathrm{co}(-K \cup \{0\}).$$

This proves that $\mu(f(K)) \leq \mu(\mathrm{co}(-K \cup \{0\})) = \mu(K)$.

Now we will prove that $I - f$ is ϕ-expansive. Let $g = I - f$. Clearly,

$$g(x) = \begin{cases} 2x & \text{if } \|x\| \leq 1, \\ (1 + \frac{1}{\|x\|})x & \text{if } \|x\| > 1. \end{cases}$$

Let $x, y \in X$. We shall consider three cases:

Case 1, If $x, y \in \mathbb{B}_1$, then $\|g(x) - g(y)\| = 2\|x - y\|$.

Case 2. Let $x \in \mathbb{B}_1$ and $y \in X \backslash \mathbb{B}_1$. Since $\|x\| \leq 1$ if and only if $-(\|y\| - \|x\|) \leq -(\|y\| - 1)$, we obtain

$$\begin{aligned} \|g(x) - g(y)\| &= \left\|2x - y - \frac{y}{\|y\|}\right\| = \left\|2(x - y) + \frac{(\|y\| - 1)y}{\|y\|}\right\| \\ &\geq 2\|x - y\| - (\|y\| - 1) \\ &\geq 2\|x - y\| - (\|y\| - \|x\|) \\ &\geq \|x - y\|. \end{aligned}$$

Case 3. Let $x, y \in X \backslash \mathbb{B}_1$. As in case 2, we have $\|g(x) - g(y)\| \geq \|x - y\|$.

On the other hand, f is not condensing because $f(\mathbb{B}_1) = \mathbb{B}_1$ and f is nonexpansive if and only if X is a Hilbert space (see [81]). Therefore, when X is not a Hilbert space, Proposition 6.6.1 does not apply.

It is well known that property (FPP) for nonexpansive mappings closely depends upon geometric properties of the Banach spaces under consideration. Even when K is a weakly compact, convex subset of X, a nonexpansive self-mapping of K does not necessarily have fixed points (see, for example, Chapter 2 in [143] where B. Sims collected examples of fixed point free nonexpansive mappings in a variety of Banach spaces, also see [115]).

The next result gives an iterative method for approaching the fixed point of a nonexpansive mappings f such that $I - f$ becomes ϕ-expansive. To show this, we need the following lemma which may be found in [242].

Lemma 7.4.1. *Suppose that* $(a_n)_{n\in\mathbb{N}}$ *is a sequence in* $[0, +\infty)$ *such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n, \quad n \ge 0,$$

where $(\gamma_n)_{n\in\mathbb{N}}$ *is a sequence in* $(0, 1)$ *and* (δ_n) *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{+\infty} \gamma_n = +\infty$,

(2) $\limsup_{n\to+\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{+\infty} |\delta_n| < +\infty$.

Then $\lim_{n\to+\infty} a_n = 0$.

Theorem 7.4.2. *Let* K *be a closed, convex subset of a Banach space* X *and let* $f : K \longrightarrow K$ *be a nonexpansive map such that:*

(a) $I - f : K \to \mathrm{R}(I - f)$ *is* ϕ*-expansive,*

(b) *there exist* $R > 0$ *and* $x_0 \in K$ *such that, for all* $x \in K \cap \partial\mathbb{B}_R(x_0)$ *and for all* $\lambda > 1$, *we have* $f(x) - x_0 \ne \lambda(x - x_0)$.

Let $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence in* $(0, 1)$ *satisfying:*

(i) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$,

(ii) $\lim_{n\to+\infty} \alpha_n = 0$,

(iii) $\sum_{n=1}^{+\infty} |\alpha_n - \alpha_{n-1}| < +\infty$ *(for example,* $\alpha_n = \frac{1}{n}$*).*

Let $x_1 \in K$ *and define* $x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)f(x_n)$ *for each positive integer n. Then the sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to the unique fixed point of* f.

Proof. It is well known that, if f is nonexpansive, then it is α-nonexpansive where $\alpha(\cdot)$ denotes Kuratowski measure of noncompactness and thus f satisfies the assumptions of Theorem 7.4.1, which implies that f has a unique fixed point $p \in K$.

We claim that $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in K. To see this, let us notice that

$$\begin{aligned} \|x_n - p\| &= \big\|\alpha_n(x_1 - p) + (1 - \alpha_n)\big(f(x_{n-1}) - p\big)\big\| \\ &\le \alpha_n\|x_1 - p\| + (1 - \alpha_n)\|x_{n-1} - p\| \\ &\le \max\big\{\|x_1 - p\|, \|x_{n-1} - p\|\big\}. \end{aligned}$$

By induction, we infer that $\|x_n - p\| \le \|x_1 - p\|$ for every $n \in \mathbb{N}$. This means that $(x_n)_{n\in\mathbb{N}}$ is bounded, which proves our claim.

On the other hand, by the definition of $(x_n)_{n\in\mathbb{N}}$, we obtain that $(f(x_n))_{n\in\mathbb{N}}$ is also a bounded sequence. Let M be an upper bound of the sequence $(\|x_1 - f(x_n)\|)_{n\in\mathbb{N}}$. Then

$$\begin{aligned}\|x_{n+1} - x_n\| &= \big\|\alpha_n x_1 + (1-\alpha_n)f(x_n) - (\alpha_{n-1}x_1 + (1-\alpha_{n-1})f(x_{n-1}))\big\| \\ &= \big\|(\alpha_n - \alpha_{n-1})x_1 + (1-\alpha_n)(f(x_n) - f(x_{n-1})) + (\alpha_{n-1} - \alpha_n)f(x_{n-1})\big\| \\ &= \big\|(\alpha_n - \alpha_{n-1})(x_1 - f(x_{n-1})) + (1-\alpha_n)(f(x_n) - f(x_{n-1}))\big\| \\ &\le M|\alpha_n - \alpha_{n-1}| + (1-\alpha_n)\|x_n - x_{n-1}\|.\end{aligned}$$

Applying Lemma 7.4.1, we conclude that $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. Finally, because $x_{n+1} - f(x_n) = \alpha_n(x_1 - f(x_n)) \to 0$ as $n \to +\infty$, the use of the inequality

$$\big\|x_n - f(x_n)\big\| \le \|x_n - x_{n+1}\| + \big\|x_{n+1} - f(x_n)\big\|$$

implies that $(x_n)_{n\in\mathbb{N}}$ is a bounded almost fixed point sequence. Finally, according to Theorem 7.4.1, the sequence $(x_n)_{n\in\mathbb{N}}$ converges to the unique fixed point of f. ☐

Remark 7.4.1. It is well known that if $f : X \to X$ is a nonexpansive mapping such that $I - f : X \to X$ is ϕ-expansive, then $I - f$ is m-accretive (see Theorem 4.4.3). Applying Theorem 6.8.1, we conclude that $I - f$ is bijective. This means, among other things, that there exists a unique $z \in X$ such that $z = f(z)$. Thus, the same technique as in the proof of Theorem 7.4.2 shows that the sequence defined in such a result converges to the unique fixed point of f.

7.5 Tychonoff-type fixed point theorems

7.5.1 Tychonoff fixed point theorem

The aim of this section is to show that the Schauder mapping method can be adapted to yield a proof to the Schauder fixed point theorem in Hausdorff locally convex topological vector spaces. For the sake of completeness, we include a proof of the Tychonoff theorem due to S. Cobzas [68, p. 5]. A similar proof appeared in [38, p. 61], but it is based on the existence of a partition of unity instead of the Schauder mapping. Another proof can be found in the treatise by N. Dunford and J. T. Schwartz [89], it is based on three lemmas and, with some minor modifications, a similar proof may be found in the book by R. E. Edwards [92].

Definition 7.5.1. Let X be a vector space and let Z be a subset of X. We say that Z has finite dimensional type if $\dim(\operatorname{span}(Z)) < +\infty$.

Proposition 7.5.1. *Let X be a Hausdorff topological vector space. Then, for every finite subset $\{a_1, a_2, \ldots, a_n\}$ of X, there exists $m \in \mathbb{N}$ with $m \le n$, such that the set $\overline{\mathrm{co}}\{a_1, a_2, \ldots, a_n\}$ is linearly homeomorphic to a compact convex subset of $\mathbb{R}^m$.*

Proof. Let $Z = \operatorname{span}\{a_1, a_2, \dots, a_n\}$ and $m = \dim Z$. It follows that Z is linearly homeomorphic to $\mathbb{R}^m$; hence, there exists a linear homeomorphism $L : Z \to \mathbb{R}^m$. Since $\Pi := \overline{\operatorname{co}}\{a_1, a_2, \dots, a_n\}$ is a compact subset of Z, its image $L(\Pi)$ is a convex compact subset of $\mathbb{R}^m$. ☐

We infer from Proposition 7.5.1 the following result which is more general than the Brouwer fixed point theorem.

Corollary 7.5.1. *Let X be a Hausdorff topological vector space and let K be a finite-dimensional compact convex subset of X. Then any continuous mapping $f : K \to K$ has a fixed point.*

Let p be a seminorm on a vector space X and K a nonempty convex subset of X. For $\varepsilon > 0$, suppose that there exists a (p, ε)-net $\{z^1, z^2, \dots, z^n\} \in K$, that is, $K \subseteq \bigcup_{i=1}^n B'_p(z^i, \varepsilon)$. For $i \in \{1, 2, \dots, n\}$, we denote by $g^i = g^i_{p,\varepsilon}$, $w = w_{p,\varepsilon}$, and $w^i = w^i_{p,\varepsilon}$ the real valued functions defined, for all $x \in K$, by

$$g^i(x) = \max(\varepsilon - p(x - z^i), 0), \quad w(x) = \sum_{i=1}^n g^i(x), \quad \text{and} \quad w^i(x) = \frac{g^i(x)}{w(x)}. \tag{7.11}$$

We define the function $\varphi = \varphi_{p,\varepsilon} : K \to K$ by

$$\varphi(x) = \sum_{i=1}^n w^i(x) z^i, \quad \text{for all } x \in K. \tag{7.12}$$

The mapping $\varphi_{p,\varepsilon}$ is called a Schauder mapping.

Lemma 7.5.1. *Let p be a continuous seminorm on a topological vector space $(X, \mathcal{T})$, K a convex subset of X, and $\varepsilon > 0$. The mappings defined by* (7.11) *and* (7.12) *have the following properties:*

1. *The functions g^i are continuous and nonnegative on K.*
2. *The function w is continuous and $\forall x \in K$, $w(x) > 0$.*
3. *The functions w^i are well defined, continuous, nonnegative, and*

$$\sum_{i=1}^n w^i(x) = 1, \quad \forall x \in K.$$

4. *The mapping φ is continuous on K and*

$$p(\varphi(x) - x) < \varepsilon, \quad \textit{for all } x \in K. \tag{7.13}$$

Proof. (a) The continuity of g^i follows from the equality $g^i(x) = \frac{1}{2}(\varepsilon - p(x - z^i) + |\varepsilon - p(x - z^i)|)$ and the continuity of p.

(b) The continuity of w is obvious. Moreover, because for all $x \in K$, there exists $j \in \{1, 2, \dots, n\}$ such that $p(x - z^j) < \varepsilon$, it follows that $w(x) \geq g^j(x) = \varepsilon - p(x - z^j) > 0$.

(c) This item follows from (a) and (b).

(d) By (b) and (c), the functions w^i are well defined and continuous. Since for all $x \in K$, $\varphi(x)$ is a convex combination of the elements $z^1, z^2, \ldots, z^n$ of K, $\varphi(x) \in K$. Note that, for all $x \in K$, we have

$$\varphi(x) - x = \sum_{i=1}^{n} w^i(x)(z^i - x),$$

so that, by (c) and the fact that $p(z^i - x) < \varepsilon$ whenever $w^i(x) > 0$, we have

$$p(\varphi(x) - x) \le \sum_{i=1}^{n} w^i(x) p(z^i - x) < \varepsilon. \qquad \square$$

Remark 7.5.1. It follows that, for each $x \in K$, $\varphi(x)$ is a convex combination of the elements $z^1, z^2, \ldots, z^n$. So, φ is a mapping from K into $\operatorname{co}\{z^1, z^2, \ldots, z^n\}$. ☐

Theorem 7.5.1 (Tychonoff). *Let X be a Hausdorff, locally convex topological vector space and K a compact, convex subset of X. Then, for any continuous map $f : K \to K$, we have* $\operatorname{Fix}(f) \neq \emptyset$.

Proof. Let $\mathcal{B}$ be a basis of 0-neighborhoods formed by open convex symmetric subsets of X. The Minkowski functional μ_B corresponding to a set $B \in \mathcal{B}$ is a continuous seminorm on X and

$$B = \{x \in X : \mu_B(x) < 1\}. \tag{7.14}$$

It follows from the compactness of K that there exist $z_B^1, \ldots, z_B^{n(B)} \in K$ such that $K \subseteq \bigcup_{i=1}^{n(B)} (z_B^i + B)$. Denote by φ_B the Schauder mapping corresponding to μ_B, $\varepsilon = 1$ and $z_B^1, \ldots, z_B^{n(B)}$. Set $K_B = \operatorname{co}\{z_B^1, \ldots, z_B^{n(B)}\}$. It follows that $f_B := \mu_B \circ f$ is a continuous mapping of the finite-dimensional convex compact set K_B into itself. So, by Corollary 7.5.1, there exists $x_B \in K_B$ such that $f_B(x_B) = x_B$.

Using again the compactness of the set K, the net $(x_B : B \in \mathcal{B})$ admits a subnet $(x_{\gamma(\alpha)} : \alpha \in \Lambda)$ converging to an element $x \in K$. Here Λ is a directed set and $\gamma : \Lambda \to \mathcal{B}$ the nondecreasing mapping defining the subnet. We shall show that x is a fixed point of f. Since the topology of the space X is separable Hausdorff, this is equivalent to

$$\forall V \in \mathcal{N}_0, \quad x - f(x) \in V. \tag{7.15}$$

For $V \in \mathcal{N}_0$, we choose an element B of $\mathcal{B}$ such that $B + B \subset V$. By the definition of the subnet, there exists $\alpha_0 \in \Lambda$ such that $\gamma(\alpha_0) \subset B$. Then, for all $\alpha > \alpha_0$, $\gamma(\alpha) \subset \gamma(\alpha_0) \subset B$ and then, by (7.13) (with $\varepsilon = 1$), the fact that $\varphi_{\gamma(\alpha)}(f(x_{\gamma(\alpha)})) = x_{\gamma(\alpha)}$, and (7.14), we get

$$\begin{aligned} p_{\gamma(\alpha)}(\varphi_{\gamma(\alpha)}(f(x_{\gamma(\alpha)})) - f(x_{\gamma(\alpha)})) < 1 &\Longrightarrow \varphi_{\gamma(\alpha)}(f(x_{\gamma(\alpha)})) - f(x_{\gamma(\alpha)}) \in \gamma(\alpha) \subseteq B \\ &\Longrightarrow x_{\gamma(\alpha)} - f(x_{\gamma(\alpha)}) \in B. \end{aligned}$$

Passing to the limit for $\alpha \geq \alpha_0$ and taking into account the continuity of f, one obtains

$$x - f(x) \in \overline{B} \subseteq B + B \subseteq V,$$

because $\overline{B} = \bigcap\{B + A : A \in \mathcal{B}\}$. Thus one sees that equation (7.15) is satisfied, which completes the proof. ☐

We recall that, if X is a normed (Banach) space, then X equipped with its weak topology, $\sigma(X, X^*)$, is a Hausdorff, locally convex topological vector space. The following result is the formulation of Tychonoff theorem in a Banach space endowed with its weak topology.

Corollary 7.5.2. *Let X be a Banach space and K a convex, weakly compact subset of X. If $f : K \to K$ is a weakly continuous map, then* $\mathrm{Fix}(f) \neq \emptyset$.

7.5.2 Weakly sequentially continuous maps

The formulation of the Tychonoff theorem in Banach spaces (Corollary 7.5.2) is certainly interesting; nevertheless, the continuity hypothesis for the weak topology is, in practice, difficult to verify. The interest in this result lies in the fact that the sequential continuity for the weak topology of a mapping is often easier to verify than the continuity.

Before stating the theorem of O. Arino, S. Gautier, and J. P. Penot [21], we first establish the following result.

Proposition 7.5.2. *Let X be a metrizable, locally convex topological vector space, K a weakly compact subset of X, and $f : K \to X$ a map. If f is weakly sequentially continuous, then f is weakly continuous.*

It should be noticed that normed vector spaces are examples of metrizable locally convex topological vector spaces.

Proof. Let F be a weakly closed subset of X and $(y_n)_{n\in\mathbb{N}}$ a sequence of points of $f^{-1}(F)$ such that $y_n \rightharpoonup y$ as $n \to +\infty$. Since f is weakly sequentially continuous, we have $f(y_n) \rightharpoonup f(y)$. Since the sequence $(f(y_n))_{n\in\mathbb{N}}$ is contained in F, and F is weakly closed, we conclude that $f(y) \in F$ or equivalently, $y \in f^{-1}(F)$. This shows that $f^{-1}(F)$ is weakly sequentially closed. Taking into account of the weak compactness of K, we see that $\overline{f^{-1}(F)}^{w}$ is weakly compact.

Let $x \in \overline{f^{-1}(F)}^{w}$. By Theorem 1.7.3 (in fact, we use a version of Eberlein–Šmulian theorem (Theorem 1.7.3) adapted to locally convex metrizable topological vector spaces [92, Theorem 8.12.4, p. 549]), there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $f^{-1}(F)$ such that $x_n \rightharpoonup x$. Since $f^{-1}(F)$ is weakly sequentially closed, we have $x \in f^{-1}(F)$. This shows that $\overline{f^{-1}(F)}^{w} = f^{-1}(F)$ and then $f^{-1}(F)$ is weakly closed. ☐

Theorem 7.5.2 (Arino–Gauthier–Penot). *Let X be a metrizable locally convex topological vector space and K a weakly compact convex subset of K. Then any weakly sequentially continuous map $f : K \to K$ has a fixed point.*

Proof. To establish this result, it suffices to show that f is weakly continuous and apply Theorem 7.5.1. The continuity for the weak topology of f follows from Proposition 7.5.2. □

The next result is a consequence of Theorem 7.5.2.

Corollary 7.5.3. *Let X be a metrizable, locally convex topological vector space, K a closed, convex subset of X, and $f : K \to K$ a map such that $f(K)$ is relatively weakly compact. If f is weakly sequentially continuous, then* $\mathrm{Fix}(f) \neq \emptyset$.

Let X be a normed space. We note that closed subsets of X for the weak topology $\sigma(X, X^*)$ and closed subsets for strong topology (topology of the norm) coincide only if the dimension of X is finite. If the dimension of X is infinite, then the topology $\sigma(X, X^*)$ is strictly coarser than the strong topology (that is, there are closed sets for the strong topology which are not closed for the weak topology). However, closed convex subsets for the weak topology and closed convex subsets for the strong topology coincide (cf. Theorem 1.7.2).

In the context of Banach spaces, Theorem 7.5.2 can be restated in the following way.

Corollary 7.5.4. *Let X be a Banach space, K a nonempty, closed, convex subset of X, and $f : K \to K$ a map such that $f(K)$ is relatively weakly compact. If f is weakly sequentially continuous, then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof. Let $C = \overline{\mathrm{co}}(f(K))$. It follows from Theorem 1.7.3 that C is a weakly compact, convex subset of X. Further, we have $f(C) \subseteq C$. To complete the proof, it suffices to apply Theorem 7.5.2 because X endowed with its weak topology $\sigma(X, X^*)$ is a Hausdorff locally convex topological vector space. □

7.6 ws-compact and ww-compact mappings

Definition 7.6.1. Let X be a Banach space and let $f : D(f) \subseteq X \to X$ be a mapping.

(a) We say that f is ws-compact if f is continuous for the strong topology and if $(x_n)_{n\in\mathbb{N}} \subseteq D(f)$ is a weakly convergent sequence in X, then $(f(x_n))_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.

(b) We say that f is ww-compact if f is continuous for the strong topology and if $(x_n)_{n\in\mathbb{N}} \subseteq D(f)$ is a weakly convergent sequence in X, then $(f(x_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence in X.

Remark 7.6.1.

(1) It is clear that each compact operator is ws-compact and, when X is reflexive, these two notions are equivalent. However, the fact that f is ws-compact does not neces-

sarily imply the compactness of f even if f is linear. It is well known that a compact linear operator from a Banach space X into a Banach space Y transforms weakly convergent sequence into norm convergent sequences. If X is not reflexive, the converse of this assertion is, in general, not true even if the space Y is reflexive. Indeed, we denote by f the injection from ℓ_1 into ℓ_2. It is clear that f is not compact. However, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in ℓ_1 which converges weakly to x, then, by Corollary 14 in [89, p. 296], $(x_n)_{n\in\mathbb{N}}$ converges strongly in ℓ_1 to x. Using the continuity of f, we see that $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in ℓ_2.

(2) Note also that weakly compact linear operators acting on Banach spaces with Dunford–Pettis property are ws-compact. Indeed, if X is a Banach space with Dunford–Pettis property, then every weakly compact linear operator from X into an arbitrary Banach space Y maps weakly convergent sequences in X onto norm convergent sequences in Y.

(3) An operator f is ww-compact if it transforms the relatively weakly compact subsets into the relatively weakly compact subsets (use Theorem 1 in [89, p. 430]).

(4) Every bounded linear operator is ww-compact.

(5) ws-compact and ww-compact operators are not necessarily weakly continuous (see, for example, Theorem 2.6 in [17]).

In applications, it often happens that we have a noncompact continuous operator on a closed convex subset of a nonreflexive Banach space into itself. When the latter is ws-compact, we have the following result obtained in [159].

Theorem 7.6.1. *Let K be a nonempty, closed, convex subset of a Banach space X. Assume that $f : K \to K$ is a ws-compact mapping. If $f(K)$ is relatively weakly compact, then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof. Let $C = \overline{\mathrm{co}}(f(K))$ (the closed convex hull of $f(K)$). Since K is a closed convex subset of X satisfying $f(K) \subseteq K$, then $C \subseteq K$ and therefore $f(C) \subseteq f(K) \subseteq \overline{\mathrm{co}}(f(K)) = C$. Thus, f maps C into itself. Since $f(K)$ is relatively weakly compact, applying Theorem 1.7.4, one sees that C is weakly compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in C, then it has a weakly convergent subsequence, say (x_{n_k}). By hypothesis, $(f(x_{n_k}))_{k\in\mathbb{N}}$ has a strongly convergent subsequence and therefore $f(C)$ is relatively compact. Now the use of Schauder fixed point theorem concludes the proof. ☐

Definition 7.6.2. Let X be a Banach space, K a nonempty subset of X, and $f : K \to X$ a continuous mapping. We say that f is weakly compact if for all bounded subset C of K, $f(C)$ is relatively weakly compact.

It is obvious that if K is bounded, then f is weakly compact means that $f(K)$ is relatively weakly compact.

Theorem 7.6.2. *Let K be a nonempty, closed, convex subset of a Banach space X with $0 \in K$ and let $f : K \to K$ be a ws-compact map. Suppose*

(a) *f is weakly compact, and*
(b) *there exists $R > 0$ such that $f(x) \neq \lambda x$ for every $\lambda > 1$ and every $x \in K \cap \partial\mathbb{B}_R$.*

Then $\mathsf{Fix}(f) \neq \emptyset$.

Proof. Since $0 \in K$, it is clear that $K \cap \mathbb{B}_R$ is a nonempty, bounded, closed, and convex subset of K. We denote by ζ the radial retraction of K on $K \cap \mathbb{B}_R$ defined by

$$\zeta(x) = \begin{cases} x & \text{if } \|x\| \leq R, \\ R\frac{x}{\|x\|} & \text{if } \|x\| > R. \end{cases}$$

The map ζ is continuous from K on $K \cap \mathbb{B}_R$. Let $f_\zeta : \mathbb{B}_R \cap K \to \mathbb{B}_R \cap K$ be the mapping defined by $f_\zeta(x) = \zeta(f(x))$. It is clear that f_ζ is ws-compact. Since $\mathbb{B}_R \cap K$ is bounded, $f(\mathbb{B}_R \cap K)$ is relatively weakly compact.

The set $f_\zeta(\mathbb{B}_R \cap K)$ is relatively weakly compact on X. Indeed, we have $f_\zeta(\mathbb{B}_R \cap K) = \zeta(f(\mathbb{B}_R \cap K)$. If $x \in f(\mathbb{B}_R \cap K)$, there are two possibilities:
(i) if $\|x\| \leq R$, then $\zeta(x) = x \in f(\mathbb{B}_R \cap K) \subseteq \mathrm{co}(f(\mathbb{B}_R \cap K) \cup \{0\})$;
(ii) if $\|x\| > R$, then $\zeta(x) = \frac{R}{\|x\|}x + (1 - \frac{R}{\|x\|})0 \in \mathrm{co}(f(\mathbb{B}_R \cap K) \cup \{0\})$.

Hence $f_\zeta(\mathbb{B}_R \cap K)) \subseteq \mathrm{co}(f(\mathbb{B}_R \cap K) \cup \{0\})$. Now, since $f(\mathbb{B}_R \cap K)$ is relatively weakly compact, also $f(\mathbb{B}_R \cap K) \cup \{0\}$ is relatively weakly compact. Applying Krein–Šmulian theorem [89, p. 434], one sees that $\mathrm{co}(f(\mathbb{B}_R \cap K) \cup \{0\})$ is relatively weakly compact. This shows that f_ζ satisfies the conditions of Theorem 7.6.1 and thus there exists $x_0 \in \mathbb{B}_R \cap K$ such that $x_0 = f_\zeta(x_0)$.

We note that
(i) if $f(x_0) \in \mathbb{B}_R \cap K$, then $x_0 = f_\zeta(x_0) = \zeta(f(x_0)) = f(x_0)$;
(ii) if $f(x_0) \notin \mathbb{B}_R \cap K$, then $x_0 = f_\zeta(x_0) = \zeta(f(x_0)) = \frac{R}{\|f(x_0)\|}f(x_0)$. Accordingly, $\|x_0\| = R$ and if we take $\lambda = \frac{\|f(x_0)\|}{R} > 1$, then $f(x_0) = \lambda x_0$, which contradicts the hypothesis (b).

We conclude that $x_0 \in \mathsf{Fix}(f)$, which completes the proof. □

Remark 7.6.2. (a) Under the assumptions of Theorem 7.6.2, if further we assume that X is reflexive, then, since every bounded subset is relatively weakly compact, the fact that f is ws-compact implies that it is completely continuous, and thus we are under the conditions of Petryshyn theorem (see [196] or Theorem 2.9 in [99]).

(b) We also notice that we can derive from Theorem 7.6.2 a nonlinear alternative of Schaefer type for ws-compact operators (cf. Theorem 7.9.1).

Definition 7.6.3. Let K be a nonempty subset of a Banach space X and let $F : X \times X \to X$ be a mapping. The family of mappings $\{F(\cdot, y), : y \in K\}$ is said to be nonlinear contractive, if for each $y \in K$, the map $F(\cdot, y) : X \to X$ is a nonlinear contraction with Φ-function ϕ.

We note that the Φ-function for each mapping of the family $\{F(\cdot, y) : y \in K\}$ is ϕ.

Lemma 7.6.1. *Let K be a nonempty subset of a Banach space X. If $F : X \times X \to X$ is continuous, and the family $\{F(\cdot,y) : y \in K\}$ is nonlinear contractive, then there exists a continuous map $J : K \to X$ such that $J(y) = F(Jy,y)$ for each $y \in K$.*

Proof. For an arbitrary fixed $y \in K$, the mapping $F(\cdot,y)$ defined by $X \ni x \mapsto F(x,y)$ is a nonlinear contraction and maps X into itself. So, according to Theorem 7.12.2, it has a unique fixed point. Let us denote by $J : K \to X$ the map which assigns to each $y \in K$ the unique point $Jy \in X$ such that $Jy = F(Jy,y)$. Thus the map J is well defined.

Consider a sequence $(y_n)_{n\in\mathbb{N}}$ in K converging to some $z \in K$, thus we have

$$\begin{aligned}\|J(y_n) - J(z)\| &= \big\|F(Jy_n,y_n) - F(Jz,z)\big\| \\ &\le \big\|F(Jy_n,y_n) - F(Jz,y_n)\big\| + \big\|F(Jz,y_n) - F(Jz,z)\big\| \\ &\le \phi(\|J(y_n) - J(z)\|) + \big\|F(Jz,y_n) - F(Jz,yz)\big\|,\end{aligned}$$

which implies

$$\|J(y_n) - J(z)\| - \phi(\|J(y_n) - J(z)\|) \le \big\|F(Jz,y_n) - F(Jz,z)\big\|.$$

Let $\tau_n = \|J(y_n) - J(z)\|$. From the continuity of F, we obtain that $\tau_n - \phi(\tau_n) \to 0$ as $n \to \infty$. The property of ϕ shows that $\tau_n \to 0$, that is, $J(y_n) \to J(z)$. Hence, the map J is continuous on K. ☐

Theorem 7.6.3. *Let X be a Banach space, and let K be a nonempty subset of X. Suppose that the two continuous operators $f : K \to X$ and $F : X \times X \to X$ satisfy*

(a) *f is ws-compact,*
(b) *the family $\{F(\cdot,y) : y \in \overline{f(K)}\}$ is nonlinear contractive,*
(c) *there exists a nonempty, weakly compact, and convex subset $\mathcal{P}$ of K such that*

$$x = F\big(x,f(z)\big) \implies x \in \mathcal{P}, \quad \text{for all } z \in \mathcal{P}.$$

Then there is a point $x \in K$ such that $x = F(x,f(x))$.

Proof. We denote by $J : \overline{f(K)} \to X$ the map which assigns to each $y \in \overline{f(K)}$ the unique point Jy in X such that $Jy = F(Jy,y)$. From Lemma 7.6.1, the map J is well defined and continuous on $\overline{f(K)}$.

For any $z \in \mathcal{P}$, by assumption (c), we conclude that there is $x = (J \circ f)(z) \in \mathcal{P}$ such that $x = F(x,f(z))$. This proves that $(J \circ f)(\mathcal{P}) \subseteq \mathcal{P}$.

Moreover, since f is ws-compact and J is continuous, the mapping $(J \circ f)$ is ws-compact. The use of the inclusion $(J \circ f)(\mathcal{P}) \subseteq \mathcal{P}$ shows that $(J \circ f)(\mathcal{P})$ is relatively weakly compact. So, by Theorem 7.6.1 we obtain that $(J \circ f)$ has at least one fixed point $x \in \mathcal{P} \subset K$, that is, $(J \circ f)(x) = x$, which implies that

$$F\big(x,f(x)\big) = F\big((J \circ f)(x),f(x)\big) = (J \circ f)(x) = x.$$

This ends the proof. ☐

We close this section with the following theorem and its corollaries.

Theorem 7.6.4. *Let X be a Banach space and let K be a nonempty, closed, convex subset of X. Let $f : X \to K$ and $g : K \to X$ be two continuous maps and set $F = f \circ g$. Assume that*
(a) *$F(K)$ is relatively weakly compact,*
(b) *f is ws-compact,*
(c) *g is ww-compact.*

Then $\mathrm{Fix}(F) \neq \emptyset$.

Proof. Let $C = \overline{\mathrm{co}}(F(K))$. Since K is a closed, convex subset of X satisfying $F(K) \subseteq K$, one has $C \subseteq K$ and therefore $F(C) \subseteq F(K) \subseteq \overline{\mathrm{co}}(F(K)) = C$. This shows that F maps C into itself. Since $F(K)$ is relatively weakly compact, by Krein–Šmulian theorem (Theorem 1.7.4), C is weakly compact, too.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in C. Using the fact that C is weakly compact and g is ww-compact, we infer that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(g(x_{n_k}))_{k\in\mathbb{N}}$ converges weakly in C. Next, since f is ws-compact, we conclude that there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$ such that the sequence $(F(x_{n_{k_j}}))_{j\in\mathbb{N}}$ converges strongly in C. Hence, F is ws-compact. Because $F(C)$ is relatively weakly compact, the use of Theorem 7.6.1 concludes the proof. ☐

Remark 7.6.3. Note that, the identity operator I of the space X belongs to $\mathcal{L}(X)$, so it is ww-compact. Hence, if we take $g = I$ in Theorem 7.6.4, we recapture Theorem 7.6.1.

Remark 7.6.4. According to Definition 1.9.8, linear Dunford–Pettis operators map weakly compact sets into norm compact sets, so they are ws-compact operators.

Corollary 7.6.1. *Let X be a Banach space and let K be a nonempty, closed, convex subset of X. Let $f : X \to K$ and $g : K \to X$ be two continuous maps and set $F = f \circ g$. Assume that*
(a) *$F(K)$ is relatively weakly compact,*
(b) *f is a linear Dunford–Pettis operator,*
(c) *g is ww-compact.*

Then $\mathrm{Fix}(F) \neq \emptyset$.

Proof. We note that, according to Remark 7.6.4, the operator f is ws-compact, so the result follows from Theorem 7.6.4. ☐

Corollary 7.6.2. *Let X be a Banach space with Dunford–Pettis property (see Section* 1.9.5*) and let K be a nonempty, closed, convex subset of X. Let $f : X \to K$ and $g : K \to X$ be two continuous maps and set $F = f \circ g$. Assume that*
(a) *f is a linear weakly compact operator,*
(b) *g is ww-compact and $g(K)$ is bounded.*

Then $\mathrm{Fix}(F) \neq \emptyset$.

Proof. Since X is a Banach space with Dunford–Pettis property and f is a weakly compact linear operator, according to Remark 1.9.2, f is a Dunford–Pettis operator. To complete the proof, it suffices to show that $F(K)$ is relatively weakly compact. This follows from the boundedness of $g(K)$ and hypothesis (a). Now the use of Corollary 7.6.1 ends the proof. ☐

7.7 Measure of weak noncompactness

7.7.1 Axiomatic approach

Let X be a Banach space. We denote by $\mathbf{W}(X)$ (resp. $\mathbf{RW}(X)$) the subset of $\mathbf{B}(X)$ consisting of all weakly (resp. relatively weakly) compact subsets of X.

Definition 7.7.1. A mapping $\mu : \mathbf{B}(X) \to [0, +\infty[$ is a measure of weak noncompactness on X if the following conditions are satisfied:

(1) the family $\ker \mu := \{M \in \mathbf{B}(X) : \mu(M) = 0\}$ is nonempty and $\ker \mu \subset \mathbf{RW}(X)$,
(2) $M_1 \subset M_2 \Longrightarrow \mu(M_1) \le \mu(M_2)$, $\forall M_1, M_2 \in \mathbf{B}(X)$
(3) $\mu(\overline{\text{co}}(M)) = \mu(M)$, $\forall M \in \mathbf{B}(X)$
(4) $\mu(M \cup \{x\}) = \mu(M)$, $\forall M \in \mathbf{B}(X)$ and $x \in X$.

The family $\ker \mu$ is called the kernel of measure of weak noncompactness $\mu(\cdot)$. We note that $\mu(\cdot)$ satisfies

(5) $\mu(\overline{M}^{w}) = \mu(M)$.

Indeed, property (5) results from the inclusions $M \subseteq \overline{M}^{w} \subseteq \overline{\text{co}}(M)$ and properties (2) and (3) above.

The measure of weak noncompactness μ is said to be homogeneous if, for all $M \in \mathbf{B}(X)$,

(6) $\mu(\lambda M) = |\lambda| \mu(M)$, $\forall \lambda \in \mathbb{R}$.

We say that μ is subadditive if

(7) $\mu(M + N) \le \mu(M) + \mu(N)$, $\forall M, N \in \mathbf{B}(X)$.

Proposition 7.7.1. *Let X be a Banach space and $\mu(\cdot)$ a measure of weak noncompactness on X. If $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence (in the sense of the inclusion) of weakly closed subsets of $\mathbf{B}(X)$ such that $\lim_{n\to+\infty} \mu(K_n) = 0$, then $K_\infty = \bigcap_{n=1}^{+\infty} K_n$ is a nonempty weakly compact subset of X.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of X such that for all $n \in \mathbb{N}$, $x_n \in K_n$. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of subsets where $S_n = \{x_k : k \ge n\}$. It is clear that $(S_n)_{n\in\mathbb{N}}$ is decreasing and $S_n \subseteq K_n$ for all $n \in \mathbb{N}$. Since singletons are weakly compact, property (4) above ensures that, for all $n \in \mathbb{N}$, $\mu(S_0) = \mu(S_n) \le \mu(K_n)$. Because $\lim_{n\to+\infty} \mu(K_n) = 0$, we have $\mu(S_0) = 0$ and therefore the set $\{x_n, n \in \mathbb{N}\}$ is relatively weakly compact. Let x be the weak limit of a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. It is clear that, for any $n \in \mathbb{N}$, $x \in K_n$ and therefore $x \in K_\infty$ which proves that $K_\infty \neq \emptyset$. Moreover, for all $n \in \mathbb{N}$, $\mu(K_\infty) \leq \mu(K_n)$ and then $\mu(K_\infty) = 0$ because $\lim_{n\to+\infty} \mu(K_n) = 0$. Hence, K_∞ is relatively weakly compact. But K_∞ is weakly closed, so it is weakly compact. ☐

Definition 7.7.2. Let $\mu(\cdot)$ be a measure of weak noncompactness on a Banach space. We say that $\mu(\cdot)$ is regular if it satisfies, in addition, conditions (6), (7), and $\ker(\mu) = \mathbf{RW}(X)$.

7.7.2 De Blasi measure of weak noncompactness

The first example of a measure of weak noncompactness was introduced by F. S. De Blasi in 1977 [75]. It is defined by

$$\omega : \mathbf{B}(X) \to [0, +\infty[, \quad A \mapsto \omega(A),$$

where

$$\omega(A) = \inf\{t > 0 : \text{there exists } K \in \mathbf{W}(X) \text{ such that } A \subseteq K + t\mathbb{B}_1\}.$$

It satisfies the following properties.

Proposition 7.7.2. *Let X be a Banach space and A and B belong to $\mathbf{B}(X)$. Then*

(1) $\ker(\omega) = \mathbf{RW}(X)$,
(2) *if* $A \subset B$, *then* $\omega(A) \leq \omega(B)$,
(3) $\omega(A) = \omega(\overline{A}^w)$,
(4) $\omega(A) = 0$ *if and only if* $\overline{A}^w$ *is weakly compact,*
(5) $\omega(A \cup B) = \max(\omega(A), \omega(B))$,
(6) $\omega(A) = \omega(\mathrm{co}(A))$,
(7) $\omega(A + B) \leq \omega(A) + \omega(B)$ *and* $\omega(A + \{a\}) = \omega(A)$,
(8) $\omega(tA) = t\omega(A)$, $t \geq 0$.

Proof. The proofs are similar to those for the abstract measure of weak noncompactness. We restrict ourselves to the proof of statement (3).

It follows from the definition of $\omega(\cdot)$ that there exist $C \in \mathbf{W}(X)$ and $t > 0$ such that $A \subseteq C + t\mathbb{B}_1$. Hence $A \subseteq \overline{\mathrm{co}}(C) + t\mathbb{B}_1$. By Theorem 1.7.4, $\overline{\mathrm{co}}(C)$ is weakly compact. Further, using Proposition 1.7.2 shows that $t\mathbb{B}_1$ weakly closed. Thus $\overline{\mathrm{co}}(C) + t\mathbb{B}_1$ is the sum of a weakly compact set and a weakly closed set, so it is weakly closed. It is clear that $\overline{A}^w \subseteq \overline{\mathrm{co}}(C) + t\mathbb{B}_1$, hence $\omega(\overline{A}^w) \leq t$, which proves that $\omega(\overline{A}^w) \leq \omega(A)$. Finally, using the inclusion $A \subseteq \overline{A}^w$ and property (2), we obtain $\omega(A) \leq \omega(\overline{A}^w)$, which ends the proof. ☐

We shall use the following lemma due to H. Radström [197].

Lemma 7.7.1. *Let A, B, and C be nonempty subsets of a Banach space X. If B is closed, convex and C is bounded such that $A + C \subseteq B + C$, then $A \subseteq B$.*

Proof. Let $a \in A$. We shall prove that a belongs also to B. Given an element c_1 in C, it is clear that $a + c_1 \in B + C$. There exist $b_1 \in B$ and $c_2 \in C$ such that $a + c_1 = b_1 + c_2$. For the same reason, there exist $b_2 \in B$ and $c_3 \in C$ with $a + c_2 = b_2 + c_3$. Repeating this process for each $n \in \mathbb{N}$ and summing the first n equations, we get

$$na + \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} b_i + \sum_{i=2}^{n+1} c_i,$$

or equivalently,

$$a = \frac{1}{n}\sum_{i=1}^{n} b_i + \frac{c_{n+1}}{n} + \frac{c_1}{n}.$$

Since B is convex, for all $n \in \mathbb{N}$, we have $\eta_n = (1/n)\sum_{i=1}^{n} b_i \in B$. Using the boundedness of C, we conclude that $\frac{c_1}{n}$ and $\frac{c_n}{n}$ converge to 0 as n tends towards infinity, and therefore η_n converges to a as n goes to $+\infty$. Since B is closed, we conclude that $a \in B$. ☐

Proposition 7.7.3. *Let X be a Banach space.*
(a) *If X is reflexive, then $\omega(\mathbb{B}_1) = 0$.*
(b) *If X is not reflexive, then $\omega(\mathbb{B}_1) = 1$.*

Proof. (a) If X is reflexive, then, according to Theorem 1.9.1, $\mathbb{B}_1$ is a weakly compact, and so $\omega(\mathbb{B}_1) = 0$.

(b) We suppose that X is not reflexive. We first note that $\mathbb{B}_1 \subseteq \{0\} + \mathbb{B}_1$ and then we have $\omega(\mathbb{B}_1) \le 1$. For the purpose of an indirect proof, suppose that $\omega(\mathbb{B}_1) < 1$. From the definition of $\omega(\cdot)$, there exist $C \in \mathbf{W}(X)$ and $\omega(\mathbb{B}_1) < t < 1$ such that $\mathbb{B}_1 \subseteq C + t\mathbb{B}_1$. Since $\overline{\text{co}}(C)$ is weakly compact (use Theorem 1.7.4), we have $\mathbb{B}_1 \subseteq \overline{\text{co}}(C) + t\mathbb{B}_1$. Let $t \in\,]0, 1[$. Taking into the convexity of $\mathbb{B}_1$, we have $(1-t)\mathbb{B}_1 + t\mathbb{B}_1 \subseteq \mathbb{B}_1$ and then $(1-t)\mathbb{B}_1 + t\mathbb{B}_1 \subseteq \overline{\text{co}}(C) + t\mathbb{B}_1$. Applying Lemma 7.7.1, we conclude that $(1-t)\mathbb{B}_1 \subseteq \overline{\text{co}}(C)$. Further, since $(1-t)\mathbb{B}_1$ is a closed convex of X, it follows from Theorem 1.7.2 that it is weakly closed. The inclusion $(1-t)\mathbb{B}_1 \subseteq \overline{\text{co}}(C)$ implies that $(1-t)\mathbb{B}_1$ is also weakly compact. Finally taking into account the fact that the mapping $x \to (1-t)^{-1}x$ is weakly continuous (because it is linear), we deduce that $\mathbb{B}_1$ is weakly compact and therefore X is reflexive which contradicts our hypothesis. Hence $\omega(\mathbb{B}_1) = 1$. ☐

Remark 7.7.1. Using the equality $\overline{\text{co}}(\partial\mathbb{B}_1) = \mathbb{B}_1$ and Proposition 7.7.3, we obtain $\omega(\mathbb{B}_1) = \omega(\overline{\text{co}}(\partial\mathbb{B}_1)) = \omega(\partial\mathbb{B}_1) = 1$. Hence Proposition 7.7.3 holds true for the unit sphere.

Proposition 7.7.4. *If A is a bounded subset of a Banach space X, then*

$$\omega(A + r\mathbb{B}_1) = \omega(A) + r\omega(\mathbb{B}_1), \quad \forall r \ge 0.$$

Proof. It is clear that $\omega(A + r\mathbb{B}_1) \le \omega(A) + r\omega(\mathbb{B}_1)$.

If X is reflexive, then $\omega(\mathbb{B}_1) = 0$. Moreover, because A and $A + r\mathbb{B}_1$ are bounded, they are relatively weakly compact and then $\omega(A) = \omega(A + r\mathbb{B}_1) = 0$.

If X is not reflexive, then, by Proposition 7.7.3, we have $\omega(\mathbb{B}_1) = 1$. First observe that we have $\omega(A + r\mathbb{B}_1) \ge r$. Otherwise, if x is any point in A, $r\mathbb{B}_1 \subseteq A - \{x\} + r\mathbb{B}_1$ and so

$$r > \omega(A + r\mathbb{B}_1) = \omega\big(A - \{x\} + r\mathbb{B}_1\big) \ge r\omega(\mathbb{B}_1) = r,$$

which is impossible. According to the definition of ω, there exist $C \in \mathbf{W}(X)$ and $t > \omega(A + r\mathbb{B}_1)$ such that $A + r\mathbb{B}_1 \subseteq C + t\mathbb{B}_1$. So, we get

$$A + r\mathbb{B}_1 \subseteq \overline{\mathrm{co}}(C) + t\mathbb{B}_1, \quad \text{or equivalently,} \quad A + r\mathbb{B}_1 \subseteq \overline{\mathrm{co}}(C) + (t - r)\mathbb{B}_1 + r\mathbb{B}_1.$$

Since $\overline{\mathrm{co}}(C)+(t-r)\mathbb{B}_1$ is strongly closed, invoking Lemma 7.7.1, we get $A \subseteq \overline{\mathrm{co}}(C)+(t-r)\mathbb{B}_1$. Moreover, since $\overline{\mathrm{co}}(C)$ is weakly compact, we have $\omega(A) \le (t-r)$. Letting t go to $\omega(A+r\mathbb{B}_1)$, we get $\omega(A) + r \le \omega(A + r\mathbb{B}_1)$. This completes the proof. □

Since $\omega(\cdot)$ is a regular measure of weak noncompactness, Proposition 7.7.1 holds also true for ω. The proof follows the same lines.

Proposition 7.7.5. *Let $(K_n)_{n\in\mathbb{N}}$ be a decreasing sequence of nonempty, weakly closed subsets of a Banach space X. If K_0 is bounded and $\lim_{n\to+\infty} \omega(K_n) = 0$, then $K_\infty = \bigcap_{n=1}^{+\infty} K_n$ is a nonempty, weakly compact subset of X.*

Remark 7.7.2. We note that the measure of weak noncompactness of De Blasi, $\omega(\cdot)$, is regular.

Let $r > 0$ and let M be a subset of a Banach space. We recall that a ball centered at M with radius r is the set

$$\mathbb{B}_r(M) := \bigcup_{x\in M} \mathbb{B}_r(x) = \{y \in X : \mathrm{d}(y, M) \le r\},$$

where $d(y, M)$ stands for the distance from y to M. It is clear that, for all $r > 0$, the set $\mathbb{B}_r(M)$ can be written in the form

$$\mathbb{B}_r(M) = M + \mathbb{B}_r = M + r\mathbb{B}_1.$$

Let M and N be two elements of $\mathbf{B}(X)$. Set

$$d(M, N) = \inf\{r > 0 : M \subset \mathbb{B}_r(N)\}$$

and let $d_H : \mathbf{B}(X) \times \mathbf{B}(X) \to \mathbb{R}^+$ be the function defined by

$$d_H(M, N) = \max\{d(M, N), d(N, M)\}.$$

It is well known that d_H is a pseudometric on $\mathbf{B}(X)$ and it is a complete metric on $\mathbf{BF}(X)$. The function d_H is called the Hausdorff distance and $d_H(M,N)$ is the Hausdorff distance between the sets M and N.

Let Π be a family of subsets of $\mathbf{B}(X)$ and $M \in \mathbf{B}(X)$. We recall that the distance from M to Π is given by

$$\delta(M,\Pi) := \inf\{d_H(M,N) : N \in \Pi\}.$$

Since $\ker\omega = \mathbf{W}(X)$, the measure of weak noncompactness, ω, is given by the formula

$$\omega(M) = \delta(M,\mathrm{W}) \quad \forall M \in \mathbf{B}(X). \tag{7.16}$$

We also notice that there exists a relationship between a regular measure of weak noncompactness and ω. Indeed, we have the following result.

Theorem 7.7.1. *Let X be a Banach and μ a regular measure of weak noncompactness on X. Then, for all $M \in \mathbf{B}(X)$, we have*

$$\mu(M) \le \mu(\mathbb{B}_1)\omega(M).$$

Proof. This result is trivial if X is reflexive. Assume now that X is not reflexive and set $r = \omega(M)$. Let $\varepsilon > 0$ be an arbitrary real number. Due to (7.16), there exists $N \in \mathbf{W}(X)$ such that $M \subseteq \mathbb{B}_{r+\varepsilon}(N)$. Since $\mathbb{B}_{r+\varepsilon}(N) = N + (r+\varepsilon)\mathbb{B}_1$, we obtain

$$\mu(M) \le \mu\big(N + (r+\varepsilon)\mathbb{B}_1\big) \le (r+\varepsilon)\mu(\mathbb{B}_1).$$

Since ε is an arbitrary real number, we conclude that $\mu(M) \le \mu(\mathbb{B}_1)\omega(M)$. ☐

It should be noticed that, following J. Appell and E. De Pascale [18], if X is a finite-dimensional Banach space and Ω is a measurable subset of $\mathbb{R}^n$, then the measure of weak noncompactness of De Blasi on the space $L^1(\Omega,X)$ can be explicitly written as

$$\omega(M) = \limsup_{\varepsilon\to 0}\left\{\sup_{\psi\in M}\left[\int_D \|\psi(t)\|_X\,dt : |D| \le \varepsilon\right]\right\} \tag{7.17}$$

for all bounded subset M of $L^1(\Omega,X)$. Here $|D|$ denotes the Lebesgue measure the subset D of Ω.

If X is a normed space and A a subset of X, then the notation $\|A\|$ means

$$\|A\| := \sup\{\|x\| : x \in A\}. \tag{7.18}$$

In the context of Banach algebras, we have the following result.

Proposition 7.7.6. *Let A and B be bounded subsets of a Banach algebra X. Then the following assertions hold:*

(a) $\omega(AB) \le \|B\|\omega(A) + \|A\|\chi(B) + \omega(A)\chi(B)$.

(b) *If X is a WC-Banach algebra, then*

$$\omega(AB) \le \|B\|\omega(A) + \|A\|\omega(B) + \omega(A)\omega(B).$$

(c) $\chi(AB) \le \|B\|\chi(A) + \|A\|\chi(B) + \chi(A)\chi(B)$.

Here $\chi(\cdot)$ is the Hausdorff measure of noncompactness.

Proof. (a) Fix arbitrary real numbers r and t such that $r > \omega(A)$ and $t > \chi(B)$. Then there exist a weakly compact set W and a finite set F such that

$$A \subset W + \mathbb{B}_r \tag{7.19}$$

and

$$B \subset F + \mathbb{B}_t \tag{7.20}$$

Let $z \in AB$. Then we can find $x \in A$ and $y \in B$ such that $z = xy$. Using (7.19) and (7.20), we deduce that there are $w \in W, f \in F, u \in \mathbb{B}_r$, and $v \in \mathbb{B}_t$ such that

$$\begin{aligned} z = xy = (w+u)(f+v) &= wf + wv + uf + uv \\ &= wf + (x-u)v + u(y-v) + uv \\ &= wf + xv + uy - uv. \end{aligned}$$

This yields

$$AB \subset WF + A\mathbb{B}_t + \mathbb{B}_r B + \mathbb{B}_r\mathbb{B}_t \subset WF + \mathbb{B}_{\|A\|t+\|B\|r+rt}.$$

It follows from Lemma 1.13.2 and the definition of $\omega(\cdot)$ that

$$\omega(AB) \le \|A\|t + \|B\|r + rt.$$

Finally, letting $r \to \omega(A)$ and $t \to \chi(B)$, we get

$$\omega(AB) \le \|A\|\chi(B) + \|B\|\omega(A) + \omega(A)\chi(B),$$

which proves (a).

(b) Suppose that X is WC-Banach algebra and fix arbitrary real numbers r and t such that $r > \omega(A)$ and $t > \omega(B)$. Then there exist weakly compact subsets W_1 and W_2 such that

$$A \subset W_1 + \mathbb{B}_r \quad \text{and} \quad B \subset W_2 + \mathbb{B}_t. \tag{7.21}$$

Let $z \in AB$. Then z may be written in the form $z = xy$ with $x \in A$ and $y \in B$. It follows from (7.21) that there exist $w \in W_1$ and $w_2 \in W_2$, $u \in \mathbb{B}_r$, and $v \in \mathbb{B}_t$ such that $x = w_1 + u$, $y = w_2 + v$. Arguing in the same way as in the proof of (a), we get

$$z = xy = (w_1 + u)(w_2 + v) = w_1 w_2 + xv + uy - uv.$$

This implies the inclusion

$$AB \subset W_1 W_2 + A\mathbb{B}_t + \mathbb{B}_r B + \mathbb{B}_t \mathbb{B}_r \subset W_1 W_2 + \mathbb{B}_{\|A\|t + \|B\|r + rt}.$$

Since X is a WC-Banach algebra (cf. Definition 1.13.2), it follows from the definition of $\omega(\cdot)$ that

$$\omega(AB) \leq \|A\|t + \|B\|r + rt.$$

Now, letting $r \to \omega(A)$ and $t \to \omega(B)$, we get

$$\omega(AB) \leq \|B\|\omega(A) + \|A\|\omega(B) + \omega(A)\omega(B).$$

(c) It is similar to the proof of (b), hence omitted. ☐

7.7.3 Banaś–Knap measure of weak noncompactness

The second example of measure of weak noncompactness was introduced by J. Banaś and Z. Knap [27] on the space $L^1([0, +\infty[, dx)$, which is very useful in applications (cf. [42, 156]). Set

$$X := L^1([0, +\infty[, dx)$$

and define the function $\gamma : \mathbf{B}(X) \to [0, +\infty[$ by

$$\gamma(M) = \gamma_1(M) + \gamma_2(M) \quad \forall M \in \mathbf{B}(X),$$

where

$$\gamma_1(M) = \limsup_{\varepsilon \to 0} \left\{ \sup_{\psi \in M} \left(\int_D |\psi(t)|\, dt : |D| < \varepsilon \right) \right\}$$

and

$$\gamma_2(M) = \lim_{T \to +\infty} \left\{ \sup_{\psi \in M} \left(\int_T^{+\infty} |\psi(t)|\, dt \right) \right\},$$

with $|D|$ denoting the Lebesgue measure of the subset D of $[0,+\infty[$. The function $\gamma(\cdot)$ satisfies the following properties.

Proposition 7.7.7. *If M, M_1, and M_2 belong to $\mathbf{B}(X)$, then*

(1) *if $M_1 \subseteq M_2$, then $\gamma(M_1) \leq \gamma(M_2)$,*
(2) $\gamma(M_1 \cup M_2) = \max\{\gamma(M_1), \gamma(M_2)\}$,
(3) $\gamma(\lambda M) = |\lambda|\gamma(M)$ *for all* $\lambda \in \mathbb{R}$,
(4) $\gamma(M_1 + M_2) \leq \gamma(M_1) + \gamma(M_2)$,
(5) $\gamma(M) = 0$ *if and only if* $M \in \mathbf{RW}(X)$,
(6) $\gamma(\mathrm{co}(M)) = \gamma(M)$,
(7) $\gamma(\overline{M}^{w}) = \gamma(M)$.

Proof. Assertions (1), (2), (3), and (4) are immediate.

(5) The fact that $\gamma(M) = 0$ implies that $M \in \mathbf{RW}(X)$ (use Theorem 1.7.6). Suppose now that $M \in \mathbf{RW}(X)$. It follows from Theorem 1.7.6(a) that $\gamma_1(M) = 0$. It remains to show that $\gamma_2(M) = 0$. Set $E_n = [T_n, +\infty[$ where T_n is a nonnegative real number. It is clear that, if $(T_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of nonnegative real numbers such that $\lim_{n\to+\infty} T_n = +\infty$, then $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence of intervals (in the sense of inclusion) such that $\bigcap_{n\geq 0} E_n = \emptyset$. According to Theorem 1.7.5, we have

$$\lim_{n\to+\infty} \sup_{\psi\in M} \int_{T_n}^{+\infty} |\psi(t)|\, dt = 0,$$

and consequently $\gamma_2(M) = 0$. Thus, we conclude that $\gamma(M) = 0$.

(6) Since $M \subset \mathrm{co}(M)$, using (1), we get $\gamma(M) \leq \gamma(\mathrm{co}(M))$. To show the opposite inequality, we consider a measurable subset D of $[0,+\infty[$ such that $|D| < \varepsilon$. For $\psi \in \mathrm{co}(M)$, there exist $(a_i)_{1\leq i\leq n}$ with $a_i \geq 0$ and $(\varphi_i)_{1\leq i\leq n}$ with $\varphi_i \in M$ such that $\sum_{i=1}^n a_i = 1$ and $\psi = \sum_{i=1}^n a_i\varphi_i$. It is clear that

$$\begin{aligned}\int_D |\psi(t)|\, dt &\leq \sum_{i=1}^n a_i \int_D |\varphi_i(t)|\, dt \leq \sum_{i=1}^n a_i \sup_{1\leq i\leq n} \int_D |\varphi_i(t)|\, dt \\ &= \sup_{1\leq i\leq n} \int_D |\varphi_i(t)|\, dt \leq \sup_{\varphi\in M} \int_D |\varphi(t)|\, dt,\end{aligned}$$

and therefore $\sup_{\psi\in\mathrm{co}(M)} \int_D |\psi(t)|\, dt \leq \sup_{\varphi\in M} \int_D |\varphi(t)|\, dt$. This shows that $\gamma_1(\mathrm{co}(M)) \leq \gamma_1(M)$. Moreover, for all $T > 0$, similar calculations give $\sup_{\psi\in\mathrm{co}(M)} \int_T^{+\infty} |\psi(t)|\, dt \leq \sup_{\varphi\in M} \int_T^{+\infty} |\varphi(t)|\, dt$. Letting T tend to $+\infty$, we get $\gamma_2(\mathrm{co}(M)) \leq \gamma_2(M)$.

Finally, we have

$$\gamma(\mathrm{co}(M)) \leq \gamma(M).$$

(7) It is clear that $\gamma(M) \leq \gamma(\overline{M}^w)$ (because $M \subset \overline{M}^w$). Conversely, using (1) and the inclusion $M \subset \overline{M}^w \subset \overline{\mathrm{co}}(M)$, we infer that $\gamma(\overline{M}^w) \leq \gamma(\overline{\mathrm{co}}(M))$. So, we just have to check that $\gamma(M) = \gamma(\overline{M})$. By (1), we already have the inequality $\gamma(M) \leq \gamma(\overline{M})$. Further, for $\psi \in \overline{M}$, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of points of M such that $\lim_{n\to+\infty} \psi_n = \psi$. Hence

$$\int_D |\psi(t)|\,dt \leq \int_D |\psi(t) - \psi_n(t)|\,dt + \int_D |\psi_n(t)|\,dt \leq \|\psi - \psi_n\|_X + \gamma_1(M).$$

It is clear that, for n large enough, $\|\psi - \psi_n\|_X$ is arbitrarily small. Hence we deduce that $\gamma_1(\overline{M}) \leq \gamma_1(M)$. Similar reasoning shows that $\gamma_2(\overline{M}) \leq \gamma_2(M)$ and so $\gamma(\overline{M}) = \gamma(M)$. Next, using (6) gives $\gamma(\overline{\mathrm{co}}(M)) = \gamma(\mathrm{co}(M)) = \gamma(M)$. Then, $\gamma(M) \leq \gamma(\overline{M}^w) \leq \gamma(M)$, which completes the proof of (7). ☐

Remark 7.7.3. The properties above show that $\gamma(\cdot)$ is a regular measure of weak noncompactness. So, Proposition 7.7.1 holds true for $\gamma(\cdot)$.

7.8 Darbo–Sadovskii-type fixed point theorems

The purpose of this section is to present some results in the spirit of Darbo and Sadovskii fixed point theorems for the weak topology in nonreflexive Banach spaces.

Definition 7.8.1. Let X be a Banach space, K a nonempty subset of X, $\mu(\cdot)$ a measure of weak noncompactness on X, and f a map from K into X such that the set $f(\mathbf{B}(X) \cap 2^K) \subseteq \mathbf{B}(X)$.

(a) We say that f is μ-k-contractive if there exists $k \in [0,1)$ such that

$$\mu(f(A)) \leq k\mu(A), \quad \forall A \in \mathbf{B}(X) \cap 2^K.$$

(b) We say that f is μ-nonexpansive if

$$\mu(f(A)) \leq \mu(A), \quad \forall A \in \mathbf{B}(X) \cap 2^K.$$

(c) We say that f is μ-condensing if

$$\mu(f(A)) < \mu(A), \quad \forall A \in (\mathbf{B}(X) \cap 2^K) \backslash \ker \mu.$$

Proposition 7.8.1. *Let X be a Banach space, K a nonempty, closed, convex element of $\mathbf{B}(X)$, $\mu(\cdot)$ a measure of weak noncompactness on X, and $f : K \to K$ a weakly continuous map. Then*

(a) *if f is μ-k-contractive for some $k \in [0,1)$, then* $\mathrm{Fix}(f) \neq \emptyset$,

(b) *if f is μ-condensing, then* $\mathrm{Fix}(f) \neq \emptyset$.

We note that assertion (a) (resp. (b)) is nothing else but Darbo (resp. Sadovskii) fixed point theorem for the weak topology. The proof of (a) (resp. (b)) may be modeled in the same way to that of Dardo (resp. Sadovskii) fixed point theorem – it suffices to replace the strong continuity by the weak continuity and the measure of noncompactness by the measure of weak noncompactness and apply Tychonoff fixed point theorem instead of Schauder fixed point theorem. Obviously, we take into account that any bounded subset is weakly bounded (cf. Proposition 1.7.2) and any closed convex subset is weakly closed (cf. Theorem 1.7.2).

Assuming that f is weakly sequentially continuous, we get the following result.

Theorem 7.8.1. *Let X be a Banach space, K a nonempty, closed, convex element of $\mathbf{B}(X)$, $\mu(\cdot)$ a measure of weak noncompactness on X, and $f : K \to K$ a weakly sequentially continuous map. Then*
(a) *if f is μ-k-contractive for some $k \in [0,1)$, then* $\mathrm{Fix}(f) \neq \emptyset$,
(b) *if f is μ-condensing, then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof. (a) Define a sequence $(K_n)_{n\in\mathbb{N}}$ of decreasing nonempty, closed, convex subsets of K by $K_0 = K$ and $K_{n+1} = \overline{\mathrm{co}}(f(K_n))$, $n = 1,2,\dots$ Proceeding as in the proof of Darbo theorem (Theorem 7.3.1), we see that, for all $n \in \mathbb{N}$, we have $K_n \subseteq K_{n-1}$ and $\mu(K_n) \leq k\mu(K_{n-1}) \leq \cdots \leq k^n\mu(K_0)$. Since $k \in [0,1[$ and K_0 is bounded, we have $\lim_{n\to+\infty}\mu(K_n) = 0$ and, by Proposition 7.7.1, we infer that $\bigcap_{n\geq 0} K_n = K_\infty$ is a nonempty, convex, weakly compact subset of X. Further, for all $n \geq 1$, we have $f(K_n) \subseteq f(K_{n-1}) \subseteq \overline{\mathrm{co}}(f(K_{n-1})) = K_n$, hence we conclude that $f(K_\infty) \subseteq K_\infty$. Since $f_{|K_\infty}$ is weakly sequentially continuous, applying Theorem 7.5.2 to the mapping $f_{|K_\infty} : K_\infty \to K_\infty$ gives the desired result.

(b) Fix an element k in K, define the set

$$\Pi := \{C \subset K : C \text{ is closed, convex, } k \in C, \text{ and } f(C) \subseteq C\}$$

and set

$$B = \bigcap_{C\in\Pi} C \quad \text{and} \quad D = \overline{\mathrm{co}}\{f(B) \cup \{k\}\}.$$

Clearly, D is a closed, convex subset of K containing k. Since $k \in B$ and $f : B \to B$, we deduce that $D \subset B$, and therefore $f(D) \subseteq f(B) \subseteq D$. Now, since $k \in D$, we conclude that $D \in \Pi$ and then $B \subseteq D$, consequently, $D = B$. Further, using the properties of $\mu(\cdot)$, we see that $\mu(D) = \mu(\overline{\mathrm{co}}\{f(B) \cup \{k\}\}) = \mu(f(B)) = \mu(f(D)) < \mu(D)$, which is a contradiction. Hence $\mu(D) = 0$ and then $B = D$ is weakly compact. Since B is weakly compact and $f : B \to B$ is weakly sequentially continuous, the use of Theorem 7.5.2 gives $\mathrm{Fix}(f) \neq \emptyset$. □

For ws-compact mappings, we have the following result.

Theorem 7.8.2. *Let X be a Banach space, K a nonempty, closed, bounded, convex subset of X, $f : K \to K$ a ws-compact mapping, and $\mu(\cdot)$ a measure of weak noncompactness on X. Then*

(a) *if f is μ-k-contractive for some $k \in [0,1)$, then* $\mathrm{Fix}(f) \neq \emptyset$,

(b) *if f is μ-condensing, then* $\mathrm{Fix}(f) \neq \emptyset$.

Proof. (a) If f is μ-k-contractive, proceeding as in the proof of Theorem 7.8.1(a), we prove that there exists a convex, weakly compact, and f-invariant subset K_∞ of K. To conclude the proof of (a), it suffices to apply Theorem 7.6.1 to $f : K_\infty \to K_\infty$.

(b) Let $k_0 \in K$ and define the set

$$\mathcal{K} = \{A \subset K : f(A) \subseteq A, k_0 \in A, A \text{ is closed and convex}\}.$$

Reasoning as in the proof of Theorem 7.8.1(b), we show that the set

$$B := \bigcap_{A \in \mathcal{K}} A = \mathrm{co}\big(f(B) \cup \{k_0\}\big)$$

is convex and belongs to $\mathcal{K}$. Moreover, we have

$$\mu(B) = \mu\big(\overline{\mathrm{co}}\{f(B) \cup \{k_0\}\}\big) = \mu\big(f(B)\big) < \mu(B),$$

and therefore B is a convex, weakly compact, and f-invariant subset of X. The result follows from Theorem 7.6.1. ☐

Assertion (b) of Theorem 7.8.2 was obtained for bounded, closed, convex subsets of X. For unbounded convex subsets of X, we get the following result.

Theorem 7.8.3. *Let K be an unbounded, closed, convex subset of a Banach space X, $\mu(\cdot)$ a measure of weak noncompactness on X, and $f : K \to K$ a ws-compact, μ-condensing map. If $f(K)$ is bounded, then* $\mathrm{Fix}(f) \neq \emptyset$.

The proof of this result requires the following lemma established in [124, p. 636].

Lemma 7.8.1. *Let X be a topological space and $f : X \to X$ a multimap with closed graph. If there exists a subset A of X such that $\overline{A}$ is compact and $f(A) \subseteq A$, then there exists a nonempty compact subset K of X such that $K \subset f(K)$.*

Proof of Theorem 7.8.3. Let $\zeta \in K$ and $A = \{f^n(\zeta), n = 0, 1, 2, \dots\}$ where $f^0(\zeta) = \zeta$. It is clear that $A = f(A) \cup \{\zeta\}$ and therefore $\mu(f(A)) = \mu(A)$. Since f is μ-condensing, we get $\mu(A) = 0$ and thus $\overline{A}^w$ is weakly compact. Using the fact that f is ws-compact, we infer that $f(A)$ is relatively compact. Since $f(f(A)) \subseteq f(A)$, according to Lemma 7.8.1, one can choose a compact set $A_0 \subset f(A_0)$ such that $A_0 \subseteq \overline{\mathrm{co}}(f(A_0))$. Define the set

$$\mathcal{K} = \{C \text{ such that } A_0 \subseteq C, \overline{\mathrm{co}}(C) = C, f(C) \subseteq C\}.$$

It is obvious that $\mathcal{K} \neq \emptyset$ $(K \in \mathcal{K})$. If Π is a chain in the ordered set $(\mathcal{K}, \subset)$, then $\bigcap_{C\in\Pi} C$ is a lower bound of Π (for the inclusion). Hence, by Zorn lemma (see, for example, [89, p. 6] or [46, p. 2]), $\mathcal{K}$ has a minimal element S. Since S is a closed convex subset of X satisfying $f(S) \subseteq S$, we conclude that $\overline{\text{co}}(f(S)) \subseteq S$. Accordingly,

$$f(\overline{\text{co}}(f(S))) \subseteq f(S) \subseteq \overline{\text{co}}(f(S)).$$

Using the inclusion $A_0 \subseteq \overline{\text{co}}(f(S))$, we conclude that $\overline{\text{co}}(f(S)) \in \mathcal{K}$. Because S is a minimal element of $\mathcal{K}$, we have $S = \overline{\text{co}}(f(S))$ and therefore $\mu(S) = \mu(\overline{\text{co}}(f(S))) = \mu(f(S))$. This implies that $\mu(S) = \mu(f(S)) = 0$ because f is μ-condensing, and then $f(S)$ is relatively weakly compact. Now using the fact that f is ws-compact and applying Theorem 7.6.1, we conclude that f has a fixed point belonging to $S \subseteq K$. □

Proposition 7.8.2. *Let X be a Banach space and let $f : X \to X$ be a ww-compact map.*
(a) *If f is k-Lipschitz, then $\omega(f(M)) \leq k\omega(M), \forall M \in \mathbf{B}(X)$.*
(b) *If f is a Φ-contraction with Φ-function ϕ, then f is ω-condensing.*

For the definition of Φ-contractions, we refer to Section 6.6.

Proof. (*a*) Let $M \in \mathbf{B}(X)$ and $r > \omega(M)$. There exist $r_0 \in [0, r[$ and a weakly compact subset K of X such that $M \subseteq K + \mathbb{B}_{r_0}$. Let $x \in M$. Then there is a $y \in K$ such that $\|x - y\| \leq r_0$. Using the fact that f is k-Lipschitz, we get $\|f(x) - f(y)\| \leq k\|x - y\| \leq kr_0$. This shows that

$$f(M) \subseteq f(K) + \mathbb{B}_{kr_0} \subseteq \overline{f(K)}^w + \mathbb{B}_{kr_0}.$$

Since f is ww-compact, Theorem 1.7.3 implies that $\overline{f(K)}^w$ is weakly compact and therefore we have $\omega(f(M)) \leq kr_0 \leq kr$. Letting $r \to \omega(M)$, we obtain $\omega(f(M)) \leq k\omega(M)$.

(*b*) Since f is a Φ-contraction mapping, it is clear that f is continuous. Thus, we only have to prove that if $M \in \mathbf{B}(X)$ such that $\omega(M) > 0$, then $\omega(f(M)) < \omega(M)$. To see this, let us consider $M \in \mathbf{B}(X)$ and an arbitrary real number $\varepsilon > 0$ such that

$$\omega(M) = \inf\{r > 0 : M \subseteq W + \mathbb{B}_r, W \in \mathbf{W}(X)\} > 0.$$

There exist $r_\varepsilon = \omega(M) + \varepsilon$ and $K \in \mathbf{W}(X)$ such that $M \subseteq K + \mathbb{B}_{r_\varepsilon}$. If $y \in f(K + \mathbb{B}_{r_\varepsilon})$, then there exists $x \in K + \mathbb{B}_{r_\varepsilon}$ such that $y = f(x)$. Since $x \in K + \mathbb{B}_{r_\varepsilon}$, there are $w \in K$ and $b \in \mathbb{B}_{r_\varepsilon}$ such that $x = w + b$. Hence,

$$\|y - f(w)\| = \|f(x) - f(w)\| \leq \phi(\|x - w\|) = \phi(\|b\|) \leq \sup_{0\leq t\leq r_\varepsilon} \phi(t).$$

Because ϕ is continuous, there exists $t_\varepsilon \in [0, r_\varepsilon]$ such that $\|y - f(w)\| \leq \sup_{0\leq t\leq r_\varepsilon} \phi(t) = \phi(t_\varepsilon)$, which implies $y \in f(K) + \mathbb{B}_{\phi(t_\varepsilon)}$. Consequently, $f(K + \mathbb{B}_{r_\varepsilon}) \subseteq f(K) + \mathbb{B}_{\phi(t_\varepsilon)}$. Since $M \subseteq K + \mathbb{B}_{r_\varepsilon}$, we have

$$f(M) \subseteq f(K + \mathbb{B}_{r_\varepsilon}) \subseteq f(K) + \mathbb{B}_{\phi(t_\varepsilon)}.$$

Using the fact that f is ww-compact, one sees that $\overline{f(K)}^{w} \in \mathbf{W}(X)$, and therefore

$$\omega(f(M)) = \inf\{r > 0 : f(M) \subseteq K + \mathbb{B}_r, K \in \mathbf{W}(X)\} \leq \phi(t_\varepsilon).$$

If there exists $\varepsilon > 0$ such that $\phi(t_\varepsilon) < \omega(M)$, we have reached to the conclusion. Otherwise, for every $\varepsilon > 0$ we have $\phi(t_\varepsilon) \geq \omega(M)$. In this case, the properties of ϕ yield that $\omega(M) \leq \phi(t_\varepsilon) < t_\varepsilon \leq \omega(M) + \varepsilon$. Consequently, $\phi(t_\varepsilon), t_\varepsilon \to \omega(M)$ as $\varepsilon \to 0$. Bearing in mind that ϕ is continuous, we get

$$\omega(M) = \lim_{\varepsilon \to 0} \phi(t_\varepsilon) = \phi(\omega(M)).$$

Hence $\omega(M) = 0$, which is a contradiction. ☐

Remark 7.8.1. It is clear that Proposition 7.8.2(b) remains true if g is a nonlinear contraction.

Let X be a Banach space. According to Theorem 1.7.1, every bounded linear operator T on X is weakly sequentially continuous. Hence the use of Remark 7.6.1(3), along with Proposition 7.8.2(a), implies the following result useful in applications.

Corollary 7.8.1. Let X be a Banach space. If T is a bounded linear operator on X, then

$$\omega(T(M)) \leq \|T\|\omega(M), \quad \text{for all } M \in \mathbf{B}(X), \tag{7.22}$$

where $\|T\|$ denotes the operator norm of T.

Theorem 7.8.4. *Let X be a Banach, K a nonempty, closed, convex subset of X, $\mu(\cdot)$ a measure of weak noncompactness on X, and $f : K \to K$ a mapping. Suppose*

(a) *f is ws-compact,*
(b) *$\mu(f(C)) < \mu(C)$ whenever C is a bounded subset of K such that $\overline{C}^{w} \notin \mathbf{W}(X)$,*
(c) *there exist $R > 0$ and $x_0 \in K$ such that $f(x) - x_0 \neq \lambda(x - x_0)$ for every $\lambda > 1$ and $x \in K \cap \partial\mathbb{B}_R(x_0)$.*

Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. Consider the set $\mathbb{B}_R^K(x_0) := \mathbb{B}_R(x_0) \cap K$. It is clear that $\mathbb{B}_R^K(x_0)$ is a nonempty, bounded, closed, and convex subset of K. Define the function ζ on K by

$$\zeta(x) = \begin{cases} x & \text{if } \|x - x_0\| \leq R, \\ \frac{R}{\|x-x_0\|}x + (1 - \frac{R}{\|x-x_0\|})x_0 & \text{if } \|x - x_0\| > R. \end{cases}$$

It is clear that ζ is a retraction of K on $\mathbb{B}_R^K(x_0)$. Let $f_\zeta : \mathbb{B}_R^K(x_0) \to \mathbb{B}_R^K(x_0)$ be the map defined by $f_\zeta(x) = \zeta(f(x))$. Because f and ζ are continuous, f_ζ is continuous, too. Furthermore, since f is ws-compact, f_ζ is also ws-compact. On the other hand, by hypothesis, if $C \subset \mathbb{B}_R^K(x_0)$ is such that $\overline{C}^w \notin \mathbf{W}(X)$, then $\mu(f(C)) < \mu(C)$.

The map f_ζ is μ-condensing. Indeed, if $x \in f(C) \subseteq f(\mathbb{B}_R^K(x_0))$, then there are two possibilities:

(i) if $\|x - x_0\| \leq R$, then $\zeta(x) = x \in f(C) \subseteq \operatorname{co}(f(C) \cup \{x_0\})$,
(ii) if $\|x - x_0\| > R$, then $\zeta(x) = \frac{R}{\|x-x_0\|}x + (1 - \frac{R}{\|x-x_0\|})x_0 \in \operatorname{co}(f(C) \cup \{x_0\})$.

Assertions (i) and (ii) imply the inclusion $f_\zeta(C) \subseteq \operatorname{co}(f(C) \cup \{x_0\})$. Now, by using the properties of f and μ, we obtain

$$\mu(f_\zeta(C)) \leq \mu(f(C)) < \mu(C),$$

which proves our claim.

Let $\mathcal{K}$ be the set defined by

$$\mathcal{K} := \{M \subseteq \mathbb{B}_R^K(x_0) : M \text{ is closed convex, } x_0 \in M, \text{ and } f_\zeta(M) \subseteq M\}.$$

It is clear that

$$B := \bigcap_{D \in \mathcal{K}} D \in \mathcal{K}.$$

Set

$$S = \overline{\operatorname{co}}\{f_\zeta(B) \cup \{x_0\}\}.$$

Since $x_0 \in B$ and $f_\zeta : B \to B$, we have $S \subseteq B$, and then $f_\zeta(S) \subseteq f_\zeta(B) \subseteq S$. Moreover, because $x_0 \in S$, we see that $S \in \mathcal{K}$, and then $S = B$. Accordingly,

$$\mu(S) = \mu(\overline{\operatorname{co}}\{f_\zeta(B) \cup \{x_0\}\}) = \mu(f_\zeta(B)) = \mu(f_\zeta(S)) < \mu(S),$$

which is a contradiction. Hence, $\mu(S) = 0$ and therefore S is a relatively weakly compact subset of X.

The above arguments show that $f_\zeta : S \to S$ satisfies the conditions of Theorem 7.6.1 and thus there exists $y_0 \in S$ such that $f_\zeta(y_0) = y_0$.

If $f(y_0) \in \mathbb{B}_R^K(x_0)$, then $y_0 = f_\zeta(y_0) = \zeta(f(y_0)) = f(y_0)$.

If $f(y_0) \notin \mathbb{B}_R^K(x_0)$, then $y_0 = f_\zeta(y_0) = \zeta(f(y_0)) = \frac{R}{\|f(y_0)-x_0\|}f(y_0) + (1 - \frac{R}{\|f(y_0)-x_0\|})x_0$.

Accordingly $\|x_0 - y_0\| = R$ and, if we take $\lambda = \frac{\|f(y_0)-x_0\|}{R} > 1$, then $f(y_0) - x_0 = \lambda(y_0 - x_0)$, which contradicts (c).

Hence, in any case, we have $f(y_0) = y_0$, which completes the proof. □

The next result gives conditions under which we can guarantee the existence of a bounded almost fixed point sequence of a mapping f defined on an unbounded set.

Theorem 7.8.5. *Let K be a nonempty, closed, convex subset of a Banach space X and let $f : K \to K$ be a mapping such that*

(a) *f is ws-compact,*

(b) *f is ω-nonexpansive,*

(c) *there exist $R > 0$ and $x_0 \in K$ such that, for all $x \in K \cap \partial\mathbb{B}_R(x_0)$ and $\lambda > 1$, we have $f(x) - x_0 \neq \lambda(x - x_0)$.*

Then there exists a bounded almost fixed point sequence $(x_n)_{n\in\mathbb{N}}$ of f. Furthermore, if $(I - f) : K \to X$ is ϕ-expansive, then f has a unique fixed point $x \in K$ and $x_n \to x$ as $n \to +\infty$..

Proof. Let $\mathbb{B}_R^K(x_0) = \{x \in K : \|x - x_0\| \leq R\}$ and let $\rho : K \to \mathbb{B}_R^K(x_0)$ be the mapping defined by

$$\rho(x) = \begin{cases} x & \text{if } \|x - x_0\| \leq R, \\ \frac{R}{\|x-x_0\|}x + (1 - \frac{R}{\|x-x_0\|})x_0 & \text{if } \|x - x_0\| > R. \end{cases}$$

The set $\mathbb{B}_R^K(x_0) \subset K$ is nonempty, closed, bounded, convex, and the mapping ρ is a continuous retraction of K on $\mathbb{B}_R^K(x_0)$. For each integer $n \geq 2$, we define the mapping $f_n : K \to K$ by

$$f_n(x) = \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)f(x).$$

It is clear that f_n is a ws-compact and ω-$(1 - \frac{1}{n})$-contractive, for any $n \geq 2$.

Now we define the maps $f_{n,\rho} : \mathbb{B}_R^K(x_0) \to \mathbb{B}_R^K(x_0)$ as $f_{n,\rho}(x) = \rho(f_n(x))$ with $n \geq 2$. The mapping $f_{n,\rho}$ is ws-compact. Moreover, $f_{n,\rho}$ is ω-$(1 - \frac{1}{n})$- contractive because f_n is ω-$(1 - \frac{1}{n})$-contractive and ρ is ω-nonexpansive. So, applying Theorem 7.8.4 we conclude that $f_{n,\rho}$ has a fixed point, say $f_{n,\rho}(x_n) = x_n$.

We shall check that $f_n(x_n) = x_n$. In order to prove this, we will prove that $\|f_n(x_n) - x_0\| \leq R$. Assume for a contradiction that $\|f_n(x_n) - x_0\| > R$. Hence

$$x_n = \rho(f_n(x_n)) = \frac{R}{\|f_n(x_n) - x_0\|}f_n(x_n) + \left(1 - \frac{R}{\|f_n(x_n) - x_0\|}\right)x_0,$$

and thus

$$\frac{R}{\|f_n(x_n) - x_0\|}(f_n(x_n) - x_0) = x_n - x_0.$$

Consequently, $x_n \in K \cap \partial\mathbb{B}_R^K(x_0)$. We also have that

$$f(x_n) - x_0 = \frac{n}{n-1}(f_n(x_n) - x_0) = \lambda_n(x_n - x_0)$$

where $\lambda_n = \frac{n}{n-1}\frac{\|f_n(x_n)-x_0\|}{R} > 1$, which contradicts condition (b). Hence we have

$$\|f_n(x_n) - x_0\| \le R,$$

and therefore

$$x_n = \rho(f_n(x_n)) = f_n(x_n).$$

Next we shall prove that $(x_n)_{n\in\mathbb{N}}$ is an almost fixed point sequence for f. First, we note that

$$\|x_0 - f(x_n)\| \le \|x_0 - f_n(x_n)\| + \|f_n(x_n) - f(x_n)\| \le R + \frac{1}{n}\|x_0 - f(x_n)\|,$$

and thus $\|x_0 - f(x_n)\| \le \frac{n}{n-1}R$. Therefore, by the following inequality:

$$\|x_n - f(x_n)\| = \frac{1}{n}\|x_0 - f(x_n)\| \le \frac{R}{n-1},$$

we obtain the boundedness of the almost fixed point sequence $(x_n)_{n\in\mathbb{N}}$ of f. Finally, if $I - f$ is ϕ-expansive, then, invoking Remark 6.6.2 (see also Corollary 6.6.2), we see that f has a unique fixed point $x \in K$ and $x_n \to x$. ☐

Remark 7.8.2. Note that, according to Lemma 6.6.3, the last assertion of Theorem 7.8.5 remains valid if we replace the hypothesis that $(I - f)$ is ϕ-expansive by the condition that $(I - f)^{-1} : R(I - T) \to K$ exists and it is uniformly continuous.

The next example illustrates the importance of assuming the ϕ-expansiveness or the uniform continuity of $(I - f)^{-1}$ in Theorems 7.8.5.

Example 7.8.1. Let X be the Banach space ℓ_1 endowed with its natural norm. Consider the mapping $f : \mathbb{B}_1 \to \mathbb{B}_1$ given by

$$f(x) = \left(1 - \sum_{i=1}^{\infty} |x_i|, x_1, x_2, x_3, \ldots\right)$$

for each $x = (x_i)_{i\in\mathbb{N}^*} \in \mathbb{B}_{\ell_1}$. The map f is a ws-compact, ω-nonexpansive, $I - f : \mathbb{B}_{\ell_1} \to \ell_1$ is injective, yet f does not have fixed points. Indeed, notice that $f = g + h$ where $g : \mathbb{B}_1 \to \ell_1$ is given by $g(x) = (1 - \sum_{i=1}^{\infty} |x_i|)e_1$ and $h : \mathbb{B}_1 \to \mathbb{B}_1$ is defined as $h(x) = \sum_{i=1}^{\infty} x_i e_{i+1}$. Clearly, g is compact and h is a ω-nonexpansive linear mapping, in particular h is ω-nonexpansive. Thus, f is also ω-nonexpansive. Clearly, f is continuous, in fact, for every $x = (x_i)_{i\in\mathbb{N}^*}$ and $y = (y_i)_{i\in\mathbb{N}^*}$ in $\mathbb{B}_1$,

$$\|f(x) - f(y)\|_{\ell_1} = \left|\sum_{i=1}^{\infty}(|y_i| - |x_i|)\right| + \sum_{i=1}^{\infty} |x_i - y_i| \le 2\|x - y\|_{\ell_1}.$$

Since ℓ_1 is a Schur space (or use Corollary 14 in [89, p. 296]) and f is continuous, we have that f is ws-compact.

Now we will prove that $I - f$ is injective. Let $x = (x_i)_{i\in\mathbb{N}^*}$ and $y = (y_i)_{i\in\mathbb{N}^*}$ in $\mathbb{B}_1$ be such that $(I - f)(x) = (I - f)(y)$. We have that

$$\left(x_1 - y_1 + \sum_{i=1}^{\infty}(|x_i| - |y_i|), x_2 - y_2 - (x_1 - y_1), x_3 - y_3 - (x_2 - y_2), \dots\right) = (0, 0, 0, \dots),$$

and then, for every positive integer i, we have $x_{i+1} - y_{i+1} = x_i - y_i$. Hence $x_i - y_i = 0$, that is, $x = y$. Finally, if $f(x) = x$ for some $x = (x_i) \in \mathbb{B}_1$, we have

$$\left(1 - \sum_{i=1}^{\infty}|x_i|, x_1, x_2, x_3, \dots\right) = (x_1, x_2, x_3, \dots),$$

and then $1 - \sum_{i=1}^{\infty}|x_i| = x_1$ and $x_i = x_{i+1}$ for every i. Hence $x_i = 0$ for every i and, consequently,

$$1 = 1 - \sum_{i=1}^{\infty}|x_i| = x_1 = 0,$$

which is not possible.

We close this section with the following result concerning mappings on WC-Banach algebras.

Theorem 7.8.6. *Let K be a nonempty, closed, bounded, convex subset of a WC-Banach algebra X and let $f, g : K \to X$ be two weakly sequentially continuous mappings such that $f(K)$ and $g(K)$ are bounded. Suppose that the mapping $h := fg$ (the product of f and g) is weakly sequentially continuous and maps K into itself. If, for any bounded subset S of K, we have $\omega(f(S)) \le k_1\omega(S)$ and $\omega(g(S)) \le k_2\omega(S)$, then the mapping h satisfies*

$$\omega(h(S)) \le k_1\|f(K)\| + k_2\|g(K)\| + k_1k_2\omega(K).$$

Here, for $C \subset X$, by $\|C\|$ we denote the real number $\sup\{\|x\| : x \in C\}$.

Moreover, if $k := k_1\|f(K)\| + k_2\|g(K)\| + k_1k_2\omega(S) < 1$, then h is ω-k-contractive and has at least one fixed point in K.

Proof. Let S be an arbitrary subset of K. According to the assumptions and Proposition 7.7.6(b), we have

$$\begin{aligned}
\omega(h(S)) &\le \omega(f(S)g(S)) \\
&\le \|f(S)\|\omega(g(S)) + \|g(S)\|\omega(f(S)) + \omega(f(S))\omega(g(S)) \\
&\le \|f(S)\|k_1\omega(S) + \|g(S)\|k_2\omega(S) + k_1k_2\omega(S)^2 \\
&\le [k_1\|f(K)\| + k_2\|g(K)\| + k_1k_2\omega(K)]\omega(S) \\
&\le k\omega(S),
\end{aligned}$$

where $k := k_1\|f(K)\| + k_2\|g(K)\| + k_1k_2\omega(K)$.

If $k < 1$, then h is ω-k-contractive. On the other hand, h transforms K into itself and it is weakly sequentially continuous on K. According to Theorem 7.8.1(a), the operator h has at least one fixed point in the set K, which completes the proof. ☐

7.9 A Schaefer-type fixed point theorem

In this section we present a result of Schaefer type for the weak topology.

Theorem 7.9.1. *Let X be a Banach space and let $f : X \to X$ be a ws-compact map. If f is ω-condensing, then*
(a) *either* $\mathrm{Fix}(f) \neq \emptyset$, *or*
(b) *the set of all solutions of $x = \lambda f(x)$, for $\lambda \in (0,1)$, is unbounded.*

Proof. Let $R > 0$ and denote by ζ the radial retraction of X onto $\mathbb{B}_R$. Since ζ is continuous, one can define the function $J_\zeta : \mathbb{B}_R \to \mathbb{B}_R$ by $J_\zeta(x) = \zeta(f(x))$. It is clear that J_ζ is continuous. Since f is ws-compact, the operator J_ζ is also ws-compact. Further, we have $J_\zeta(\mathbb{B}_R) = \zeta(f(\mathbb{B}_R))$. Hence, for every $x \in \mathbb{B}_R$, we have two possibilities: either $\|f(x)\| \leq R$ or $\|f(x)\| > R$.

- If $\|f(x)\| \leq R$, then $\zeta(f(x)) = f(x) \in f(\mathbb{B}_R) \subseteq \mathrm{co}(f(\mathbb{B}_R) \cup \{0\})$.
- If $\|f(x)\| > R$, then $\zeta(f(x)) = \frac{R}{\|f(x)\|} f(x) = \frac{R}{\|f(x)\|} f(x) + (1 - \frac{R}{\|f(x)\|})0$ is an element of $\mathrm{co}(f(\mathbb{B}_R) \cup \{0\})$.

This proves $J_\zeta(\mathbb{B}_R) \subseteq \mathrm{co}(f(\mathbb{B}_R) \cup \{0\})$. Using the properties of ω and the fact that f is ω-condensing, we obtain $\omega(J_\zeta(\mathbb{B}_R)) \leq \omega(f(\mathbb{B}_R) \cup \{0\}) < \omega(\mathbb{B}_R)$, hence J_ζ is ω-condensing. By Theorem 7.8.3, there exists $x_0 \in \mathbb{B}_r$ such that $x_0 = J_\zeta(x_0)$. Consequently, either $f(x_0) \in \mathbb{B}_R$ or $f(x_0) \notin \mathbb{B}_R$.

- If $f(x_0) \in \mathbb{B}_R$, then $x_0 = J_\zeta(x_0) = \zeta(f(x_0)) = f(x_0)$ and hence f has a fixed point.
- If $f(x_0) \notin \mathbb{B}_R$, then $x_0 = J_\zeta(x_0) = \zeta(f(x_0)) = \frac{R}{\|f(x_0)\|} f(x_0)$ and therefore x_0 is a solution of the equation $x = \lambda f(x)$ where $\lambda = \frac{R}{\|f(x_0)\|} \in (0,1)$ and $\|x_0\| = R$.

Accordingly, either for some $R > 0$ we obtain a solution $f(x) = x$, or for each $R > 0$ we obtain an eigenvector of norm R for some eigenvalue in $(0,1)$. In the second case, the set of eigenvectors is unbounded because R may be arbitrary large. ☐

Corollary 7.9.1. *Let X be a Banach space and $f : X \to X$ a ws-compact map. If, for each $M \in \mathbf{B}(X)$, $f(M)$ is relatively weakly compact, then*
(a) *either* $\mathrm{Fix}(f) \neq \emptyset$, *or*
(b) *the set of all solutions of $x = \lambda f(x)$, for $\lambda \in (0,1)$, is unbounded.*

Proof. This result is a consequence of Theorem 7.9.1 because operators which transform elements of $\mathbf{B}(X)$ into elements of $\mathbf{RW}(X)$ are ω-condensing. ☐

7.10 Convex-power condensing mappings

Let X be a Banach space, K a nonempty closed convex subset of X, $f : K \to K$ a mapping, and $x_0 \in K$. For any $M \subset K$, we set

$$\begin{aligned} &f^{(1,x_0)}(M) = f(M), \\ &f^{(n,x_0)}(M) = f\big(\overline{\text{co}}(f^{(n-1,x_0)}(M) \cup \{x_0\})\big), \quad \text{for } n = 2, 3, \ldots \end{aligned}$$

Definition 7.10.1. Let X be a Banach space, K a nonempty, closed, convex subset of X and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $f : K \to K$ be a bounded mapping (that is, it maps bounded sets into bounded ones), $x_0 \in K$, and n_0 a nonnegative integer. We say that f is μ-convex-power condensing about x_0 and n_0 if, for each bounded set $M \subset K$ with $\mu(M) > 0$, we have

$$\mu(f^{(n_0,x_0)}(M)) < \mu(M). \qquad \square$$

It is obvious that any μ-condensing map $f : K \to K$ is μ-convex-power condensing about any $x_0 \in K$ and 1.

Remark 7.10.1. The concept of convex-power condensing mappings was introduced in [224] using the Kuratowski measure of noncompactness.

Theorem 7.10.1. *Let X be a Banach space and $\mu(\cdot)$ a regular measure of weak noncompactness on X. Let K be a nonempty, closed, convex subset of X, $x_0 \in K$, and n_0 a nonnegative integer. Let $f : K \to K$ be a μ-convex-power condensing map about x_0 and n_0. If f is weakly sequentially continuous and $f(K)$ is bounded, then* $\text{Fix}(f) \neq \emptyset$.

Proof. Define the set

$$\mathcal{K} = \{A \subset K,\ \overline{\text{co}}(A) = A, x_0 \in A,\ \text{and } f(A) \subset A\}.$$

The set $\mathcal{K}$ is nonempty because $K \in \mathcal{K}$.

Set

$$B = \bigcap_{A \in \mathcal{K}} A.$$

Now we shall show that, for any nonnegative integer n, we have

$$(\mathcal{P}(n)) \quad B = \overline{\text{co}}(f^{(n,x_0)}(B) \cup \{x_0\}).$$

To see this, we proceed by induction. Clearly, B is a closed, convex subset of K and $f(B) \subset B$. Thus, $B \in \mathcal{K}$. This implies that $\overline{\text{co}}(f(B) \cup \{x_0\}) \subset B$, and therefore

$$f\big(\overline{\text{co}}(f(B) \cup \{x_0\})\big) \subset f(B) \subset \overline{\text{co}}(f(B) \cup \{x_0\}).$$

Accordingly, $\overline{\mathrm{co}}(f(B) \cup \{x_0\}) \in \mathcal{K}$. Hence $B \subset \overline{\mathrm{co}}(f(B) \cup \{x_0\})$. So we conclude that $B = \overline{\mathrm{co}}(f(B) \cup \{x_0\})$. This shows that $\mathcal{P}(1)$ holds.

Let n be fixed nonnegative integer, and suppose $\mathcal{P}(n)$ holds. Composing by f, we obtain

$$f^{(n+1,x_0)}(B) = f(f^{(n,x_0)}(B) \cup \{x_0\}) = f(B),$$

and therefore

$$\overline{\mathrm{co}}(f^{(n+1,x_0)}(B) \cup \{x_0\}) = \overline{\mathrm{co}}(f(B) \cup \{x_0\}) = B,$$

which means that

$$\overline{\mathrm{co}}(f^{(n,x_0)}(B) \cup \{x_0\}) = B.$$

Using the properties of $\mu(\cdot)$, we obtain

$$\mu(B) = \mu(\overline{\mathrm{co}}(f^{(n_0,x_0)}(B) \cup \{x_0\})) = \mu(f^{(n_0,x_0)}(B)) < \mu(B),$$

because f is μ-convex-power condensing. This is a contradiction. Hence $\mu(B) = 0$, and therefore B is relatively weakly compact. Since $f : B \to B$ is weakly sequentially continuous, the result follows from Theorem 7.5.2. □

As an easy consequence of Theorem 7.10.1 we have

Corollary 7.10.1. *Let X be a Banach space and $\mu(\cdot)$ a regular measure of weak noncompactness on X. Let K be a nonempty, closed, convex subset of X. Assume $f : K \to K$ is a sequentially weakly continuous map with $F(K)$ bounded. If f is μ-condensing, then f has a fixed point.*

We give the following result which we shall use later.

Proposition 7.10.1. *Let X be a Banach space and let $\mu(\cdot)$ be a regular measure of weak noncompactness on X. Let $f : X \to X$ be a μ-convex-power condensing map about x_0 and n_0. Let $g : X \to X$ be the map defined on X by $g(x) = f(x + x_0) - x_0$. Then, g is μ-convex-power condensing about 0 and n_0 with respect to μ. Moreover, f has a fixed point if g does.*

Proof. Let $S \in \mathbf{B}(X)$ be a bounded subset of X such that $\mu(S) > 0$.

We claim that, for all integers $n \geq 1$,

$$g^{(n,0)}(S) \subset f^{(n,x_0)}(S + x_0) - x_0. \tag{7.23}$$

To see this, we shall proceed by induction. Since $g(S) = f(S + x_0) - x_0$, we have

$$g^{(1,0)}(S) = g(S) = f(S + x_0) - x_0 = f^{(1,x_0)}(S + x_0) - x_0.$$

Let $n \geq 1$ be an integer and assume that for all $p \in \{1, 2, \ldots, n-1\}$, we have

$$g^{(p,0)}(S) \subset f^{(p,x_0)}(S + x_0) - x_0.$$

Hence

$$g^{(n-1,0)}(S) \subset f^{(n-1,x_0)}(S + x_0) - x_0,$$

and then

$$g^{(n-1,0)}(S) \cup \{0\} \subset \overline{\mathrm{co}}\{f^{(n-1,x_0)}(S + x_0), x_0\} - x_0.$$

This yields $\overline{\mathrm{co}}\{g^{(n-1,x_0)}(S), 0\} \subset \overline{\mathrm{co}}\{f^{(n-1,x_0)}(S + x_0), x_0\} - x_0$. Accordingly,

$$\begin{aligned} g^{(n,x_0)}(S) &= g(\overline{\mathrm{co}}(g^{(n-1,0)}(S) \cup \{0\})) \\ &\subset g(\overline{\mathrm{co}}(f^{(n-1,x_0)}(S + x_0) \cup \{x_0\} - x_0)) \\ &\subset f(\overline{\mathrm{co}}(f^{(n-1,x_0)}(S + x_0) \cup \{x_0\} - x_0)) \\ &= f^{(n,x_0)}(S + x_0) - x_0. \end{aligned}$$

This proves our claim. Consequently,

$$\begin{aligned} \mu(g^{(n,0)}(S)) &= \mu(f^{(n,x_0)}(S + x_0) - x_0) \\ &= \mu(f^{(n,x_0)}(S + x_0)) \\ &< \mu(S + x_0) \leq \mu(S). \end{aligned}$$

This establishes the first statement. The second statement is straightforward. □

7.11 Leray–Schauder-type fixed point theorems

The object of this section is to present some Leray–Schauder-type fixed point theorems involving the weak topology.

Theorem 7.11.1. *Let K be a nonempty, closed, convex subset of a Banach space X, and let $U \subseteq K$ be weakly open relative to K, and $p \in U$. Suppose $\overline{U}^w$ is a weakly compact subset of K and $f : \overline{U}^w \to K$ is a weakly sequentially continuous map. Then*

(a) *either* $\mathrm{Fix}(f) \neq \emptyset$, *or*

(b) *there are a point $u \in \partial_K U$ (the weak boundary of U in K) and $\lambda \in (0, 1)$ such that $u = \lambda f(u) + (1 - \lambda)p$.*

In this theorem we suppose that U is weakly open relative to K, which means that U is open for the topology induced on K by the weak topology of X. In other words, there exists a weakly open subset O of X such that $U = O \cap K$.

Proof. Suppose assertion (b) does not hold and f does not possess fixed points in $\partial_K U$ (otherwise (a) will be satisfied). Let A be the set defined by

$$A = \{x \in \overline{U}^w : x = tf(x) + (1-t)p \text{ for some } t \in [0,1]\}.$$

We note that $A \neq \emptyset$ because $p \in A$ (take $t = 0$) and $A \subseteq \overline{U}^w$. The subset A is weakly compact. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of A such that $x_n \rightharpoonup x$ (it is clear that $x \in \overline{U}^w$). By definition of A, for each $n \in \mathbb{N}$, there exists $t_n \in [0,1]$ such that $x_n = t_n f(x_n) + (1-t_n)p$. Since $(t_n)_{n\in\mathbb{N}}$ is contained in the compact set $[0,1]$, there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} t_{n_k} = t \in [0,1]$. Using the fact that $f : \overline{U}^w \to K$ is weakly sequentially continuous, we get $x_{n_k} = t_{n_k} f(x_{n_k}) + (1-t_{n_k})p \rightharpoonup tf(x) + (1-t)p$, and then $x \in A$. Thus A is weakly sequentially closed.

Let $x \in \overline{U}^w$ be adherent to A for the weak topology. By Theorem 1.7.3, $\overline{A}^w$ is weakly compact, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of A such that $x_n \rightharpoonup x$, hence $x \in A$. We infer that A is weakly closed. Since $\overline{U}^w$ is weakly compact, we conclude that A is weakly compact.

Since $(X, \sigma(X, X^*))$ is a Hausdorff locally convex topological vector space, it is completely regular and assertion (b) is not satisfied, so we have $A \cap (K\backslash U) = \emptyset$. By Theorem 1.1.2, there exists a weakly continuous map $\zeta : K \to [0,1]$ with $\zeta(x) = 1$ if $x \in A$ and $\zeta(x) = 0$ if $x \in K\backslash U$. Now define the map

$$P(x) = \begin{cases} \zeta(x)f(x) + (1-\zeta(x))p, & x \in \overline{U}^w, \\ p, & x \in K\backslash\overline{U}^w. \end{cases}$$

Since $\partial_K U = \partial_K \overline{U}^w$, it follows from the weak continuity of ζ and the sequential continuity of f that P is weakly sequentially continuous. Further, we have $P(K) \subseteq \overline{\text{co}}(f(\overline{U}^w) \cup \{p\})$.

Set $H := \overline{\text{co}}(f(\overline{U}^w) \cup \{p\})$. Using the sequential continuity of f and Theorems 1.7.3 and 1.7.4, we see that H is a convex, weakly compact subset of X and $P(H) \subseteq H$. It follows from Proposition 7.5.2 that the function $P : H \to H$ is weakly continuous. According to Corollary 7.5.2, P has a fixed point $z \in K$. If $z \notin U$, then we have $\zeta(z) = 0$ and then $z = p$, which contradicts the fact that $p \in U$. This implies that $z \in U$ and $z = \zeta(z)f(z) + (1-\zeta(z))p$, proving that $z \in A$. Thus we have $\zeta(z) = 1$ and then $f(z) = z$. This completes the proof. ☐

The next result is a variant of Theorem 7.11.1 in the case where K is bounded, the assumption that $\overline{U}^w$ *is relatively weakly compact* will be replaced by the requirement that the *map f is μ-condensing.*

Theorem 7.11.2. *Let K be a nonempty, bounded, closed, convex subset of a Banach space X and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $U \subseteq K$ be a weakly open subset relative to K, $p \in U$ and let $f : \overline{U}^w \to K$ be a weakly sequentially continuous and μ-condensing map. Then*

(a) *either* Fix$(f) \neq \emptyset$, *or*
(b) *there exist* $u \in \partial_K U$ *and* $\lambda \in \,]0,1[$ *such that* $u = \lambda f(u) + (1-\lambda)p$ *where* $\partial_K U$ *is the weak boundary of* U *in* $\overline{K}^w$.

Proof. Assume assertion (b) is false and A does not have a fixed point in $\partial_K U$ (otherwise we have nothing to prove). Define the set

$$A = \{x \in \overline{U}^w : x = tf(x) + (1-t)p \text{ for some } t \in [0,1]\}.$$

It is clear that $A \neq \emptyset$ is nonempty and bounded because $p \in A$ and $A \subset \overline{U}^\omega$. Since $A \subseteq \mathrm{co}(f(A) \cup \{0\})$ and f is μ-condensing, we have

$$\mu(A) \leq \mu(\overline{\mathrm{co}}(\{0\} \cup f(A))) = \mu(f(A)) < \mu(A),$$

which is a contradiction. Hence $\mu(A) = 0$, and therefore A is relatively weakly compact. Proceeding as in the proof of Theorem 7.11.1, one sees that A is weakly closed and then it is weakly compact. As in the proof of Theorem 7.11.1, we may define a map P by

$$P(x) = \begin{cases} \zeta(x)f(x) + (1-\zeta(x))p, & x \in \overline{U}^w, \\ p, & x \in K \backslash \overline{U}^w, \end{cases}$$

where $\zeta : K \to [0,1]$ is a weakly continuous function such that $\zeta(x) = 1$ if $x \in A$ and $\zeta(x) = 0$ if $x \in K \backslash U$. It is checked in the proof of Theorem 7.11.1 that P is weakly sequentially continuous. Let V be a subset of K. By definition of P, we have $P(V) \subset \mathrm{co}(f(V) \cup \{p\})$, and therefore

$$\mu(P(V)) \leq \mu(\mathrm{co}(f(V) \cup \{p\})) \leq \mu(f(V)) < \mu(V).$$

Thus P is μ-condensing. It follows from Theorem 7.8.1(b) that there exists $z \in K$ such that $P(z) = z$. If $z \notin U$, we have $\zeta(z) = 0$ and then $z = p$, which is a contradiction because $p \in U$. This yields $z \in U$ and then $z = \zeta(z)f(z) + (1-\zeta(z))p$, which proves that $z \in A$. This implies that $\zeta(z) = 1$ and then $f(z) = z$, completing the proof. ☐

Remark 7.11.1. It is obvious that Theorem 7.11.2 holds true for μ-k-contractive mappings because μ-k-contractive maps are μ-condensing.

Theorem 7.11.3. *Let* K *be a nonempty, closed, convex subset of a Banach space* X, $U \subseteq K$ *an open subset and* $p \in U$. *Let* $f : \overline{U} \to K$ *be a ws-compact map. If* $f(\overline{U})$ *is relatively weakly compact, then*
(a) *either* Fix$(f) \neq \emptyset$, *or*
(b) *there exist an element* $x \in \partial_K U$ *(the boundary of* U *in* K*) and* $\lambda \in (0,1)$ *such that* $x = \lambda f(x) + (1-\lambda)p$.

Proof. As in the proofs of the preceding two theorems, we suppose that (b) is not satisfied and f has no fixed point on ∂U, otherwise we have nothing to do. Hence we have

$$x \neq \lambda f(x) + (1-\lambda)p \text{ for all } x \in \partial U, \quad \text{for some } \lambda \in [0,1].$$

Since $p \in U$, the set

$$A = \{x \in \overline{U} : x = tf(x) + (1-t)p \text{ for some } t \in [0,1]\}$$

is nonempty because $p \in A$. Since $A \cap \partial U = \emptyset$, the continuity of f implies that A is closed. By Theorem 1.1.2, there exists a continuous function $\zeta : \overline{U} \to [0,1]$ with $\zeta(A) = \{1\}$ and $\zeta(\partial U) = \{0\}$. Proceeding as in the proof of Theorem 7.11.1, we may define the mapping P by

$$P(x) = \begin{cases} \zeta(x)f(x) + (1-\zeta(x))p, & x \in \partial U, \\ p, & x \in K\backslash\overline{U}. \end{cases} \tag{7.24}$$

It is clear that P is continuous. The use of Theorem 7.6.1 completes the proof, so it suffices to check that P is ws-compact and $P(K)$ is relatively weakly compact. To see this, let $(x_n)_{n\in\mathbb{N}}$ be a weakly converging sequence in K. Depending on whether or not $(x_n)_{n\in\mathbb{N}}$ lies in $\overline{U}$ for n large enough, we distinguish two cases:

(a) There exists some $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ ($n \geq n_0 \implies x_n \in \overline{U}$). In this case, the sequence $(x_n)_{n\in\mathbb{N}}$ lies in $\overline{U}$ and converges weakly in U. Since f is ws-compact, the sequence $(f(x_n))_{n\geq n_0}$ has a strongly convergent subsequence $(f(x_{n_k}))_{k\geq 0}$ such that $f(x_{n_k}) \to y$ in K. Since $[0,1]$ is compact, we can extract from $(\zeta(x_{n_k}))_{k\geq 0}$ a convergent subsequence $(\zeta(x_{n_{k_r}}))_{r\geq 0}$. We note that the sequence $(\zeta(x_{n_{k_r}}))_{r\geq 0}$ satisfies $P(x_{n_{k_r}}) = (\zeta(x_{n_{k_r}})f(x_{n_r})) + (1-\zeta(x_{n_{k_r}}))p$. Thus letting r go to $+\infty$, we get $ty + (1-t)p \in K$.

(b) If $(x_n)_{n\in\mathbb{N}}$ is such that for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $x_{n_k} \notin \overline{U}$, then we may consider a subsequence $(x_{n_k})_{k\geq 0}$ of $K\backslash\overline{U}$ such that $P(x_{n_k}) = p \to p$ in K.

It follows from (a) and (b) that P is ws-compact. To check that $P(K)$ is relatively weakly compact, we use the fact that $f(\overline{U})$ is relatively weakly compact, together with an argument similar to that used to prove that P is ws-compact. Under these conditions, Theorem 7.6.1 guarantees the existence of some $z \in K$ such that $P(z) = z$. Since $p \in U$, we have $z \in U$, and therefore

$$z = \zeta(z)f(z) + (1-\zeta(z))p.$$

This proves that $z \in A$, that is, $\zeta(z) = 1$ and thus $z = f(z)$. □

Theorem 7.11.4. *Let X be a Banach space, and let $\mu(\cdot)$ be a regular measure of weak noncompactness on X. Let Q and C be closed, convex subsets of X with $Q \subset C$. In addition, let U be a weakly open subset relative to Q with $x_0 \in U$ and such that U is weakly open in C.*

Suppose $f : X \to X$ *is a weakly sequentially continuous and a* μ*-convex-power condensing map about* x_0 *and* n_0 *(*n_0 *is a nonnegative integer). If, moreover,* $f(\overline{U}^w)$ *is bounded and* $f(\overline{U}^w) \subset C$*, then*

(a) *either* $\mathrm{Fix}(f) \neq \emptyset$*, or*

(b) *there exist an element* $x \in \partial_Q U$ *and* $\lambda \in (0,1)$ *such that* $x = \lambda f(x)$*.*

Proof. Let us first note that if we replace Q, C, U and f by $Q - x_0$, $C - x_0$, $U - x_0$, and $\tilde{f}$ (where $\tilde{f}(x) = f(x + x_0)$), respectively, then, using Proposition 7.10.1, we may assume that $0 \in U$ and f is μ-convex-power condensing about 0 and n_0. Suppose (b) does not hold and f does not have a fixed point on $\partial_Q U$ (otherwise we have nothing to do since (a) is true). Let

$$A = \{x \in \overline{U}^w : x = tf(x) \text{ for some } t \in [0,1]\}.$$

It is clear that A is nonempty because $0 \in U$.

The set A is weakly sequentially closed. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of A which converges weakly to some $x \in \overline{U}^w$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence of $[0,1]$ such that $x_n = t_n f(x_n)$. By passing to a subsequence, if necessary, we may assume that $(t_n)_{n\in\mathbb{N}}$ converges to some $t \in [0,1]$. Since f is weakly sequentially continuous, $f(x_n) \rightharpoonup f(x)$ and therefore $t_n f(x_n) \rightharpoonup tf(x)$. Hence, $x = tf(x)$, which proves that $x \in A$. Thus, A is weakly sequentially closed.

The set A is relatively weakly compact. Indeed, suppose $\mu(A) > 0$. Clearly,

$$A \subset \mathrm{co}(f(A) \cup \{0\}). \tag{7.25}$$

Arguing by induction, we see that, for each integer $n \geq 1$, we have

$$A \subset \mathrm{co}(f^{(n,0)}(A) \cup \{0\}). \tag{7.26}$$

Indeed, fix an integer $n \geq 1$ and suppose that (7.26) holds true. Then

$$f(A) \subset f(\mathrm{co}(f^{(n,0)}(A) \cup \{0\})) = f^{(n+1,0)}(A) \tag{7.27}$$

and therefore

$$\mathrm{co}(f(A) \cup \{0\}) \subset \mathrm{co}(f(f^{(n+1,0)}(A) \cup \{0\})). \tag{7.28}$$

Combining (7.25) and (7.28), we obtain

$$A \subset \mathrm{co}(f^{(n+1,0)}(A) \cup \{0\}), \tag{7.29}$$

and then (7.26) is proved. In particular, we have $A \subset \mathrm{co}(f^{(n_0,0)}(A) \cup \{0\})$. Thus

$$\mu(A) \leq \mu(\mathrm{co}(f^{(n_0,0)}(A) \cup \{0\})) = \mu(f(A)) < \mu(A),$$

which is a contradiction. Hence, $\mu(A) = 0$, and therefore $\overline{A}^w$ is weakly compact. This proves our claim.

Now let $x \in \overline{A}^w$. Since $\overline{A}^w$ is weakly compact, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A which converges weakly to x. Since A is weakly sequentially closed, we have $x \in A$. Thus, $\overline{A}^w = A$. Hence, A is weakly closed and therefore weakly compact. From our assumptions, we have $A \cap \partial Q_U = \emptyset$. Because X endowed with the weak topology is a locally convex Hausdorff topological space, there exists a weakly continuous mapping $\zeta : \overline{U}^w \to [0,1]$ with $\zeta(A) = \{1\}$ and $\zeta(\partial_Q U) = \{0\}$. Let

$$P(x) = \begin{cases} \zeta(x)f(x), & x \in \overline{U}^w, \\ 0, & x \in X \backslash \overline{U}^w. \end{cases}$$

Since f is weakly sequentially continuous, P is also weakly sequentially continuous. Further, if S is a subset of C, then $P(S) \subset \operatorname{co}(P(S) \cup \{0\})$, and therefore,

$$\begin{aligned} P^{(2,0)}(S) &= P(\overline{\operatorname{co}}(P(S) \cup \{0\})) \subseteq P(\overline{\operatorname{co}}(f(S) \cup \{0\})) \\ &\subseteq \overline{\operatorname{co}}(f(\overline{\operatorname{co}}(f(S) \cup \{0\}) \cup \{0\})) \\ &= \overline{\operatorname{co}}(f^{(2,0)}(S) \cup \{0\}). \end{aligned}$$

Proceeding by induction, for each nonnegative integer n, we have

$$\begin{aligned} P^{(n,0)}(S) &= P(\overline{\operatorname{co}}(P^{(n-1,0)}(S) \cup \{0\})) \subseteq P(\overline{\operatorname{co}}(f^{(n-1,0)}(S) \cup \{0\})) \\ &\subseteq \overline{\operatorname{co}}(f(\overline{\operatorname{co}}(f^{(n-1,0)}(S) \cup \{0\}) \cup \{0\})) \\ &= \overline{\operatorname{co}}(f^{(n,0)}(S) \cup \{0\}). \end{aligned}$$

Next, if $\mu(S) > 0$, then the properties of $\mu(\cdot)$ imply

$$\mu(P^{(n_0,0)}(S)) \leq \mu(\overline{\operatorname{co}}(f^{(n_0,0)}(S) \cup \{0\})) = \mu(f^{(n_0,0)}(S) \cup \{0\}) < \mu(S).$$

Thus, $P : X \to X$ is weakly sequentially continuous, $P(C) \subset C$, and P is μ-convex-power condensing about 0 and n_0. By Theorem 7.10.1, there exists $x \in C$ such that $P(x) = x$. Now $x \in U$ since $0 \in U$. Consequently, $x = \zeta(x)f(x)$ and so $x \in A$. This implies that $\zeta(x) = 1$ and so $f(x) = x$, which completes the proof. □

7.12 Fixed point theorems for a sum of mappings

7.12.1 Krasnosel'skii-type fixed point theorems

The next result, due to M. A. Krasnosel'skii [146], deals with the fixed points for a sum of a compact operator and a strict contraction. Krasnosel'skii's proof of this result is a combination of the Banach contraction principle and Schauder fixed point theorem.

This result is often used to discuss the solutions of perturbed nonlinear problems. M. A. Krasnosel'skii observed that a fixed point problem of the form

$$f(x) + g(x) = x \tag{7.30}$$

can be reduced to the problem $(I - g)(x) = f(x)$. If the operator $(I - g)$ is invertible with continuous inverse, then solving problem (7.30) is equivalent to solving the following one:

$$(I - g)^{-1}f(x) = x. \tag{7.31}$$

It follows from the compactness of f and the continuity of $(I - g)^{-1}$ that the operator $(I - g)^{-1} \circ f$ is compact. Since problems (7.30) and (7.31) are equivalent – to solve (7.30), it suffices to solve (7.31). This may be performed by using Corollary 7.2.3. M. A. Krasnosel'skii fixed point theorem is a simple prototype where the perturbation g is assumed to be a contraction. This result holds also true for more general perturbations. In this section we shall present some fixed point theorems of Krasnosel'skii type involving the weak topology and various kind of perturbations.

Originally, Krasnosel'skii fixed point theorem was stated in the following form.

Proposition 7.12.1 (Krasnosel'skii). *Let X be a Banach space and let K be a nonempty, closed, bounded, convex subset of X. Let f and g be two mappings from K into X. Suppose that*

(a) *f is compact,*
(b) *g is k-contractive for some $k \in [0, 1)$,*
(c) *$f(K) + g(K) \subseteq K$.*

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

The proof of this proposition is based on the next lemma and Schauder fixed point theorem (Theorem 7.2.3).

Remark 7.12.1. It should be noticed that, in Proposition 7.12.1, the hypothesis $f(K) + g(K) \subseteq K$ is very strong. In [244], Proposition 7.12.1 was established where the hypothesis (c) was replaced by the weaker hypothesis $(f + g)(K) \subseteq K$. Later, this hypothesis has been weakened by Burton [54] who replaced (c) by the condition

$$\big(x = g(x) + f(y), y \in K\big) \implies x \in K.$$

In the rest of this chapter, depending on the case, we shall use either of these three conditions.

Lemma 7.12.1. *Let X be a normed space and let K be a nonempty subset of X. If $g : K \to X$ is a k-contractive mapping for some $k \in [0, 1)$, then the mapping $(I - g)$ is a homeomorphism from K onto $(I - g)(K)$.*

Proof. We note that the operator $(I - g)$ is continuous. For $x, y \in X$, we have

$$\begin{aligned}\|(I-g)(x)-(I-g)(y)\| &\geq \|x-y\| - \|g(x)-g(y)\| \\ &\geq (1-k)\|x-y\|.\end{aligned}$$

Hence $(I - g)$ is one-to-one and then invertible with inverse $(I-g)^{-1} : (I-g)(K) \to K$. The continuity of $(I-g)^{-1}$ follows from the previous inequality. □

Theorem 7.12.1. *Let X be a Banach space and let K be a nonempty, closed, bounded, and convex subset of X. Let $f : K \to X$ and $g : X \to X$ be two weakly sequentially continuous mappings. Suppose*
(i) *$f(K)$ is relatively weakly compact,*
(ii) *g is k-contractive for some $k \in [0,1)$,*
(iii) $(x = g(x) + f(y), y \in K) \Longrightarrow x \in K$.

Then $\mathrm{Fix}(f+g) \neq \emptyset$.

Proof. Since g is k-contractive, it follows from Lemma 7.12.1 that $(I-g)$ is a homeomorphism from X onto $(I-g)(X)$. Let y be a fixed point in K. The map which assigns to each $x \in X$ the value $g(x)+f(y)$ defines a contractive map from X into X. Hence, by the Banach contraction principle, the equation $x = g(x) + f(y)$ has a unique solution $x \in X$. Using hypothesis (iii), we infer that $x \in K$. Hence $x = (I-g)^{-1}f(y) \in K$. This implies

$$(I-g)^{-1}f(K) \subset K. \tag{7.32}$$

The rest of the proof consists in showing that the map $(I-g)^{-1}f$ satisfies the hypotheses of Theorem 7.5.2. Let

$$C := \overline{\mathrm{co}}\big((I-g)^{-1}f(K)\big).$$

(a) The operator $(I-g)^{-1}f$ transforms C into itself. Indeed, since K is closed and convex, it follows from (7.32) that $C \subset K$ and then

$$(I-g)^{-1}f(C) \subset (I-g)^{-1}f(K) \subset \overline{\mathrm{co}}\big((I-g)^{-1}f(K)\big) = C.$$

(b) The subset C is weakly compact. To see this, suppose that C is not weakly compact (that is $\omega(C) > 0$). Before going further, we first prove the equality

$$(I-g)^{-1}f = f + g(I-g)^{-1}f. \tag{7.33}$$

Indeed, using the equality $I = I - g + g$, we obtain

$$(I-g)^{-1}f = (I-g+g)(I-g)^{-1}f = f + g(I-g)^{-1}f,$$

which proves (7.33).

Now the use of (7.33) and properties (1), (2), and (5) of Proposition 7.7.2 gives

$$\begin{aligned}\omega(C) &= \omega(\overline{\mathrm{co}}((I-g)^{-1}f(K))) = \omega((I-g)^{-1}f(K))\\ &\le \omega(f(K)) + \omega(g(I-g)^{-1}f(K)).\end{aligned}$$

Since $f(K)$ is relatively weakly compact, the preceding equation becomes

$$\omega(C) \le \omega(g(I-g)^{-1}f(K)). \tag{7.34}$$

Next, let $r > 0$ and let $W \in \mathbf{W}(X)$ be such that $(I-g)^{-1}f(K) \subset W + \mathbb{B}_r$. Since g is k-contractive, one may write

$$g(I-g)^{-1}f(K) \subset g(W) + \mathbb{B}_{kr} \subset \overline{g(W)}^{w} + \mathbb{B}_{kr}.$$

Because g is weakly sequentially continuous, we have that $g(W)$ is relatively weakly compact (use Proposition 7.5.2). Using the properties of $\omega(\cdot)$, we deduce that

$$\omega(g(I-g)^{-1}f(K)) \le k\omega((I-g)^{-1}f(K)). \tag{7.35}$$

Combining (7.34) and (7.35), we obtain

$$\omega(C) \le k\omega((I-g)^{-1}f(K)) = k\omega(C) < \omega(C),$$

which is a contradiction. Hence $\omega(C) = 0$, and therefore C is weakly compact.

(c) The operator $(I-g)^{-1}f : C \to C$ is weakly sequentially continuous. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in C which converges weakly to u. Since C is weakly compact and $(I-g)^{-1}f(C) \subset C$, we infer that $(I-g)^{-1}f(C)$ is relatively weakly compact. Thus, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that

$$(I-g)^{-1}f(u_{n_k}) \rightharpoonup \rho \quad \text{as } n \to +\infty.$$

Taking into account equation (7.33) and the weak sequential continuity of f and g, we get $\rho = g(\rho) + f(u)$ and therefore $\rho = (I-g)^{-1}f(u)$. This yields

$$(I-g)^{-1}f(u_{n_k}) \rightharpoonup (I-g)^{-1}f(u) \quad \text{as } n \to +\infty.$$

Now we shall show that $(I-g)^{-1}f(u_n) \rightharpoonup (I-g)^{-1}f(u)$. If it is not the case, then there exists a weak neighborhood N^w of $(I-g)^{-1}f(u)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that $(I-g)^{-1}f(u_{n_k}) \notin N^w$ for all $k \ge 1$. Evidently, the subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converges weakly to u. Arguing as above, we may extract a subsequence $(u_{n_{k_j}})_{j\in\mathbb{N}}$ of $(u_{n_k})_{k\in\mathbb{N}}$ such that $(I-g)^{-1}f(u_{n_{k_j}}) \rightharpoonup (I-g)^{-1}f(u)$. This is a contradiction because, for all $j \ge 1$, $(I-g)^{-1}f(u_{n_{k_j}}) \notin N^w$. Finally, we conclude that the map $(I-g)^{-1}f$ is weakly sequentially continuous.

To conclude the proof, it suffices to apply Theorem 7.5.2 to the map $(I-g)^{-1}f$ defined from C into C. □

The next result is a consequence of the previous theorem.

Corollary 7.12.1. *Let X be a Banach space, K a nonempty, closed, bounded, convex subset of X and $f : K \to X$ and $g : X \to X$ two weakly sequentially continuous mappings. Suppose*

(a) *$f(K)$ is relatively weakly compact,*
(b) *g is nonexpansive,*
(c) *if $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements of K such that $(I-g)(x_n) \rightharpoonup y$, then $(x_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence,*
(d) *if $\lambda \in (0,1)$ and $x = \lambda g(x) + f(y)$ where $y \in K$, then $x \in K$.*

Then $\mathrm{Fix}(f+g) \neq \emptyset$.

Proof. For each $\lambda \in (0,1)$, the operators f and λg satisfy the conditions of Theorem 7.12.1 and then $f+\lambda g$ has a fixed point in K, say x_λ. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of $(0,1)$ such that $\lambda_n \to 1$ as $n \to +\infty$ and let $(x_n)_{n\in\mathbb{N}}$ be the sequence of points of K such that

$$f(x_n) + \lambda_n g(x_n) = x_n. \tag{7.36}$$

Using hypothesis (a) (passing eventually to a subsequence), we may assume that $(f(x_n))_{n\in\mathbb{N}}$ converges weakly to some $y \in K$. Accordingly,

$$(I - \lambda_n g)(x_n) \rightharpoonup y.$$

Since $K \in \mathbf{B}(X)$, the set $\{x_n : n \in \mathbb{N}\}$ is bounded and then $\{g(x_n) : n \in \mathbb{N}\}$ is also bounded. Hence

$$\|(x_n - g(x_n)) - (x_n - \lambda_n g(x_n))\| = (1-\lambda_n)\|g(x_n)\| \to 0 \quad \text{as } n \to \infty.$$

This implies $x_n - g(x_n) \rightharpoonup y$. Next, by (c), the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges weakly to some $x \in K$. Finally, it follows from (7.36) and the weak sequential continuity of f and g that $f(x) + g(x) = x$. □

Note that Theorem 6.3.1 may be stated in the context of Banach spaces as follows.

Theorem 7.12.2. *Let X be a Banach space and let $f : X \to X$ be a mapping. If f is a nonlinear contraction, then* $\mathrm{Fix}(f) \neq \emptyset$.

The following elementary lemma is required below.

Lemma 7.12.2. *Let X be a normed space and let K be a nonempty subset of X. If $g : K \to X$ is a nonlinear contraction with Φ-function ϕ, then $(I-g)$ is a homeomorphism from K onto $(I-g)(K)$.*

Proof. Since g is a nonlinear contraction, there exists a continuous nondecreasing map $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\phi(t) < t$ for each $t > 0$ such that

$$\|g(x) - g(y)\| \leq \phi(\|x - y\|) \quad \text{for all } x, y \in K.$$

Letting x and y be two distinct points of K, we have

$$\begin{aligned}\|(I - g)x - (I - g)y\| &= \|(x - y) - (g(x) - g(y))\| \geq \|x - y\| - \|g(x) - g(y)\| \\ &\geq \|x - y\| - \phi(\|x - y\|) > 0.\end{aligned}$$

Thus $(I - g)$ is one-to-one on K and then $(I - g)^{-1}$ exists and transforms $(I - g)(K)$ into K. To show the continuity of $(I - g)^{-1}$, we suppose that there exist a point x and a sequence $(x_n)_{n\in\mathbb{N}}$ of points of K such that $(I - g)(x_n) \to (I - g)(x)$ and $\lim_{n\to+\infty} \sup_{k\geq n} \|x_k - x\| = a$. It follows from the inequality $\|(I - g)(x_n) - (I - g)(x)\| \geq \|x_n - x\| - \phi(\|x_n - x\|)$ that $0 \geq a - \phi(a)$. Since $\phi(a) < a$, we conclude that $a = 0$. This proves the continuity of the operator $(I - g)^{-1}$. ☐

Theorem 7.12.3. *Let X be a Banach space and let K be a nonempty, closed, convex, subset of X Let $f : K \to X$ and $g : X \to X$ be two weakly sequentially continuous mappings. We suppose*

(a) *$f(K)$ is relatively weakly compact,*
(b) *g is a nonlinear contraction with Φ-function ϕ,*
(c) $(x = g(x) + f(y), y \in K) \Longrightarrow x \in K$.

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

Proof. Let M be a bounded subset of K with $\omega(M) > 0$. Since $f(M)$ is relatively weakly compact, using the properties of ω, we obtain

$$\omega((f + g)(M)) \leq \omega(f(M)) + \omega(g(M)) = \omega(g(M)).$$

It follows from Proposition 7.8.2(b) that g is ω-condensing, and so $\omega((f + g)(M)) < \omega(M)$, which proves that $f + g$ is ω-condensing. Since $f + g$ is weakly sequentially continuous, to complete the proof, it suffices to apply Theorem 7.8.1(b). ☐

Theorem 7.12.4. *Let X be a Banach space and let K be a nonempty, closed, bounded, convex subset of X. Let $f : K \to X$ and $g : X \to X$ be two mappings. Suppose*

(a) *f is ws-compact and $f(K)$ is relatively weakly compact,*
(b) *g is ww-compact and k-contractive for some $k \in [0, 1)$,*
(c) $(x = g(x) + f(y), y \in K) \Longrightarrow x \in K$.

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

Remark 7.12.2. Some fixed point theorems for a sum of two mappings with the lack of compactness were established for weakly continuous maps (see, for example, [34]).

In general, the weak continuity is not easy to be verified. In Theorem 7.12.4 the weak continuity is not required.

Proof. Since g is k-contractive, $I - g$ is continuous. Arguing as in the proof of Theorem 7.12.1, we see that $(I-g)^{-1}f$ is continuous on K and

$$(I-g)^{-1}f(K) \subset K. \tag{7.37}$$

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of K defined by

$$K_0 = K \quad \text{and} \quad K_{n+1} = \overline{\text{co}}((I-g)^{-1}f(K_n)).$$

It is a decreasing sequence of nonempty, closed, convex subsets of K. Using (7.33), one sees that

$$(I-g)^{-1}f(K_n) \subseteq f(K_n) + g(I-g)^{-1}f(K_n) \subseteq f(K_n) + g(K_{n+1}).$$

Since $(K_n)_{n\in\mathbb{N}}$ is decreasing, we get

$$(I-g)^{-1}f(K_n) \subseteq f(K_n) + g(K_n)$$

and then

$$\omega((I-g)^{-1}f(K_n)) \le \omega(f(K_n)) + \omega(g(K_n)).$$

Because $f(K)$ is relatively weakly compact, we have $\omega(f(K_n)) = 0$. Thus

$$\omega((I-g)^{-1}f(K_n)) \le \omega(g(K_n)).$$

Let $r > 0$ and $Y \in \mathbf{W}(X)$ be such that $K_n \subseteq Y + \mathbb{B}_r$. Thus, we have

$$g(K_n) \subseteq g(Y) + \mathbb{B}_{kr} \subseteq \overline{g(Y)}^{w} + \mathbb{B}_{kr}.$$

Since g is ww-compact, it follows from Proposition 7.8.2(a) that $\omega(g(K_n)) \le k\omega(K_n)$. Hence $\omega(K_{n+1}) \le k\omega(K_n)$. By a finite induction over n, we obtain $\omega(K_{n+1}) \le k^n\omega(K)$ and therefore $\lim_{n\to+\infty}\omega(K_n) = 0$. Applying Proposition 7.7.5, we conclude that $K_\infty := \bigcap_{n=1}^{\infty} K_n$ is a nonempty, weakly compact, convex subset of K and $(I-g)^{-1}f(K_\infty) \subseteq K_\infty$. Consequently, $(I-g)^{-1}f(K_\infty)$ is relatively weakly compact. Further, the properties of f and the continuity of $(I-g)^{-1}$ imply that $(I-g)^{-1}f$ is ws-compact. Now the use of Theorem 7.6.1 to the mapping $(I-g)^{-1}f_{|K_\infty} : K_\infty \to K_\infty$ finishes the proof. ☐

Theorem 7.12.5. *Let X be a Banach space, K a nonempty, closed, bounded, convex subset of X and $\mu(\cdot)$ a regular measure of weak noncompactness on X. Let $f : K \to X$ and $g : K \to X$ be two mappings satisfying*

(a) *f is ws-compact,*
(b) *there exists $\gamma \in [0,1)$ such that $\mu(f(S) + g(S)) \leq \gamma\mu(S)$ for all $S \subseteq K$,*
(c) *g is k-contractive for some $k \in [0,1)$,*
(d) *$(x = g(x) + f(y), y \in K) \Longrightarrow x \in K$.*

Then Fix$(f + g) \neq \emptyset$.

Proof. Since g is contractive, using the same arguments as in the proof of Theorem 7.12.4, we prove that $(I-g)^{-1}f$ is well defined and continuous on K, and $(I-g)^{-1}f(K) \subseteq K$.

We define a sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of K by $K_0 = K$ and $K_{n+1} = \overline{\mathrm{co}}((I-g)^{-1}f(K_n))$. It is clear that $(K_n)_{n\in\mathbb{N}}$ is decreasing (in the sense of inclusion) and satisfies

$$(I-g)^{-1}f(K_n) \subseteq f(K_n) + g(K_n). \tag{7.38}$$

Using (7.38), together with the properties $\mu(\cdot)$, we obtain

$$\mu(K_{n+1}) = \mu(\overline{\mathrm{co}}((I-g)^{-1}f(K_n))) = \mu((I-g)^{-1}f(K_n)) \leq \mu(f(K_n) + g(K_n)).$$

Hypotheses (a) and (b) yield $\mu(K_{n+1}) \leq \gamma\mu(K_n)$. Proceeding by induction, for each $n \in \mathbb{N}$, we get $\mu(K_n) \leq \gamma^n\mu(K)$, and therefore conclude that $\lim_{n\to+\infty}\mu(K_n) = 0$. Applying Proposition 7.7.1, we infer that $K_\infty := \bigcap_{n=1}^{+\infty} K_n$ is a nonempty, convex, weakly compact subset of K. On the other hand, since $(I-g)^{-1}$ is continuous and f is ws-compact, $(I-g)^{-1}f$ is also ws-compact on K, and so it is ws-compact on K_∞. Now it suffices to apply Theorem 7.6.1 to the map $(I-g)^{-1}f : K_\infty \to K_\infty$. □

Theorem 7.12.6. *Let X be a Banach space and let K be a nonempty, closed, bounded, convex subset of X. Let f and g be two mappings from K into X such that*
(a) *f is ws-compact and $f(K)$ is relatively weakly compact,*
(b) *g is nonexpansive and ω-condensing,*
(c) *$I - g$ is ϕ-expansive,*
(d) *$f(K) + g(K) \subseteq K$.*

Then Fix$(f + g) \neq \emptyset$.

Proof. We first prove that the mapping $(I-g)^{-1}f$ is well defined on K and $(I-g)^{-1}f(K) \subseteq K$. To see this, let x and y be two distinct points of K. Since $I-g$ is ϕ-expansive, we have

$$\|(I-g)(x) - (I-g)(y)\| \geq \phi(\|x-y\|) > 0,$$

which shows that $(I-g)$ is one-to-one, and so it is invertible. Let $y \in K$ and define the map h_y on K by $h_y(x) = g(x) + f(y)$. It is clear that h_y is nonexpansive. Hypothesis (c) yields

$$\begin{aligned}\|(I-h_y)(x) - (I-h_y)(z)\| &= \|x - z + h_y(z) - h_y(x)\| \\ &= \|x - z + g(z) - g(x)\|\end{aligned}$$

$$\begin{aligned} &= \big\|(I-g)(x)-(I-g)(z)\big\| \\ &\geq \phi(\|x-z\|). \end{aligned}$$

Thus $I - h_y$ is ϕ-expansive. So, according to Proposition 6.6.1, h_y has a unique fixed point $x \in K$. Hence,

$$f(y) = (I-g)(x) \quad \text{or, again,} \quad D\big((I-g)^{-1}\big) = (I-g)(K) \subseteq f(K).$$

Consequently, the mapping $(I-g)^{-1}f : K \to K$ is well defined.

We claim that $(I-g)^{-1}f$ satisfies the hypotheses of Theorem 7.6.1. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of $(I-g)(K)$ such that $x_n \to x_0$. Put $y_n = (I-g)^{-1}(x_n)$ and $y_0 = (I-g)^{-1}(x_0)$. It is clear that $(I-g)y_n = x_n$ and $(I-g)y_0 = x_0$. Since $I-g$ is ϕ-expansive, one can write

$$\phi(\|y_n - y_0\|) \leq \big\|(I-g)(y_n)-(I-g)(y_0)\big\| = \|x_n - x_0\|,$$

and then

$$\lim_{n\to+\infty} \phi(\|y_n - y_0\|) \leq \lim_{n\to+\infty} \|x_n - x_0\| = 0. \tag{7.39}$$

If the sequence $(\|y_n - y_0\|)_{n\in\mathbb{N}}$ does not converge to 0, there exists a subsequence $(\|y_{n_s} - y_0\|)_{s\in\mathbb{N}}$ of $(\|y_n - y_0\|)_{n\in\mathbb{N}}$ such that $\|y_{n_s} - y_0\| \to r > 0$.

If ϕ is continuous, then we have

$$\lim_{s\to+\infty} \phi(\|y_{n_s} - y_0\|) = \phi(r) > 0.$$

Otherwise, ϕ must be nondecreasing and then

$$0 < \phi(r/2) \leq \lim_{s\to+\infty} \phi(\|y_{n_s} - y_0\|).$$

In the two cases, according to (7.39), we must have $\lim_{s\to+\infty} \phi(\|y_{n_s} - y_0\|) = 0$, which proves that $\|y_n - y_0\| \to 0$ as $n \to +\infty$. Hence

$$\big\|(I-g)^{-1}(x_n) - (I-g)^{-1}(x_0)\big\| \to 0,$$

which proves that $(I-g)^{-1}$ is continuous.

Now using the fact that f is ws-compact, together with continuity of $(I-g)^{-1}$, we conclude that the mapping $(I-g)^{-1}f$ is ws-compact as well.

We claim that the subset $(I-g)^{-1}f(K)$ is relatively weakly compact. We first note that $(I-g)^{-1}f(K)$ is bounded and $(I-g)^{-1}f(K) \subseteq K$. If $(I-g)^{-1}f(K)$ is not relatively weakly compact, then the use of (7.33) and the properties of ω give

$$\begin{aligned} \omega\big((I-g)^{-1}f(K)\big) &= \omega\big((f + g(I-g)^{-1}f)(K)\big) \\ &\leq \omega\big(f(K)\big) + \omega\big(g(I-g)^{-1}f(K)\big) \end{aligned}$$

$$\begin{aligned} &= \omega(g \circ (I-g)^{-1} f(K)) \\ &< \omega((I-g)^{-1} f(K)), \end{aligned}$$

which is a contradiction. This yields $\omega((I-g)^{-1} f(K)) = 0$, and therefore $(I-g)^{-1} f(K)$ is relatively weakly compact, as claimed.

To complete the proof, it suffices to apply Theorem 7.6.1 to the map $(I-g)^{-1} f$. ☐

Theorem 7.12.7. *Let X be a Banach space and let K be a nonempty, closed, bounded, convex subset of X. Let $f : K \to X$ and $g : X \to X$ be two continuous mappings such that*
(a) *f is ws-compact,*
(b) *g is pseudocontractive and $I - g$ is ϕ-expansive,*
(c) *$f + g$ is ω-condensing,*
(d) $(x = g(x) + g(y), y \in K) \implies x \in K$.

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

Proof. According to the proof of Theorem 7.12.6, the map $h := (I-g)^{-1} \circ f : K \to K$ is well defined and ws-compact (the proof uses hypothesis (a) and the fact that $I - g$ is ϕ-expansive). Let $z \in K$ and define the set

$$A := \{D \subset K : D \text{is closed convex}, z \in D, \text{and } h(D) \subset D\},$$

and put

$$B = \bigcap_{D \in A} D.$$

Let C be the subset of K defined by $C := \overline{\mathrm{co}}(h(B) \cup \{z\})$. Because $z \in B$ and $h(B) \subseteq B$, we infer that $C \subset B$, and therefore

$$h(C) \subset h(B) \subset C. \tag{7.40}$$

Since $z \in C$, we conclude that $C \in A$, and therefore $C = B$. On the other hand, by the properties of $\omega(\cdot)$, condition (c), (7.40), and relation (7.33), we obtain

$$\begin{aligned} \omega(C) &= \omega(\overline{\mathrm{co}}(h(B) \cup \{z\})) \\ &= \omega(h(B)) = \omega(h(C)) = \omega((I-g)^{-1} \circ f(C)) \\ &= \omega(f(C) + g \circ (I-g)^{-1} \circ f(C)) \le \omega(f(C) + g(C)) \\ &< \omega(C), \end{aligned}$$

which is a contradiction. This proves that $\omega(C) = 0$, and then $C \in \mathbf{RW}(X)$. Since C is weakly closed, it is weakly compact.

Now using (7.40), we see that h maps C into itself and $h(C)$ is relatively weakly compact. Since h is ws-compact, applying Theorem 7.6.1, we conclude that there exists z in C (thus in K) such that $z = h(z) = (I - g)^{-1} \circ f(z)$. This ends the proof. □

Corollary 7.12.2. *Let K be a nonempty, bounded, closed, convex subset of a Banach space X and let $f : K \to X$ and $g : X \to X$ be two continuous mappings. Suppose*

(a) *g is ω-k-contractive for some $k \in [0,1)$,*
(b) *g is pseudocontractive and $I - g$ is ϕ-expansive,*
(c) *f is ws-compact and ω-s-contractive for some $s \in [0, 1 - k)$,*
(d) *$(x = g(x) + g(y), y \in K) \implies x \in K$.*

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

Proof. According to Theorem 7.12.7, to prove this corollary, it suffices to check that $f+g$ is ω-condensing. Indeed, since g is ω-k-contractive and f is ω-s-contractive, for any subset S of K with $\omega(S) > 0$, we have

$$\begin{aligned}\omega(f(S) + g(S)) \le \omega(g(S)) + \omega(f(S)) &\le k\omega(S) + s\omega(S).\\ &< k\omega(S) + (1-k)\omega(S)\\ &= \omega(S),\end{aligned}$$

Consequently, $f + g$ is ω-condensing. □

Definition 7.12.1. Let X be a Banach space, K a nonempty subset of X, and f a mapping from K into X. Let $x \in K$. We say that f is demiclosed at x if, for each sequence $(x_n)_{n\in\mathbb{N}}$ of points of K such that $x_n \rightharpoonup x$ and $f(x_n) \to y$, one has $f(x) = y$. The operator f is said to be demiclosed on K if it is demiclosed at each point of K.

Theorem 7.12.8. *Let X be a Banach space and let K be a nonempty, closed, bounded, convex subset of X such that $0 \in K$. Let $f : K \to K$ and $g : X \to X$ be two continuous mappings. Suppose*

(a) *g is pseudocontractive,*
(b) *f is ws-compact,*
(c) *$f + g$ is ω-condensing,*
(d) *$(x = \lambda f(x) + g(y), y \in K, \lambda \in (0,1)) \implies x \in K$,*
(e) *$I - (f + g) : K \to X$ is demiclosed at* 0.

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

Proof. Let $n \in \mathbb{N}^*$ and consider the following mappings $f_n = (1 - \frac{1}{n})f : K \to K$ and $g_n = (1 - \frac{1}{n})g : X \to X$.

We claim that g_n is pseudocontractive.

Indeed, because g is pseudocontractive, the operator $(I - g)$ is accretive and thus there exists $j(x - y) \in J(x - y)$ (see Definition 3.2.2) such that

$$\langle (x-g(x))-(y-g(y)), j(x-y)\rangle \geq 0.$$

Hence

$$\begin{aligned}\langle (x-g_n(x))-(y-g_n(y)), j(x-y)\rangle &= \left\langle x-g(x)+\frac{1}{n}x-\left(y-g(y)+\frac{1}{n}y\right), j(x-y)\right\rangle\\ &= \langle (x-g(x))-(y-g(y)), j(x-y)\rangle + \frac{1}{n}\|x-y\|^2\\ &\geq \frac{1}{n}\|x-y\|^2.\end{aligned}$$

Consequently, g_n is pseudocontractive, which proves our claim.

On the other hand, the same inequality yields

$$\begin{aligned}\|(I-g_n)(x)-(I-g_n)(y)\|\|x-y\| &\geq \langle (x-g_n(x))-(y-g_n(y)), j(x-y)\rangle\\ &\geq \frac{1}{n}\|x-y\|^2\end{aligned}$$

and therefore

$$\|(I-g_n)(x)-(I-g_n)(y)\| \geq \frac{1}{n}\|x-y\|.$$

Thus, if we call $\psi(t)=\frac{t}{n}$, we obtain

$$\|(I-g_n)(x)-(I-g_n)(y)\| \geq \psi(\|x-y\|).$$

This proves that $(I-g_n)$ is ψ-expansive.

Next, consider an arbitrary subset S of K such that $\omega(S)>0$. Using the properties of $\omega(\cdot)$, we obtain

$$\begin{aligned}\omega(f_n(S)+g_n(S)) &= \omega\left(\left(1-\frac{1}{n}\right)f(S)+\left(1-\frac{1}{n}\right)g(S)\right)\\ &\leq \omega(\mathrm{co}((f(S)+g(S))\cup\{0\}))\\ &\leq \omega(f(S)+g(S))\\ &< \omega(S).\end{aligned}$$

Hence, f_n+g_n is ω-condensing. Thus, the mapping f_n+g_n satisfies the hypotheses of Theorem 7.12.7. So, for any integer $n\geq 2, f_n+g_n$ has a fixed point $x_n\in K$.

We claim that the sequence $(x_n)_{n\geq 1}$ is weakly convergent. Let us suppose that $(x_n)_{n\geq 1}$ is not weakly convergent, hence $\{x_n : n\geq 1\}$ is not relatively weakly compact. Using hypothesis (c),

$$\begin{aligned}\omega(\{x_n : n\in\mathbb{N}^*\}) &= \omega\left(\left\{\left(1-\frac{1}{n}\right)f(x_n)+\left(1-\frac{1}{n}\right)g(x_n), n\in\mathbb{N}^*\right\}\right)\\ &\leq \omega(\mathrm{co}(\{f(x_n)+g(x_n) : n\in\mathbb{N}^*\})\cup\{0\})\end{aligned}$$

$$
\begin{aligned}
&= \omega(\{f(x_n) + g(x_n) : n \in \mathbb{N}^*\}) \\
&< \omega(\{x_n : n \in \mathbb{N}^*\}).
\end{aligned}
$$

Hence, we reach a contradiction, and the claim is proved.

So, without loss of generality, we may assume that $x_n \rightharpoonup x \in K$. By definition of x_n, we have

$$x_n = \left(1 - \frac{1}{n}\right)(f(x_n) + g(x_n)),$$

and therefore

$$(f(x_n) + g(x_n)) = \frac{1}{1 - \frac{1}{n}} x_n, \quad \forall n \geq 2.$$

Because K is bounded, we infer that the set $\{x_n : n \geq 2\}$ is bounded, and therefore the set $\{f(x_n) + g(x_n) : n \geq 2\}$ is also bounded. Using once again the equality

$$x_n = \left(1 - \frac{1}{n}\right)(f(x_n) + g(x_n)) = f(x_n) + g(x_n) - \frac{1}{n}(f(x_n) + g(x_n)),$$

we obtain

$$\|(I - (f + g))(x_n)\| = \|x_n - (f(x_n) + g(x_n))\| = \frac{1}{n}\|f(x_n) + g(x_n)\|,$$

and therefore

$$\lim_{n \to +\infty} (I - (f + g))(x_n) = 0.$$

Since $I - (f + g)$ is demiclosed at 0, we have $(I - (f + g))(x) = 0$, that is, x is a fixed point of $f + g$. □

Definition 7.12.2. Let X be a Banach space and $f : D(f) \subseteq X \to X$ a mapping. We see that f is weakly–strongly continuous if, for every sequence $(x_n)_{n \in \mathbb{N}}$ of $D(f)$, the following property holds true:

$$(x_n \rightharpoonup x \text{ and } x \in D(f)) \implies f(x_n) \to f(x).$$

Theorem 7.12.9. *Let X be a Banach space and K a nonempty, closed, bounded, convex subset of X. Let $f : K \to X$ and $g : X \to X$ be two mappings such that*

(a) *f is weakly–strongly continuous and $f(K)$ is relatively weakly compact,*

(b) *g is nonexpansive and ww-compact,*

(c) *if $(x_n)_{n \in \mathbb{N}}$ is a sequence of K such that $((I - g)x_n)_{n \in \mathbb{N}}$ is weakly convergent, then the sequence $(x_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence,*

(d) *$I - g$ is demiclosed,*
(e) *$f(x) + g(y) \in K$, for all $x, y \in K$.*

Then $\mathrm{Fix}(f+g) \neq \emptyset$.

Proof. Suppose first that $0 \in K$. By (e), for each $\lambda \in (0,1)$ and $x, y \in K$, we have

$$\lambda f(x) + \lambda g(x) \in K. \tag{7.41}$$

Thus the mappings λf and λg satisfy the conditions of Theorem 7.12.4. Hence, for all $\lambda \in (0,1)$, there is an $x_\lambda \in K$ such that $\lambda f(x_\lambda) + \lambda g(x_\lambda) = x_\lambda$. Choose a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n \to 1$ as $n \to +\infty$ and consider the corresponding sequence $(x_n)_{n\in\mathbb{N}}$ of elements of K satisfying

$$\lambda_n f(x_n) + \lambda_n g(x_n) = x_n. \tag{7.42}$$

Using the fact that $f(K)$ is weakly compact and passing eventually to a subsequence, we may assume that $(f(x_n))_{n\in\mathbb{N}}$ converges weakly to some $y \in K$. Hence

$$(I - \lambda_n g)(x_n) \rightharpoonup y \quad \text{as } n \to +\infty.$$

The boundedness of K implies that $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence, then so is $(g(x_n))_{n\in\mathbb{N}}$. Accordingly,

$$\|(x_n - g(x_n)) - (x_n - \lambda_n g(x_n))\| = (1-\lambda_n)\|g(x_n)\| \to 0.$$

This implies that

$$x_n - g(x_n) \rightharpoonup y \quad \text{as } n \to +\infty.$$

By (c), the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges weakly to some $x \in K$. Because f is weakly–strongly continuous, $(f(x_{n_k}))_{k\in\mathbb{N}}$ converges strongly to $f(x)$. Hence we conclude that

$$(I - \lambda_{n_k} g)(x_{n_k}) \rightharpoonup f(x) \quad \text{as } k \to +\infty.$$

Arguing as above, we get $(I - g)(x_n) \rightharpoonup f(x)$. The demiclosedness of $I - g$ yields $f(x) + g(x) = x$.

To end the proof, it remains to consider the case $0 \notin K$. In such a case, let $x_0 \in K$ be fixed and consider the set $K_0 := \{x - x_0 : x \in K\}$. Let $f_0 : K_0 \to X$ and $g_0 : K_0 \to X$ be the mappings defined by $f(x - x_0) = f(x) + (1/2)x_0$ and $g(x - x_0) = g(x) + (1/2)x_0$. Applying the result of the first case to f and g, we conclude that there exists $x \in K$ such that $f_0(x - x_0) + g_0(x - x_0) = x - x_0$, that is, $f(x) + g(x) = x$. ☐

Remark 7.12.3.

(a) We note that in Theorem 7.12.9 no additional assumption on the Banach space X is required.

(b) If we suppose that X is reflexive, then the weak–strong continuity of f implies the compactness of f. Hence, under this hypothesis, assumption (c) in Theorem 7.12.9 is always satisfied.

(c) If X is assumed to be uniformly convex, then, according to F. E. Browder [53], the operator $(I - g)$ is demiclosed.

In the light of Remark 7.12.3, the following corollaries are consequences of Theorem 7.12.9.

Corollary 7.12.3. *Let K be a nonempty, bounded, closed, convex subset of a reflexive Banach space X. Let $f : K \to X$ and $g : K \to X$ be two maps such that*

(a) *f is weakly–strongly continuous,*

(b) *g is nonexpansive and $(I - g)$ is demiclosed,*

(c) *$f(x) + g(y) \in K$, for all $x, y \in K$.*

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

Corollary 7.12.4. *Let K be a nonempty, bounded, closed, convex subset of a uniformly convex Banach space X. Let $f : K \to X$ and $g : K \to X$ be two maps such that*

(a) *f is weakly–strongly continuous,*

(b) *g is nonexpansive,*

(c) *$f(x) + g(y) \in K$, for all $x, y \in K$.*

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

Theorem 7.12.10. *Let X be a Banach space and K a nonempty, bounded, closed, convex subset of X such that $0 \in K$. Let $f : K \to K$ and $g : X \to X$ be two maps such that*

(a) *$f(K)$ is relatively weakly compact,*

(b) *f is ws-compact,*

(c) *g is nonexpansive and ω-condensing,*

(d) *the operator $I - (f + g) : K \to X$ is demiclosed at 0,*

(e) *$f(K) + g(K) \subseteq K$.*

Then $\mathrm{Fix}(f + g) \neq \emptyset$.

Remark 7.12.4. We note that condition (a) in Theorem 7.12.10 does not necessarily yield that f is weakly completely continuous, that is, f is weakly continuous and maps bounded subsets into relatively weakly compact ones. To see this, consider the classical Banach space $(\ell_2, \|\cdot\|_2)$ and define the map $f : \ell_2 \to \ell_2$ by

$$f(x) = \begin{cases} \frac{x}{\|x\|_2} & \text{if } \|x\|_2 \geq 1, \\ x & \text{if } \|x\|_2 \leq 1. \end{cases}$$

It is clear that f is continuous and maps bounded sets into relatively weakly compact sets ($f(X) \subseteq \mathbb{B}_1$). However, f fails to be weakly continuous. Indeed, if $(e_n)_{n\in\mathbb{N}}$ denotes the classical Schauder basis of ℓ_2, then $e_n + e_1 \rightharpoonup e_1$ as $n \to +\infty$. But, for each $n \geq 2$, we have $\|e_n + e_1\|_2 = \sqrt{2}$ and therefore

$$f(e_n + e_1) = \frac{e_n + e_1}{\sqrt{2}} \rightharpoonup \frac{e_1}{\sqrt{2}} \neq e_1 = f(e_1).$$

This means that f is not weakly continuous.

Proof of Theorem 7.12.10. Let $n \in \mathbb{N}^*$ and consider the mappings

$$f_n = \left(1 - \frac{1}{n}\right) f : K \to K \quad \text{and} \quad g_n = \left(1 - \frac{1}{n}\right) g : X \to X.$$

It is clear that f_n is ws-compact. Moreover, since g is nonexpansive and ω-condensing, g_n is ω-$(1 - \frac{1}{n})$-contractive and so $(I - g_n)$ is ϕ-expansive (see Remark 6.6.1). Furthermore, because $0 \in K$, we have $f_n(K) + g_n(K) \subseteq K$. Hence, applying Theorem 7.12.6, we infer that, for each $n \in \mathbb{N}^*$, the operator $f_n + g_n$ has a fixed point $u_n \in K$.

We claim that the sequence $(u_n)_{n\in\mathbb{N}^*}$ is weakly convergent. To see this, we first note that $(u_n)_{n\in\mathbb{N}^*}$ is a bounded sequence. Let us suppose that $(u_n)_{n\in\mathbb{N}^*}$ is not weakly convergent, and consequently the set $\{u_n : n \in \mathbb{N}^*\}$ is not relatively weakly compact. Since g is ω-condensing, we have

$$\begin{aligned} \omega(\{u_n : n \in \mathbb{N}\}) &= \omega\left(\left\{\left(1 - \frac{1}{n}\right) f(u_n) + \left(1 - \frac{1}{n}\right) g(u_n) : n \in \mathbb{N}\right\}\right) \\ &\leq \omega\left(\left\{\left(1 - \frac{1}{n}\right) f(u_n) : n \in \mathbb{N}\right\}\right) + \omega\left(\left\{\left(1 - \frac{1}{n}\right) g(u_n) : n \in \mathbb{N}\right\}\right) \\ &\leq \omega(\mathrm{co}(\{f(u_n) : n \in \mathbb{N}\} \cup \{0\})) + \omega(\mathrm{co}(\{g(u_n) : n \in \mathbb{N}\} \cup \{0\})) \\ &\leq \omega(\{f(u_n) : n \in \mathbb{N}\}) + \omega(\{g(u_n) : n \in \mathbb{N}\}) \\ &= \omega(g(\{u_n : n \in \mathbb{N}\})) \\ &< \omega(\{u_n : n \in \mathbb{N}\}), \end{aligned}$$

which is a contradiction. So, the set $\{u_n : n \in \mathbb{N}^*\}$ is relatively weakly compact, which proves our claim. Thus $(u_n)_{n\in\mathbb{N}^*}$ has a subsequence (which we denote again by $(u_n)_{n\in\mathbb{N}^*}$) such that $u_n \rightharpoonup u \in K$. Next, using the equation

$$u_n = \left(1 - \frac{1}{n}\right)(f + g)(u_n) = f(u_n) + g(u_n) - \frac{1}{n} f(u_n) - \frac{1}{n} g(u_n),$$

we obtain the estimate

$$\begin{aligned}\|(I-(f+g))(u_n)\| = \|u_n-(f(u_n)+g(u_n))\| &= \frac{1}{n}\|f(u_n)+g(u_n)\|\\ &\leq \frac{1}{n}(\|f(u_n)\|+\|g(u_n)-g(0)\|+\|g(0)\|)\\ &\leq \frac{1}{n}(\|f(u_n)\|+\|u_n\|+\|g(0)\|).\end{aligned}$$

Since $(u_n)_{n\in\mathbb{N}^*}$ is bounded, using hypothesis (a) implies that the sequence $(\|f(u_n)\|)_{n\in\mathbb{N}^*}$ is bounded and therefore $\lim_{n\to+\infty}(I-(f+g))(u_n)=0$. Now, using the demiclosedness of $I-(f+g)$ at 0, we obtain $(I-(f+g))(u)=0$, which ends the proof. □

Corollary 7.12.5. *Let K be a nonempty, bounded, closed, convex subset of a Banach space X with $0\in K$. Let $f:K\to K$ and $g:X\to X$ be two continuous mappings. Suppose*
(a) *f is weakly–strongly continuous and $f(K)$ is relatively weakly compact,*
(b) *g is nonexpansive and ω-condensing,*
(c) *$f(M)+g(M)\subseteq M$,*
(d) *$I-g:K\to X$ is demiclosed.*

Then, $\operatorname{Fix}(f+g)\neq\emptyset$.

Remark 7.12.5. It is well known that a mapping $g:X\to X$ is ww-compact if and only if it maps relatively weakly compact sets into relatively weakly compact sets. Accordingly, every ω-condensing map is ww-compact. So, condition (b) of Theorem 7.12.9 is more general than condition (b) of Corollary 7.12.9. However, in this corollary, we do not have to assume condition (c) of Theorem 7.12.9.

Proof of Corollary 7.12.5. Hypothesis (a) implies that f satisfies conditions (a) and (b) of Theorem 7.12.10. Thus, we have only to check that $I-(f+g)$ is demiclosed at 0. To see this, consider a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that $x_n\rightharpoonup x$ and suppose that $x_n-(f(x_n)+g(x_n))\to 0$. Since f is weakly–strongly continuous, we have $f(x_n)\to f(x)$ and therefore $x_n-g(x_n)\to f(x)$. Finally, since $I-g$ is demiclosed, we derive that $x-g(x)=f(x)$, which ends the proof. □

Theorem 7.12.11. *Let X be a Banach space and let K be a nonempty, closed, bounded, convex subset of X. Let $f:K\to X$ and $g:X\to X$ be two weakly sequentially continuous maps such that*
(a) *g is pseudocontractive, continuous, and ω-k-contractive for some $k\in[0,1)$,*
(b) *$I-g$ is ϕ-expansive,*
(c) *f is ω-s-contractive for some $s\in[0,1-k)$,*
(d) *$(x=g(x)+f(y), y\in K) \implies x\in K$.*

Then $\operatorname{Fix}(f+g)\neq\emptyset$.

Proof. We have first to check that the mapping $(I-g)^{-1}f$ is well defined on K and $(I-g)^{-1}f(K)\subseteq K$. Because $I-g$ is ϕ-expansive, reasoning as in the proof of Theorem 7.12.6, we show that $I-g$ is invertible.

The domain of $(I-g)^{-1}$ contains the range of $f(K)$. Indeed, take $y\in K$ and consider $f(y)$. We have to check if there exists some $x\in K$ such that $(I-g)(x)=f(y)$. Define the map $h_y:X\to X$ by $h_y(x)=g(x)+f(y)$. Because g is continuous, pseudocontractive, h_y is also a continuous pseudocontractive map. Since $I-g$ is ϕ-expansive, we have

$$\begin{aligned}\|(I-h_y)(x)-(I-h_y)(z)\| &= \|x-z+h_y(z)-h_y(x)\|\\ &= \|x-z+g(z)-g(x)\|\\ &= \|(I-g)(x)-(I-g)(z)\|\\ &\ge \phi(\|x-z\|).\end{aligned}$$

Hence $I-h_y:X\to X$ is accretive, continuous, and ϕ-expansive. By Theorem 4.4.3, we see that $I-h_y$ is m-accretive and then, by Theorem 6.8.1, the map $I-h_y$ is surjective. Hence $\mathsf{R}(I-h_y)=X$ and, consequently, there exists $x\in X$ such that $0=x-h_y(x)$, that is, $x=g(x)+f(y)$. Using hypothesis (d), one sees that x belongs to K. Hence, $f(y)=(I-g)^{-1}(x)$ and, consequently, the map $(I-g)^{-1}\circ f:K\to K$ is well defined.

The map $(I-g)^{-1}f:K\to K$ is weakly sequentially continuous and ω-condensing. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of K such that $x_n\rightharpoonup x$. The set $\{x_n:n\in\mathbb{N}\}$ is relatively weakly compact and, since f is weakly sequentially continuous, $\{f(x_n):n\in\mathbb{N}\}$ is also relatively weakly compact. Using equality (7.33) and the fact that g is ω-condensing, we obtain

$$\begin{aligned}\omega(\{(I-g)^{-1}f(x_n):n\in\mathbb{N}\}) &= \omega(\{f(x_n):n\in\mathbb{N}\}+\{g(I-g)^{-1}f(x_n):n\in\mathbb{N}\})\\ &\le \omega(\{g(I-g)^{-1}f(x_n):n\in\mathbb{N}\})\\ &< \omega(\{(I-g)^{-1}f(x_n):n\in\mathbb{N}\}).\end{aligned}$$

Hence we reach a contradiction. Therefore, $\omega(\{(I-g)^{-1}f(x_n):n\in\mathbb{N}\})=0$ and then $\{(I-g)^{-1}f(x_n):n\in\mathbb{N}\}$ is relatively weakly compact. There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$(I-g)^{-1}f(x_{n_k})\rightharpoonup y.$$

In view of equality (7.33) and the fact that f and g are weakly sequentially continuous, we obtain $y=g(y)+f(x)$, and then $y=(I-g)^{-1}f(x)$. This proves

$$(I-g)^{-1}f(x_{n_k})\rightharpoonup (I-g)^{-1}f(x).$$

Proceeding as in part (c) of the proof of Theorem 7.12.1, we deduce that

$$(I-g)^{-1}f(x_n)\rightharpoonup (I-g)^{-1}f(x).$$

Hence, $(I - g)^{-1}f : K \to K$ weakly sequentially continuous.

The map $(I-g)^{-1}f$ is ω-condensing. Indeed, let S be a subset of K such that $\omega(S) > 0$ and set $J := (I - g)^{-1}f$. Using equality (7.33), we conclude that

$$\omega(J(S)) = \omega(f(S) + g(J(S))).$$

The properties of ω and the assumptions on f and g imply

$$\omega(J(S)) \leq \omega(f(S)) + \omega(g(J(S))) \leq s\omega(S) + k\omega(J(S)),$$

and therefore

$$\omega(J(S)) \leq \frac{s}{1-k}\omega(S).$$

Since $0 \leq s < 1 - k$, the operator J is ω-condensing. Now the use of Theorem 7.8.1(b) completes the proof. □

Remark 7.12.6. If in the previous theorem we assume that $f(K)$ is weakly relatively compact, then we can replace the assumption "g is ω-contractive" by "g is ω-condensing." Under these conditions, assumption (c) in Theorem 7.12.11 is redundant.

Theorem 7.12.12. *Let K be a nonempty, bounded, closed, convex subset of a Banach space X and let $f : K \to X$ and $g : X \to X$ be two mappings such that*

(a) *f is ws-compact and ω-k-contractive for some $k \in [0,1)$,*

(b) *g is a nonlinear contraction with Φ-function ϕ such that, for all $r > 0$, $\phi(r) < (1-k)r$,*

(c) *g is ww-compact,*

(d) $(x = f(x) + g(y), y \in K) \implies x \in K.$

Then $\operatorname{Fix}(f + g) \neq \emptyset$.

Proof. Following the proof of Theorem 7.12.3, the map $\Pi := (I - g)^{-1}f$ is well defined, continuous, and satisfies $\Pi(K) \subseteq K$. To complete the proof, it suffices to show that Π has a fixed point in K. Since f is ws-compact, using the continuity of $(I - g)^{-1}$, we conclude that Π is ws-compact. Moreover, for each $S \subseteq K$ with $\omega(S) > 0$, the use of (7.33) implies $\Pi(S) \subseteq f(S) + (g\Pi)(S)$, and, by Proposition, 7.8.2(b), we have $\omega(g\Pi(S)) \leq \phi(\omega(\Pi(S)))$. Since f is ω-k-contractive, we get

$$\omega(\Pi(S)) \leq \omega(f(S)) + \omega(g\Pi(S)) \leq k\omega(S) + \phi(\omega(\Pi(S))). \tag{7.43}$$

If $k = 0$, then (7.43) becomes $\omega(\Pi(S)) \leq \phi(\omega(\Pi(S))) < \omega(\Pi(S))$, which is a contradiction. Hence $\omega(\Pi(S)) = 0$.

If $k \neq 0$, then, by (c), (7.43) becomes $\omega(\Pi(S)) < k\omega(S) + (1-k)\omega(\Pi(S))$ or, equivalently, $\omega(\Pi(S)) < \omega(S)$.

Hence, in each of these two cases, Π is ω-condensing. Now applying Theorem 7.8.3 to the operator $\Pi : K \to K$, we conclude that Π has a fixed point in K, which completes the proof. ☐

Definition 7.12.3. Let X be a Banach space. A mapping $g : X \to X$ is called Φ-Lipschitz if there exists a continuous nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|g(x) - g(y)\| \le \phi(\|x - y\|) \quad \text{for all } x, y \in X.$$

The function ϕ is called a Φ-function of g.

It is obvious that every Lipschitz mapping is Φ-Lipschitz. The converse is, in general, not true. For example, for $x \in \mathbb{R}$, take $g(x) = \sqrt{|x|}$ and consider $\phi(r) = \sqrt{r}$, $r > 0$. It is clear that ϕ is continuous and nondecreasing. Moreover, easy calculations show that g is subadditive, that is, for all $x, y \in \mathbb{R}$, $g(x+y) \le g(x) + g(y)$. Using the subadditivity of g, we get $|g(x) - g(y)| \le g(x - y) = \phi(|x - y|)$ for all $x, y \in \mathbb{R}$. Hence f is Φ-Lipschitz with Φ-function ϕ. However, it is well known that the function g is not Lipschitz on $\mathbb{R}$.

Lemma 7.12.3. *Let f be a Φ-Lipschitz map defined on a Banach space X with Φ-function ϕ. If f is ww-compact, then for each bounded subset M of X we have $\omega(f(M)) \le \phi(\omega(M))$.*

Proof. Let M be a bounded subset of X and $r > \omega(M)$. Then there is a weakly compact subset W of X such that $M \subset W + \mathbb{B}_r$. Since f is Φ-Lipschitz with a Φ-function ϕ,

$$f(M) \subset f(W) + \mathbb{B}_{\phi(r)} \subset \overline{f(W)}^w + \mathbb{B}_{\phi(r)}.$$

Because f is ww-compact, $\overline{f(W)}^w$ is weakly compact. Hence, $\omega(f(M)) \le \phi(r)$. Letting $r \to \omega(M)$ and using the continuity of ϕ, we obtain $\omega(f(M)) \le \phi(\omega(M))$. ☐

Lemma 7.12.4. *Let K be a nonempty, bounded, closed subset of a Banach algebra X and let $f, g : X \to X$ be Φ-Lipschitz mappings with Φ-functions ϕ_f and ϕ_g, respectively. If, for each $r > 0$, we have $\|K\|\phi_f(r) + \phi_g(r) < r$, then $(\frac{I-g}{f})^{-1} : K \to X$ exists and is continuous.*

For the definition of $\|K\|$, we refer to (7.18).

Proof. Let $y \in K$ be fixed and let h_y be the map defined on X by $X \ni x \mapsto h_y(x) = f(x) \cdot y + g(x)$. For all $x, z \in X$, we have

$$\begin{aligned}\|h_y(x) - h_y(z)\| &\le \|f(x) - f(z)\|\|y\| + \|g(x) - g(z)\| \\ &\le \|K\|\phi_f(\|x - y\|) + \phi_g(\|x - y\|).\end{aligned}$$

Hence, h_y is a nonlinear contraction with a contraction function $\psi(r) = \|K\|\phi_f(r) + \phi_g(r)$, $r > 0$. Now Theorem 6.3.1 guarantees that there exists a unique point ζ such that $h_y(\zeta) = \zeta$, i. e., $y = (\frac{I-g}{f})(\zeta)$. Hence the map $G := (\frac{I-g}{f})^{-1} : K :\to X$ is well defined. Now we show that $G : K \to X$ is continuous. To see this, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K converging to a point x. Since K is closed, then $x \in K$. First notice that, for each $z \in K$, we have

$$G(z) = (g \circ G)(z) + \big((f \circ G)(z)\big) \cdot z.$$

Hence

$$\begin{aligned}\big\|G(x_n) - G(x)\big\| &\le \big\|(g \circ G)(x_n) - (g \circ G)(x)\big\| + \big\|\big((f \circ G)(x_n)\big) \cdot x_n - \big(f \circ G(x)\big) \cdot x\big\| \\ &\le \|(g \circ G)(x_n) - (g \circ G)(x)\| + \big\|(f \circ G)(x)\big\| \|x_n - x\| \\ &\quad + \|(f \circ G)(x_n) - (f \circ G(x))\| \|x_n\| \\ &\le \phi_g\big(\big\|G(x_n) - G(x)\big\|\big) + \phi_f\big(\big\|G(x_n) - G(x)\big\|\big)\|K\| + \big\|f \circ G(x)\big\| \|x_n - x\|,\end{aligned}$$

and therefore

$$\limsup_{n\in\mathbb{N}} \big\|G(x_n) - G(x)\big\| \le \phi_g\Big(\limsup_{n\in\mathbb{N}} \big\|G(x_n) - G(x)\big\|\Big) + \phi_f\Big(\limsup_{n\in\mathbb{N}} \big\|G(x_n) - G(x)\big\|\Big)\|K\|.$$

Let $r = \limsup_{n\in\mathbb{N}} \|G(x_n) - G(x)\|$. The preceding equation can be written as $r \le \|K\|\phi_f(r) + \phi_g(r)$, which is a contradiction. This shows that $\lim_{n\to+\infty} \|G(x_n) - G(x)\| = 0$ and, consequently, G is continuous on K, which ends the proof. ☐

Theorem 7.12.13. *Let X be a Banach algebra and let $\mu(\cdot)$ be a measure of weak noncompactness on X. Let K be a nonempty, closed, convex subset of X and let $f, g : X \to X$ and $h : K \to X$ be weakly sequentially continuous mappings. Suppose*

(a) *f and g are Φ-Lipschitz maps with Φ-functions ϕ_f and ϕ_g, respectively,*

(b) *the set $h(K)$ is bounded and the map $(\frac{I-g}{f})^{-1}h$ is μ-condensing on K,*

(c) $(x = f(x)h(y) + g(x), y \in K) \implies x \in K.$

If, for each $r > 0$, we have $\|h(K)\|\phi_f(r) + \phi_g(r) < r$, then $\mathrm{Fix}(fh + g) \neq \emptyset$.

We note that, in Theorem 7.12.13, K does not need to be bounded.

Proof. According to Lemma 7.12.4, the map $\rho := (\frac{I-g}{f})^{-1}h : K \to X$ is well defined and is continuous. It follows from (c) that $\rho(K) \subset K$. Let $z \in K$ and set

$$\mathcal{F} = \{M : z \in M \subset K, M \text{ is a closed, convex set, and} \rho(M) \subset M\}.$$

Clearly, $\mathcal{F} \neq \emptyset$ because $K \in \mathcal{F}$. Put $\Pi := \bigcap_{M\in\mathcal{F}} M$. It is clear that $z \in \Pi \subset K$, Π is closed, convex, and ρ-invariant. Since $\mathrm{co}(\{\rho(\Pi) \cup z\}) \subset \Pi$, we have

$$\rho\big(\mathrm{co}(\{\rho(\Pi) \cup z\})\big) \subset \rho(\Pi) \subset \mathrm{co}\big(\{\rho(\Pi) \cup z\}\big).$$

This shows that $\mathrm{co}(\{\rho(\Pi) \cup z\}) \in \mathcal{F}$. Thus we deduce that $\mathrm{co}(\{\rho(\Pi) \cup z\}) = \Pi$. Using the properties of $\mu(\cdot)$, we obtain

$$\mu(\Pi) = \mu\big(\mathrm{co}(\{\rho(\Pi) \cup z\})\big) = \mu\big(\{\rho(\Pi) \cup z\}\big) = \mu\big(\rho(\Pi)\big).$$

From our assumptions, we get $\mu(\Pi) = 0$, and therefore, Π is a nonempty, weakly compact, convex subset of X. Since Π is closed and convex, by Theorem 1.7.2, it is weakly closed and therefore it is weakly compact.

We claim that the map $\rho : \Pi \to \Pi$ is weakly sequentially continuous. To see this, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of Π which converges weakly to some $x \in K$. Because $\rho(\Pi)$ is relatively weakly compact, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\rho(x_{n_k}) \rightharpoonup y$. Using the fact that $\rho(x_{n_k}) = f(\rho(x_{n_k}))h(\rho(x_{n_k})) + g(\rho(x_{n_k}))$ and the weak sequential continuity of f, g, and h we conclude that $y = f(y)h(y) + g(y)$. Hence $y = \rho(x)$. Consequently, $\rho(x_{n_k}) \rightharpoonup y = \rho(x)$, which proves our claim.

To conclude the proof, we only have to show that $\rho(x_n) \rightharpoonup \rho(x)$. This part of the proof is similar to the last part of the proof of step (c) of Theorem 7.12.1. So, it is omitted.

Finally, it suffices to apply Theorem 7.5.2 to the operator $\rho : \Pi \to \Pi$. ☐

Evidently, if $g = 0$ in Theorem 7.12.13, then we get the following corollary.

Corollary 7.12.6. *Let X be a Banach algebra and let $\mu(\cdot)$ be a measure of weak noncompactness on X. Let K be a nonempty, closed, convex subset of X, and let $f : X \to X$ and $h : K \to X$ be weakly sequentially continuous maps. Suppose*
(a) *the maps f is Φ-Lipschitz with Φ-functions ϕ_f,*
(b) *the set $h(K)$ is bounded and the map $(\frac{I}{f})^{-1}h$ is μ-condensing on K,*
(c) $(x = f(x)h(y), y \in K) \implies x \in K.$

If $\|h(K)\|\phi_f(r) < r$ for $r > 0$, then $\mathrm{Fix}(fh) \neq \emptyset$.

Corollary 7.12.7. *Let X be a WC-Banach algebra and let K be a nonempty, closed, convex subset of X. Let $f, g : X \to X$ and $h : K \to X$ be weakly sequentially continuous maps. Suppose*
(a) *f and g are Φ-Lipschitz mappings with Φ-functions ϕ_f and ϕ_g, respectively,*
(b) *the set $h(K)$ is relatively weakly compact,*
(c) $(x = f(x)h(y) + g(x), y \in K) \implies x \in K.$

If $\|h(K)\|\phi_f(r) + \phi_g(r) < r$ for $r > 0$, then $\mathrm{Fix}(fh + g) \neq \emptyset$.

Proof. According to Theorem 7.12.13, it suffices to show that $\rho := (\frac{I-g}{f})^{-1}h : K \to X$ maps bounded sets into weakly compact sets. To see this, let M be a bounded subset of K. It is easily seen that

$$\rho(M) \subset g(\rho(M)) + f(\rho(M))h(M).$$

Using Lemma 7.12.3 and Proposition 7.7.6, together with the weak compactness of $h(K)$, one can write

$$\begin{aligned}\omega(\rho(M)) &\leq \omega(g(\rho(M)) + f(\rho(M))h(M))\\ &\leq \omega(g(\rho(M))) + \omega(f(\rho(M))h(M))\\ &\leq \phi_g(\omega(\rho(M))) + \|h(K)\|\phi_f(\omega(\rho(M))).\end{aligned}$$

Since, for $r > 0$, $\|h(K)\|\phi_f(r) + \phi_g(r) < r$, we infer that $\omega(\rho(M)) = 0$. This proves that $\rho(M)$ is relatively weakly compact. The result follows from Theorem 7.12.13. ☐

Theorem 7.12.14. *Let X be a Banach algebra and let $\mu(\cdot)$ be a measure of weak noncompactness on X. Let K be a nonempty, closed, convex subset of X and let $f, g : X \to X$ and $h : K \to X$ be continuous maps. Suppose*

(a) *f and g are ww-compact Φ-Lipschitz mappings with Φ-functions ϕ_f and ϕ_g, respectively,*
(b) *h is ws-compact,*
(c) *the set $h(K)$ is bounded and the map $(\frac{I-g}{f})^{-1}h$ is μ-condensing on K,*
(d) $(x = f(x)h(y) + g(x), y \in K) \implies x \in K.$

If $\|h(K)\|\phi_f(r) + \phi_g(r) < r$ for $r > 0$, then $\mathrm{Fix}(fh + g) \neq \emptyset$.

Proof. It follows from Lemma 7.12.4 that the map $\rho := (\frac{I-g}{f})^{-1}h : K \to X$ is well defined and continuous. By assumption (c), we know that $\rho(K) \subset K$. Let $z \in K$ and set

$$\mathcal{F} = \{M : z \in M \subset K, M \text{ is a closed, convex set, and} \rho(M) \subset M\}.$$

Clearly, $\mathcal{F} \neq \emptyset$ because $K \in \mathcal{F}$. Set $\Pi := \bigcap_{M \in \mathcal{F}} M$. Arguing as in the proof of Theorem 7.12.13, we see that Π is a nonempty, relatively weakly compact, convex subset of X. Since Π is closed, the use of Theorem 1.7.2 shows that Π is weakly closed. Consequently, Π is weakly compact. Moreover, it is clear that $\rho(\Pi) \subset \Pi$. Since h is ws-compact, we infer that ρ is ws-compact.

To conclude the proof, it suffices to apply Theorem 7.6.1 to the operator $\rho : \Pi \to \Pi$. ☐

Theorem 7.12.15. *Let X be a WC-Banach algebra and let K be a nonempty, bounded, closed, convex subset of X. Let f, g, and h be three mappings such that $f, g : X \to X$ and $h : K \to X$. Suppose*

(a) *f and g are ww-compact and Φ-Lipschitz with Φ-functions ϕ_f and ϕ_g, respectively,*
(b) *h is ws-compact and $h(K)$ is relatively weakly compact,*
(c) $(x = f(x)h(y) + g(x), y \in K) \implies x \in K,$
(d) *$\|h(K)\|\phi_f(r) + \phi_g(r) < r$ for all $r > 0$.*

Then $\mathrm{Fix}(f \cdot h + g) \neq \emptyset$.

Proof. According to Lemma 7.12.4, the map $\rho := (\frac{I-g}{f})^{-1}h : K \to X$ is well defined and continuous. Next, the use of hypothesis (c) shows that $\rho(K) \subset K$. The map ρ is ww-compact because $(\frac{I-g}{f})^{-1}$ is continuous (Lemma 7.12.4) and h is ws-compact. Now, using the expression of ρ, we obtain $\rho = (f \circ \rho) \cdot h + g \circ \rho$, and therefore

$$\omega(\rho(K)) \leq \omega((f \circ \rho)(K) \cdot h(K)) + \omega((g \circ \rho)(K)).$$

So, by Proposition 7.7.6(b), we have

$$\begin{aligned}\omega(\rho(K)) \leq \|h(K)\|\omega((f\circ\rho)(K)) + \|(f\circ\rho)(K)\|\omega(h(K))\\ + \omega((f\circ\rho)(K))\omega(h(K)) + \omega((g\circ\rho)(K)).\end{aligned} \tag{7.44}$$

Bearing in mind that f is Φ-Lipschitz, we infer that set $(f\circ\rho)(K)$ is bounded. Next, since g is Φ-Lipschitz and ww-compact, the use of Lemma 7.12.3 implies that

$$\omega(g\circ\rho(K)) \leq \phi_g(\omega(\rho(K)). \tag{7.45}$$

Since $h(K)$ is relatively weakly compact, we have $\omega(h(K)) = 0$. Using assumption (a) and Lemma 7.12.3, we obtain

$$\|h(K)\|\omega(f\circ\rho(K)) \leq \|h(K)\|\phi_f(\omega(\rho(K))). \tag{7.46}$$

Combining (7.44) together with (7.45), (7.46), and assumption (d), we get

$$\omega(\rho(K)) \leq \|h(K)\|\phi_f(\omega(\rho(K))) + \phi_g(\omega(\rho(K))) < \omega(\rho(K)),$$

which is a contradiction. Hence $\omega(\rho(K)) = 0$, and therefore $\rho(K)$ is relatively weakly compact.

Now applying Theorem 7.6.1, we conclude that there exists $x \in K$ such that $\rho(x) = x$. Making use of the definition of ρ, we get $x = f(x)h(x) + g(x)$, which ends the proof. □

We conclude this section with the following theorem and its corollaries.

Theorem 7.12.16. *Let X be a Banach space, K a nonempty, closed, bounded, convex subset of X and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $f : X \to K$, $g : K \to X$, and $B : K \to X$ be continuous maps, and set $F = f\circ g$. Suppose that*
(a) *f is ws-compact,*
(b) *there exists $\gamma \in [0,1)$ such that $\mu(F(S) + B(S)) \leq \gamma\mu(S)$ for all $S \subset K$ with $\mu(S) > 0$,*
(c) *g is ww-compact,*
(d) *B is a nonlinear contraction with Φ-function ϕ,*
(e) *$(x = B(x) + F(y), y \in K) \Longrightarrow x \in K$.*

Then $\operatorname{Fix}(F + B) \neq \emptyset$.

Proof. Since B is a nonlinear contraction with Φ-function ϕ, by Lemma 7.12.2, $I - B$ is a homeomorphism from K onto $(I - B)(K)$. Let y be a point in K and define the map $K \ni x \mapsto B(x) + F(y)$. It is clear that it is a nonlinear contraction with Φ-function ϕ. Hence, according to Theorem 7.12.2, the equation $z = B(z) + F(y)$ has a unique solution $x \in X$. Using hypothesis (e), we infer that $x \in K$, and therefore $x = (I - B)^{-1}F(y) \in K$. Hence,

$$(I - B)^{-1}F(K) \subset K. \tag{7.47}$$

We define a sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of K by

$$K_0 = K \quad \text{and} \quad K_{n+1} = \overline{\mathrm{co}}((I-B)^{-1}F(K_n)).$$

It is clear that $(K_n)_{n\in\mathbb{N}}$ is a sequence of nonempty, closed, convex subsets of K. Moreover, the inclusion (7.47) shows that $(K_n)_{n\in\mathbb{N}}$ is decreasing (in the sense of inclusion). Furthermore, elementary calculations using equation (7.33) give

$$(I-B)^{-1}F(K_n) \subset F(K_n) + B(\overline{\mathrm{co}}((I-B)^{-1}F(K_n))) \subset F(K_n) + B(K_{n+1}).$$

Since $(K_n)_{n\in\mathbb{N}}$ is decreasing, we get $(I-B)^{-1}F(K_n) \subset F(K_n)+B(K_n)$. Thus, assumption (b) yields

$$\mu(K_{n+1}) \le \mu(F(K_n) + B(K_n)) \le \gamma\mu(K_n).$$

By induction, we get $\mu(K_{n+1}) \le \gamma^n\mu(K)$, and therefore $\lim_{n\to+\infty}\mu(K_n) = 0$ because $\gamma \in [0,1)$. Hence, it follows from the generalized Cantor intersection theorem that $K_\infty = \bigcap_{n\ge 0} K_n$ is a nonempty, convex, weakly compact subset of K. Moreover, we have $(I-B)^{-1}F(K_\infty) \subset K_\infty$ and, consequently, the set $(I-B)^{-1}F(K_\infty)$ is relatively weakly compact.

Letting $(x_n)_{n\in\mathbb{N}}$ be a sequence in K_∞, it has a weakly convergent subsequence which we denote again by $(x_n)_{n\in\mathbb{N}}$. Using hypothesis (c), one sees that there exists a subsequence denoted by $(x_{n_k})_{k\in\mathbb{N}}$ such that $(g(x_{n_k}))_{k\in\mathbb{N}}$ is weakly convergent. Next, using the fact that f is ws-compact, we conclude that there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$ such that $(f(g(x_{n_{k_j}})))_{j\in\mathbb{N}}$ converges strongly in K_∞. Because $(I-B)^{-1}$ is continuous, the sequence $(I-B)^{-1}(f(g(x_{n_{k_j}})))_{j\in\mathbb{N}}$ converges strongly in K_∞. Hence, the map $(I-B)^{-1}F$ is ws-compact. So, invoking the fact that $(I-B)^{-1}F(K_\infty)$ is relatively weakly compact and Theorem 7.6.1, we conclude that $\mathrm{Fix}((I-B)^{-1}F) \neq \emptyset$, which concludes the proof. ☐

As an immediate consequence of Theorem 7.12.16 and Remark 7.6.4, we have

Corollary 7.12.8. *Let X be a Banach space, K a nonempty, closed, bounded, convex subset of X and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $f : X \to X$, $g : K \to X$, and $B : K \to X$ be continuous maps, and set $F = f \circ g$. Suppose that*

(a) *f is a Dunford–Pettis operator,*
(b) *there exists $\gamma \in [0,1)$ such that $\mu(F(S) + B(S)) \le \gamma\mu(S)$ for all $S \subset K$ with $\mu(S) > 0$,*
(c) *g is ww-compact and $g(K)$ is bounded,*
(d) *B is a nonlinear contraction with Φ-function ϕ,*
(e) *$(x = B(x) + F(y), y \in K) \Longrightarrow x \in K$.*

Then $\mathrm{Fix}(F + B) \neq \emptyset$.

Note that if X is Banach space with Dunford–Pettis property, then, according to Remark 1.9.2, every weakly compact linear operator on X is a Dunford–Pettis operator. This yields the following result.

Corollary 7.12.9. *Let X be a Banach space with Dunford–Pettis property, K a nonempty, closed, bounded, convex subset of X, and $\mu(\cdot)$ a measure of weak noncompactness on X. Let $f : X \to K$, $g : K \to X$, and $B : K \to X$ be continuous maps, and set $F = f \circ g$. Suppose that*

(a) *f is a weakly compact linear operator,*
(b) *there exists $\gamma \in (0,1)$ such that $\mu(F(S) + B(S)) \leq \gamma\mu(S)$ for all $S \subset K$ with $\mu(S) > 0$,*
(c) *g is ww-compact and $g(K)$ is bounded,*
(c) *B is a nonlinear contraction with Φ-function ϕ,*
(d) *$(x = B(x) + F(y), y \in K) \Longrightarrow x \in K$.*

Then $\mathrm{Fix}(F + B) \neq \emptyset$.

Remark 7.12.7. Note that, according to Remark 6.3.1, if B is a contractive mapping, then Theorem 7.12.16 and Corollaries 7.12.8 and 7.12.9 hold true.

7.12.2 Krasnosel'skii–Leray–Schauder-type fixed point theorems

The purpose of this section is to present some results of Leray–Schauder type involving two operators.

Theorem 7.12.17. *Let X be a Banach space, and let Q and C be two closed, bounded, convex subsets of X such that $Q \subseteq C$. Let U be a weakly open subset relative to Q with $0 \in U$ and $\overline{U}^w$ a weakly compact subset of Q. Let $f : \overline{U}^w \to X$ and $g : X \to X$ be two weakly sequentially continuous maps. Suppose*

(a) *$f(\overline{U}^w)$ is relatively weakly compact,*
(b) *g is a nonlinear contraction with Φ-function ϕ,*
(c) *$(x = g(x) + f(y), y \in \overline{U}^w) \implies x \in C$.*

Then

(i) *either* $\mathrm{Fix}(f + g) \neq \emptyset$, *or*
(ii) *there is $x \in \partial_Q U$ (the weak boundary of U in Q) and $\lambda \in (0,1)$ such that $x = \lambda f(x) + \lambda g(x/\lambda)$.*

Proof. Let y be a fixed point in $\overline{U}^w$ and define the map $h_y : X \to X$ by $h_y(x) = g(x) + f(y)$. For all $x_1 x_2 \in C$, $x_1 \neq x_2$, we have

$$\|h_y(x_1) - h_y(x_2)\| = \|g(x_1) + f(y) - g(x_2) - f(y)\| = \|g(x_1) - g(x_2)\| \leq \phi(\|x_1 - x_2\|).$$

Hence h_y is a nonlinear ϕ-contraction. By Theorem 7.12.2, there exists a unique point $x \in X$ such that $h_y(x) = x$ or, equivalently, $x = g(x) + f(y)$. The use of hypothesis (c) shows that $x \in C$, and therefore $f(\overline{U}^w) \subseteq (I - g)(C)$. Further, Lemma 7.12.2 ensures that $(I - g)^{-1}$ exists and is continuous on $(I - g)(X)$. Thus, the mapping $(I - g)^{-1}f : \overline{U}^w \to C$ is well defined.

The subset $(I - g)^{-1}f(\overline{U}^w)$ is bounded. Indeed, since $f(\overline{U}^w)$ is relatively weakly compact, it is weakly bounded and then strongly bounded. Using the fact that g is a nonlinear ϕ-contraction, we see that $(I - g)^{-1}f(\overline{U}^w)$ is also bounded.

The subset $(I - g)^{-1}f(\overline{U}^w)$ is relatively weakly compact. To see this, assume that it is not the case, then $\omega((I - g)^{-1}f(\overline{U}^w)) > 0$. Using (7.33) and the fact that $f(\overline{U}^w)$ is relatively weakly compact, we may write

$$\begin{aligned}\omega((I - g)^{-1}f(\overline{U}^w)) &= \omega(f(\overline{U}^w) + g(I - g)^{-1}f(\overline{U}^w)) \\ &\leq \omega(f(\overline{U}^w)) + \omega(g(I - g)^{-1}f(\overline{U}^w)) \\ &= \omega(g(I - g)^{-1}f(\overline{U}^w)).\end{aligned} \tag{7.48}$$

Let $\tau > \omega((I - g)^{-1}f(\overline{U}^w))$. Then there exist $0 \leq \tau_0 < \tau$ and $W \in \mathbf{W}(X)$ such that $(I - g)^{-1}f(\overline{U}^w) \subseteq W + \mathbb{B}_{\tau_0}$. If $x \in (I - g)^{-1}f(\overline{U}^w)$, then there exists $y \in W$ such that $\|y - x\| \leq \tau_0$. Since g is a nonlinear ϕ-contraction, we have

$$\|g(x) - g(y)\| \leq \phi(\|x - y\|) \leq \phi(\tau_0). \tag{7.49}$$

This implies that $g(x) - g(y) \in \mathbb{B}_{\phi(\tau_0)}$ and therefore $g(x) \in g(W) + \mathbb{B}_{\phi(\tau_0)}$. Hence

$$g(I - g)^{-1}f(\overline{U}^w) \subseteq g(W) + \mathbb{B}_{\phi(\tau_0)}. \tag{7.50}$$

Since W is weakly compact, it follows from Proposition 7.5.2 that $g : W \to X$ (the restriction of g to W) is weakly continuous and then $g(W)$ is weakly compact. By (7.50) and the definition of $\omega(\cdot)$, we have $\omega((I - g)^{-1}f(\overline{U}^w)) \leq \phi(\tau_0) \leq \phi(\tau)$ because ϕ in nondecreasing. Using the continuity of ϕ and letting τ go to $\omega((I - g)^{-1}f(\overline{U}^w))$, we get

$$\omega((I - g)^{-1}f(\overline{U}^w)) \leq \phi(\omega((I - g)^{-1}f(\overline{U}^w))). \tag{7.51}$$

Combining (7.48) and (7.51), we obtain

$$\omega((I - g)^{-1}f(\overline{U}^w)) \leq \phi(\omega((I - g)^{-1}f(\overline{U}^w))) < \omega((I - g)^{-1}f(\overline{U}^w)),$$

which is a contradiction. Hence, we have $\omega((I - g)^{-1}f(\overline{U}^w)) = 0$ and therefore $(I - g)^{-1}f(\overline{U}^w)$ is relatively weakly compact.

The operator $(I - g)^{-1}f : \overline{U}^w \to C$ is weakly sequentially continuous. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{U}^w$ which converges weakly to u. Since $(I - g)^{-1}f(\overline{U}^w)$ is weakly compact, there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that

$$(I-g)^{-1}f(u_{n_k}) \rightharpoonup v.$$

The use of (7.33) and the weak sequential continuity of f and g yield $v = (I-g)^{-1}f(u)$, and therefore $(I-g)^{-1}f(u_{n_k}) \rightharpoonup (I-g)^{-1}f(u)$. Arguing as in the proof of step (c) of Theorem 7.12.1, we show that $(I-g)^{-1}f(u_n) \rightharpoonup (I-g)^{-1}f(u)$, which proves our claim.

The map $(I-g)^{-1}f$ satisfies the hypotheses of Theorem 7.11.1 with $p = 0$. So we have

(i) either $(I-g)^{-1}f$ has a fixed point in $\overline{U}$, or

(ii) there exist $x \in \partial_Q U$ and $\lambda \in]0,1[$ such that $x = \lambda(I-g)^{-1}f(x)$.

It is clear that if assertion (i) is satisfied, then $f+g$ admits a fixed point in $\overline{U}$. Conversely, if assertion (ii) is satisfied, then there exist $x \in \partial_Q U$ and $\lambda \in (0,1)$ such that $f(x) = (I-g)(x/\lambda)$, and therefore $x = \lambda f(x) + \lambda g(x/\lambda)$, which completes the proof. ☐

Theorem 7.12.18. *Let X be a Banach space and let U be an open subset of X with $0 \in U$ and $\overline{U}$ its closure. Let $f : U \to X$ and $g : X \to X$ be two maps. Suppose*

(a) *$f(\overline{U})$ is relatively weakly compact,*

(b) *f is ws-compact,*

(c) *g is ww-compact and k-contractive for some $k \in [0,1)$.*

Then

(i) *either the equation $x = g(x) + f(x)$ has a solution in $\overline{U}$, or*

(ii) *there exist $x \in \partial U$ and $\lambda \in (0,1)$ such that $x = \lambda f(x) + \lambda g(x/\lambda)$.*

Proof. By (c), we know that $\Pi := (I-g)^{-1}f : \overline{U} \to X$ is well defined (see the proof of Theorem 7.12.11). Since $(I-g)^{-1}$ is continuous, it follows from (b) that Π is ws-compact. Further, Proposition 7.8.2(a), together with equation (7.33) and the properties of $\omega(\cdot)$, implies

$$\omega(\Pi(\overline{U})) \le \omega(f(\overline{U})) + k\omega(\Pi(\overline{U})) = k\omega(\Pi(\overline{U}))$$

because $f(\overline{U})$ is relatively weakly compact. Since $k \in [0,1)$, we have $\omega(\Pi(\overline{U})) \le k\omega(\Pi(\overline{U}))$, which is a contradiction. Hence $\omega(\Pi(\overline{U})) = 0$ and therefore $\Pi(\overline{U})$ is relatively weakly compact.

To conclude the proof, it suffices to apply Theorem 7.11.3 with $p = 0$ to the operator Π and to argue as in the last part of the proof of Theorem 7.12.17. ☐

Theorem 7.12.19. *Let X be a Banach space and let U be an open subset of X with $0 \in U$ and $\overline{U}$ its closure. Let $f : \overline{U} \to X$ and $g : X \to X$ be two continuous maps. Suppose*

(a) *$f(\overline{U})$ is relatively weakly compact,*

(b) *f is ws-compact,*

(c) *g is pseudocontractive,*

(d) *$I-g$ is ϕ-expansive where ϕ is either strictly increasing or $\lim_{r\to\infty}\phi(r) = +\infty$ and g is ω-condensing.*

Then

(i) *either the equation* $x = g(x) + f(x)$ *has a solution in* $\overline{U}$, *or*

(ii) *there exist* $x \in \partial U$ *and* $\lambda \in (0,1)$ *such that* $x = \lambda f(x) + \lambda g(x/\lambda)$.

Proof. Arguing as in the proof of Theorem 7.12.6, we show that $I - g$ invertible, $(I-g)^{-1}$ is continuous, and $f(\overline{U})$ is contained in the domain of $(I-g)^{-1}$. Denote by $\Pi := (I-g)^{-1}f$ the map from $\overline{U}$ into X. Since $(I-g)^{-1}$ is continuous and f is ws-compact, we conclude that Π is ws-compact. Using (7.33), we can write

$$\omega(\Pi(\overline{U})) \leq \omega(f(\overline{U})) + \omega(g(I-g)^{-1}f(\overline{U})) = \omega(g\Pi(\overline{U})) < \omega(\Pi(\overline{U})),$$

which is a contradiction. So, we have $\omega((I-g)^{-1}f(\overline{U})) = 0$, and therefore $(I-g)^{-1}f(\overline{U})$ is relatively weakly compact.

To complete the proof it suffices to apply Theorem 7.11.3 with $p = 0$ to Π and to argue as in the last part of the proof of Theorem 7.12.17. ☐

7.12.3 Krasnosel'skii–Schaefer-type fixed point theorems

The goal of this section is to present some fixed point theorems of Schaefer type involving two operators.

Theorem 7.12.20. *Let* X *be a Banach space, and let* $f, g : X \to X$ *be two continuous mappings. Suppose*

(a) f *maps bounded sets into relatively weakly compact sets,*

(b) f *is ws-compact,*

(c) g *is nonexpansive and* ω*-condensing,*

(d) $I - g$ *is* ϕ*-expansive where* ϕ *is either strictly increasing or* $\lim_{r\to+\infty} \phi(r) = +\infty$.

Then

(i) *either* $\mathrm{Fix}(f+g) \neq \emptyset$, *or*

(ii) *the set* $\{x \in X : x = \lambda f(x) + \lambda g(x/\lambda)\}$ *is unbounded for* $\lambda \in (0,1)$.

Proof. As in the proof of Theorem 7.12.6, we need to prove the existence of a fixed point for the operator $(I-g)^{-1}f$. Arguing as in the proof of Theorem 7.12.6, we show that $I-g$ is invertible. To see that the domain of $(I-g)^{-1}$ contains the range of f, we show that $D((I-g)^{-1}) = X$. This is equivalent to $(I-g)(X) = X$. Since g is nonexpansive, $I-g$ is accretive and continuous, and its domain is X. By Theorem 4.4.3, $I - g$ is m-accretive. Because $I-g$ is ϕ-expansive, using Theorem 6.8.1, we conclude that $I-g$ is surjective, and hence $(I-g)(X) = X$. Consequently, $(I-g)^{-1}f : X \to X$ is well defined. To complete the proof, we only have to check that the map $(I-g)^{-1}f$ satisfies the hypotheses of Theorem 7.9.1. It is clear that the operator $(I-g)^{-1}f$ is continuous.

We claim that $(I-g)^{-1}f$ is weakly compact. Let us first check that, for each $S \in \mathbf{B}(X)$, we have $(I-g)^{-1}f(S) \in \mathbf{B}(X)$. To see this, let $S \in \mathbf{B}(X)$ and $x, y \in (I-g)^{-1}f(S)$. There exist $z_1, z_2 \in S$ such that $x = (I-g)^{-1}f(z_1), y = (I-g)^{-1}f(z_2)$ and then $x - g(x) = f(z_1)$, $y - g(y) = f(z_2)$. Since $I - g$ is ϕ-expansive, we can write

$$\begin{aligned}\phi(\|x-y\|) &\le \|(x-g(x))-(y-g(y))\| \\ &= \|f(z_1)-f(z_2)\| \\ &\le \operatorname{diam}(f(S)) < +\infty.\end{aligned}$$

If $(I-g)^{-1}f(S)$ is not a bounded set, there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $(I-g)^{-1}f(S)$ such that $\|x_n - y_n\| \to +\infty$ as $n \to +\infty$. Hence,

$$\phi(\|x_n-y_n\|) \le \operatorname{diam}(f(S)).$$

If ϕ is such that $\lim_{r\to+\infty}\phi(r) = +\infty$, then necessarily $\operatorname{diam}(f(S)) = +\infty$, which is a contradiction.

Otherwise, if ϕ is strictly increasing, then ϕ has an inverse ϕ^{-1} on $[0,+\infty[$, which is strictly increasing as well. Then

$$\|x_n-y_n\| \le \phi^{-1}(\operatorname{diam}(f(S))) < +\infty,$$

which gives another contradiction. Hence, in any case, the set $(I-g)^{-1}f(S)$ is a bounded.

Suppose that $(I-g)^{-1} \circ f(S)$ is not relatively weakly compact. Using (7.33) and the properties of $\omega(\cdot)$, we get

$$\begin{aligned}\omega((I-g)^{-1}f(S)) &= \omega(f(S)+g(I-g)^{-1}f(S)) \\ &\le \omega(f(S)) + \omega(g\circ(I-g)^{-1}f(S)) \\ &= \omega(g(I-g)^{-1}f(S)) \\ &< \omega((I-g)^{-1}f(S)),\end{aligned}$$

which is a contradiction. Hence $\omega((I-g)^{-1}f(S)) = 0$ and therefore the set $(I-g)^{-1}f(S)$ is relatively weakly compact. So, $(I-g)^{-1}f$ maps bounded sets into relatively weakly compact ones, that is, $(I-g)^{-1}f$ is weakly compact, which proves our claim.

Moreover, using of the same arguments as in the proof of Theorem 7.12.6, we see that the operator $(I-g)^{-1}f$ is ws-compact.

Hence, $(I-g)^{-1}f$ satisfies the hypotheses of Theorem 7.9.1, so we have
(i) either $(I-g)^{-1}f$ has a fixed point in X, or
(ii) the set $\{x \in X : x = \lambda(I-g)^{-1}f\}$ is unbounded for $\lambda \in (0,1)$.

It is clear that, if assertion (i) is satisfied, then $f+g$ has a fixed point in X. Conversely, if assertion (ii) is satisfied, then, since the equation $x = \lambda(I-g)^{-1}f$ is equivalent to $x =$

$\lambda f(x) + \lambda g(x/\lambda)$, we conclude that $\{x \in X : x = \lambda f(x) + \lambda g(x/\lambda)\}$ is unbounded for $\lambda \in (0,1)$. ☐

Remark 7.12.8. In assumption (d) of Theorem 7.12.20, we have imposed that ϕ is either strictly increasing or that $\lim_{r\to+\infty} \phi(r) = +\infty$ because otherwise we cannot guarantee the boundedness of the set $(I - g)^{-1} f(S)$ whenever S is a bounded subset of X.

Corollary 7.12.10. *Let X be a Banach space, and let $f, g : X \to X$ be two continuous mappings. Suppose*

(a) *f maps bounded sets into relatively weakly compact sets,*
(b) *f is ws-compact,*
(c) *g is a ww-compact separate contraction.*

Then

(i) *either* Fix$(f + g) \neq \emptyset$, *or*
(ii) *the set $\{x \in X : x = \lambda f(x) + \lambda g(\frac{x}{\lambda})\}$ is unbounded for $\lambda \in (0,1)$.*

This corollary is a consequence of Theorem 7.12.20 and Proposition 7.8.2(c) since separate contractions are, in particular, nonlinear contractions. Moreover, if they are ww-compact, then, by Proposition 7.8.2(c), they are ω-condensing.

Theorem 7.12.21. *Let X be a Banach space, and let $f, g : X \to X$ be two continuous mappings. Suppose*

(a) *f is ws-compact and ω-k-contractive for some $k \in [0,1)$,*
(b) *g is a nonlinear contraction with Φ-function ϕ such that, for all $r > 0$, $\phi(r) < (1 - k)r$ and $\lim_{r\to+\infty}(r - \phi(r)) = +\infty$,*
(c) *g is ww-compact.*

Then

(i) *either* Fix$(f + g) \neq \emptyset$, *or*
(ii) *the set $\{x \in X : x = \lambda f(x) + \lambda g(\frac{x}{\lambda})\}$ is unbounded for $\lambda \in (0,1)$.*

Proof. We know that, under the hypotheses of the theorem, the map $(I - g)^{-1}$ is well defined and continuous from X into itself. Further, the continuity of $(I - g)^{-1}$ and the fact that f is ws-compact imply that $\Pi = (I - g)^{-1} f$ is ws-compact.

We claim that Π maps $\mathbf{B}(X)$ into itself. Indeed, for any set $S \in \mathbf{B}(X)$ and $u, v \in (I - g)^{-1} f(S)$, there exist x, y in S such that $u = (I - g)^{-1} f(x)$ and $v = (I - g)^{-1} f(y)$, or again $u - g(u) = f(x)$ and $v - g(v) = f(y)$. Since S bounded, assumption (b) allows writing

$$\|u - v\| - \phi(\|u - v\|) \leq \|f(x) - f(y)\| \leq \operatorname{diam}(f(S)) < +\infty.$$

If $\Pi(S)$ is not bounded, then there exist two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} \|u_n - v_n\| = +\infty.$$

Using again assumption (b), we obtain $\|u_n - v_n\| - \varphi(\|u_n - v_n\|) \to +\infty$. This contradicts the fact that $\operatorname{diam}(S) < +\infty$. This proves our claim.

Let $S \in \mathbf{B}(X)$ be such that $\omega(S) > 0$. Using (7.33), we obtain $\Pi(S) \subseteq f(S) + g(\Pi(S))$. Moreover, Proposition 7.8.2(b) implies that $\omega(g(\Pi(S))) \leq \phi(\omega(\Pi(S)))$. Using the fact f is ω-k-contractive, we obtain

$$\omega(\Pi(S)) \leq \omega(f(S)) + \omega(g\Pi(S)) \leq k\omega(S) + \phi(\omega(\Pi(S))) < k\omega(S) + (1-k)\omega(\Pi(S)).$$

Hence $\omega(\Pi(S)) < \omega(S)$, and therefore Π is ω-condensing.

To complete the proof, it suffices to apply Theorem 7.9.1 to the map Π and to argue as at the end of the proof of Theorem 7.12.20. ☐

Theorem 7.12.22. *Let X be a Banach space and let $f, g : X \to X$ be two continuous mappings. Suppose*

(a) *g is pseudocontractive and $I - g$ is ϕ-expansive where ψ is either strictly increasing or* $\lim_{r\to+\infty} \phi(r) = +\infty$,

(b) *g is a ω-k-contractive for some $k \in [0,1)$,*

(c) *f is a ω-s- contractive for some $s \in [0, 1-k)$,*

(d) *f is ws-compact.*

Then

(i) *either* $\operatorname{Fix}(f+g) \neq \emptyset$, *or*

(ii) *the set $\{x \in X : x = \lambda f(x) + \lambda g(x/\lambda)\}$ is unbounded for some $\lambda \in (0,1)$.*

Proof. Arguing as in the proof Theorem 7.12.11, we show that $(I-g)^{-1}$ is well defined and continuous from X into itself. Since f is ws-compact, the map $(I-g)^{-1}f$ is also ws-compact. Let S be bounded subset of X such that $\omega(S) > 0$. Using (7.33), we can write

$$\begin{aligned}\omega((I-g)^{-1}f(S)) &\leq \omega(f(S)) + \omega(g(I-g)^{-1}f(S)) \\ &\leq s\omega(S) + k\omega((I-g)^{-1}f(S)),\end{aligned}$$

and therefore $\omega((I-g)^{-1}f(S)) \leq \frac{s}{1-k}\omega(S)$. Since $s < 1-k$, we conclude that $(I-g)^{-1}f$ is ω-condensing. Finally, applying Theorem 7.9.1 to the operator $(I-g)^{-1}f$ and arguing as at the end of the proof Theorem 7.12.20, we derive the desired result. ☐

Theorem 7.12.23. *Let X be a Banach space, and let f and g be two mappings from X into itself. Suppose*

(a) *f maps bounded sets into relatively weakly compact sets,*

(b) *f is ws-compact,*

(c) *g is pseudocontractive and ω-condensing,*

(d) *$I - g$ is ϕ-expansive where ϕ is either strictly increasing or* $\lim_{r\to+\infty} \phi(r) = +\infty$.

Then

(i) *either* $\mathrm{Fix}(f+g) \neq \emptyset$, *or*
(ii) *the set* $\{x \in X : x = \lambda f(x) + \lambda g(x/\lambda)\}$ *is unbounded for some* $\lambda \in (0,1)$.

Proof. Following the proof of Theorem 7.12.22, the map $\Pi := (I-g)^{-1}f : X \to X$ is well defined and ws-compact. Let S be a bounded subset of X such that $\omega(S) > 0$. As in the proof of the preceding theorem, the use of (7.33) implies

$$\omega(\Pi(S)) \leq \omega(f(S)) + \omega(g\Pi(S)) = \omega(g\Pi(S)) < \omega(\Pi(S)),$$

which is a contradiction. Hence $\omega(\Pi(S)) = 0$, and therefore the set $\Pi(S)$ is relatively weakly compact. To conclude the proof, it suffices to apply Theorem 7.9.1 to the map Π and argue as at the end of the proof of Theorem 7.12.20 to derive the desired result. □

7.13 Bibliographical notes

The results of Section 7.2 are classic and are related to the fixed point theorems of L. E. J. Brouwer and J. Schauder. For other proofs of the Brouwer fixed point theorem, we refer, for example, to the book by K. Goebel and W. A. Kirk [115] or T. Stuckless's thesis [222]. Regarding Schauder fixed point theorem, besides this manuscript, the reader can find some detailed presentations in the books [3, 115, 141] or [244].

Section 7.3 deals with set contractive mappings. We present an axiomatic definition of the measure of noncompactness and give as example the Kuratowski measure of noncompactness [148] and the Hausdorff measure of noncompactness [118]. We present further the notion of a Lipschitz mapping with respect to a measure of noncompactness and give some results of Darbo–Sadovskii type. Theorem 7.3.1 is due to G. Darbo [73]. Proposition 7.3.8 is due to D. Ariza-Ruiz, J. Garcia-Falset [22], and it generalizes Proposition 7.3.7 [183]. Proposition 7.3.8 was used by D. Ariza-Ruiz and J. Garcia-Falset to establish Theorems 7.3.2, 7.3.3, and 7.3.4, which provide extensions of Darbo fixed point theorem [22]. Theorem 7.3.5, due to V. N. Sadovskii [210], is a generalization of Darbo fixed point theorem. The concept of μ-quasicondensing maps is introduced in [22], and it is used to establish Theorems 7.3.6 and 7.3.7 [22] which generalize Sadovskii fixed point theorem.

In Section 7.4 we give some fixed point results for mappings f such that $I-f$ is ϕ-expansive. Most of these results come from the paper [108].

The results of Section 7.5 are classical and deal with Tychonoff fixed point theorem (generalization of Schauder fixed point theorem to Hausdorff locally convex topological vector spaces). Proposition 7.5.1 may be found in [68]. Based on this result, S. Cobzaş [68] established Corollary 6.3.2, which is a more general form of the Brouwer fixed point theorem. Theorem 7.5.1 is due to Tychonoff [230]. To prove Tychonoff theorem, we followed the exposition of S. Cobzaş [68]. Excellent proofs and a lot of information concerning this result may be found in the book by N. Dunford and J. T. Schwartz [89, Section V.10] and in Chapter 3 of the book by R. E. Edwards [92]. Proposition 7.5.2 is due to D. O'Regan [193]. It shows that in metrizable, locally convex topological vector spaces there is an equivalence between weakly sequentially continuous mappings and weakly continuous maps. Based on Proposition 7.5.2 and Eberlein–Šmulian theorem, O. Arino, S. Gautier, and J. P. Penot [21] established Theorem 7.5.2, which is the formulation of Tychonoff fixed point theorem for weakly sequentially continuous mappings. Theorem 7.6.1 is the Tychonof-type theorem for ws-compact mappings, which was obtained in [159], and Theorem 7.6.2 is due to J. Garcia-Falset [99]. For the axiomatic definition of the measure of weak noncompactness, the reader is referred, for example, to [26, 28, 29].

In Section 7.6 we discuss fixed points results involving ws-compact an ww-compact mappings. These two classes of operators were introduced in [159]. Theorem 7.6.1, established in [159], is a generalization of Schauder fixed point theorem to ws-compact operators. Theorem 7.6.2, established in [99], is a sharpening of Theorem 7.6.1. Theorem 7.6.3 may be found in [236]. Theorem 7.6.4 and its corollaries using ws-compact an ww-compact operators were established in [1].

Section 7.8 deals with the axiomatic definition of the measure of weak noncompactness on Banach spaces and mappings which are contractive with respect to a measure of weak noncompactness. The measures of weak noncompactness of F. S. De Blasi [75] (see also [93]) and that of J. Banaś and Z. Knap [27] were introduced as examples of measures of weak noncompactness. Proposition 7.8.1 is a rewriting of Darbo and Sadovskii theorems in a Banach space equipped with its weak topology. Theorem 7.8.1 is the analogue of Proposition 7.8.1 for weakly sequentially continuous mappings. Assertion (a) is from D. O'Regan [189] and assertion (b) was obtained by J. Garcia-Falset and K. Latrach in [101]. Theorem 7.8.2 is the analogue of Theorem 7.8.1 for ws-compact operators. Assertion (a) was obtained in [159] and assertion (b) was taken from [103]. Theorem 7.8.4 was taken from J. Garcia-Falset [98]. Theorem 7.12.16 and its corollaries were taken from [1]. Theorems 7.8.4 and 7.8.5 were obtained respectively in [98] and [108]. Theorem 7.8.6 is a fixed point theorem on WC-Banach algebras, and is taken from [30].

In Section 7.9, we present two results of Schaefer type for ws-compact mappings. Our results are from [5].

In Section 7.10 we give a fixed point result for weak power condensing mappings with respect to a measure of weak noncompactness. Our results are from [5].

In Section 7.11, we give some nonlinear alternatives of Leray–Schauder type involving the weak topology. Theorem 7.11.2 is due to J. Garcia-Falset and K. Latrach [101], and Theorem 7.11.3 may be found in [83], while Theorem 7.11.4 is from [5].

In Section 7.12, we present numerous fixed point theorems of Krasnosel'skii type using the weak topology for various kinds of perturbation in bounded and unbounded subsets of Banach spaces. To write this section, we used, among others, the following works [1, 30, 35, 55, 101, 103, 158, 159, 228, 235].

8 Solvability of nonlinear boundary value problems

This chapter aims to present some examples of nonlinear operational equations to illustrate the field of applications of the results presented in Chapters 6 and 7. It seems useful to propose a complete study of some nonlinear integral or integro-differential equations such as Hammerstein's equations, equations arising in neutron transport equations, or other types of equations which motivated a lot of abstract results on fixed point theory (see, for example, [4, 12, 13, 41, 55, 83, 98, 101–103, 109, 156, 158, 159, 235]). Although an abundant literature concerning these equations exists (dispersed in various sources), the examples constituting this chapter are recent.

8.1 A Dirichlet problem

Let us assume that Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We will further assume that

$$\rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}), \quad \rho(0) = 0,$$

and there exist $C > 0$ and $\gamma \in \mathbb{R}^+$ with $\gamma > 1$ such that

$$\rho'(r) \geq C|r|^{\gamma-1} \quad \text{for each } r \in \mathbb{R} \setminus \{0\}.$$

On the other hand, consider a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $|f(s,x)| \leq a(s) + b|x|$, where $a \in L^1(\Omega)$ and $b \geq 0$. Then, according to Theorem 1.15.1, its superposition operator $\mathcal{N}_f : L^1(\Omega) \to L^1(\Omega)$ is continuous. We study the existence of solutions in $L^1(\Omega)$ for the *Dirichlet problem*

$$\begin{cases} \Delta\rho(u(x)) - \lambda u(x) = f(x, u(x)), & x \in \Omega, \\ \rho(u(x)) = 0, & x \in \partial\Omega. \end{cases} \tag{8.1}$$

Remark 8.1.1. The above Ω denotes an open bounded domain in $\mathbb{R}^n$. We consider the differential operator

$$Lu = -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i}\right) + \sum_i \frac{\partial}{\partial x_i}(a_i u) + au,$$

where $a_{ij}, a_i \in C^1(\overline{\Omega})$, $a \in L^\infty(\Omega)$,

$$a \geq 0, \quad a + \sum_i \frac{\partial a_i}{\partial x_i} \geq 0 \quad \text{a. e.},$$

and for some positive constant α,

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$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{a. e. } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

If we take

$$D(A) = \{u \in W_0^{1,1}(\Omega) : Lu \in L^1(\Omega)\}$$

where Lu is understood in the sense of distributions and define $Au = Lu$ for $u \in D(A)$, then A is an m-accretive operator in $L^1(\Omega)$ and ϕ-expansive, since there exists $D > 0$ such that

$$D\|u\|_{1,1} \leq \|Au\|_1 \tag{8.2}$$

for each $u \in D(A)$ (cf. [49, Theorem 8]). Hence one can take $\phi(t) = Dt$. Now, by using Remark 6.8.1, we can conclude that the operator $A^{-1} : L^1(\Omega) \to D(A)$ exists and is continuous. Thus, by Proposition 6.8.2, we can guarantee that A^{-1} is m-accretive.

Theorem 8.1.1. *For each $\lambda > 0$, Problem (8.1) has a solution in $L^1(\Omega)$.*

Proof. Following [32, p. 117] (see also [49]), the operator

$$\begin{aligned} &D(P) = \{u \in L^1(\Omega) : \rho(u) \in W_0^{1,1}(\Omega), \Delta\rho(u) \in L^1(\Omega)\},\\ &P(u) = \Delta\rho(u), \quad u \in D(P), \end{aligned}$$

is m-dissipative. Now, let us consider the following operator:

$$\begin{aligned} &D(Q) = \{u \in W_0^{1,1}(\Omega) : \Delta u \in L^1(\Omega)\},\\ &Q(u) = \Delta u, \quad u \in D(Q). \end{aligned}$$

From Example 8.1.1 we know that Q is m-dissipative, ϕ-expansive, and therefore its inverse operator

$$Q^{-1} : L^1(\Omega) \to D(Q)$$

exists and is continuous.

We claim that the superposition operator defined by

$$S : L^1(\Omega) \to L^1(\Omega), \quad u \mapsto S(u)(x) := \rho^{-1}(u(x)),$$

is well defined and continuous. Let us observe that $\rho^{-1} \in \mathcal{C}(\mathbb{R})$, and there exist $k_0 > 0$ and $k_1 \in \mathbb{R}$ such that

$$|\rho^{-1}(v)| \leq k_0|v|^{\frac{1}{\gamma}} + k_1,$$

for each $v \in \mathbb{R}$. This, along with the facts that $\gamma > 1$ and Ω is bounded, allows us to obtain, by Hölder inequality, that if $u \in L^1(\Omega)$, then $S(u) \in L^1(\Omega)$. This shows that S is well defined.

In order to obtain the continuity of S, we argue as follows. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\Omega)$ which converges to a vector u in $L^1(\Omega)$. We have to prove that $\lim_{n\to+\infty} S(u_n) = S(u)$. We know that, given a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$, there exists a subsequence $(u_{n_{k_j}})_{j\in\mathbb{N}}$ of $(u_{n_k})_{k\in\mathbb{N}}$ such that
(a) $u_{k_s}(x) \to u(x)$ a. e. on Ω,
(b) $|u_{k_s}(x)| \le h(x)$ for all $s \in \mathbb{N}$ and a. e. on Ω with $h \in L^1(\Omega)$.

Because $\rho^{-1} \in \mathcal{C}(\mathbb{R})$, it is clear that

$$\rho^{-1}(u_{k_s}(x)) - \rho^{-1}(u(x)) \to 0 \quad \text{a. e. on } \Omega.$$

On the other hand,

$$\left|\rho^{-1}(u_{k_s}(x))\right| \le k_0 \left|u_{k_s}(x)\right|^{\frac{1}{\gamma}} + k_1 \le k_0 \left|h(x)\right|^{\frac{1}{\gamma}} + k_1 \quad \text{a. e. on } \Omega.$$

By Hölder inequality, the right-hand side of the above inequality is an integrable function, hence, applying the Lebesgue dominated convergence theorem, we conclude the continuity of S. As a consequence of the above facts, we may introduce an operator T defined by

$$T : L^1(\Omega) \to L^1(\Omega), \quad u \mapsto T(u) = S\big(Q^{-1}(u)\big).$$

Now, we will see that $T(u) \in D(P)$ for every $u \in L^1(\Omega)$. Indeed, we know that $T(u) \in L^1(\Omega)$ and have $\rho(T(u)) = Q^{-1}(u) \in D(Q)$. Hence

$$\rho\big(T(u)\big) \in W_0^{1,1}(\Omega) \quad \text{and} \quad \Delta\rho\big(T(u)\big) \in L^1(\Omega),$$

i. e., $T(u) \in D(P)$.

The above argument says that T is the inverse operator, in $L^1(\Omega)$, of P. By Proposition 6.8.2, we may assert that T is m-dissipative on $L^1(\Omega)$.

The operator T is ws-compact. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence of $L^1(\Omega)$, from (8.2), we deduce that $(Q^{-1}(x_n))_{n\in\mathbb{N}}$ is a bounded sequence of $W^{1,1}(\Omega)$. Because the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, the set $\{Q^{-1}(x_n) : n \in \mathbb{N}\}$ is relatively compact in $L^1(\Omega)$. Thus the sequence $(Q^{-1}(x_n))_{n\in\mathbb{N}}$ has a strongly convergent subsequence $(Q^{-1}(x_{n_k}))_{k\in\mathbb{N}}$. Using the continuity of S, we conclude that $(T(x_{n_k}))_{k\in\mathbb{N}}$ is a strongly convergent sequence in $L^1(\Omega)$. This shows that T is ws-compact.

To find a solution to problem (8.1), it suffices to show that the operator defined by

$$K : L^1(\Omega) \to L^1(\Omega), \quad u \mapsto K(u) := \lambda T(u) + N_f\big(T(u)\big)$$

has a fixed point since if u is a fixed point of K, then $v := T(u)$ is a solution of problem (8.1). We may argue as follows. Since $\mathcal{N}_f$ is a continuous mapping, we infer that $N_f \circ T$ is a ws-compact operator. It transforms bounded sets into relatively compact sets. Accordingly, the operator $K : L^1(\Omega) \to L^1(\Omega)$ is continuous and ws-compact, and so transforms bounded sets into relatively compact sets.

In order to conclude, via Theorem 7.6.2, it is enough to check that there exists $R > 0$ such that $K(\mathbb{S}_R) \subseteq \mathbb{B}_R$. Indeed, since $|\rho^{-1}(v)| \le k_0|v|^{\frac{1}{\gamma}} + k_1$, for each $v \in \mathbb{R}$, we get

$$\left|T(u(x))\right| \le k_0\left|Q^{-1}(u(x))\right|^{1/\gamma} + k_1,$$

for every $u \in L^1(\Omega)$. Therefore, by using Hölder and (8.2) inequalieties, it is not difficult to see that there exist $M, s \ge 0$ such that, for every $u \in L^1(\Omega)$, the inequality

$$\|\lambda Tu\|_1 \le M\|u\|_1^{1/\gamma} + s$$

holds. Next, using the fact that $|f(s,x)| \le a(s) + b|x|$, we get that, for every $u \in L^1(\Omega)$, there exist $L_1, L_2 > 0$ satisfying

$$\left\|N_f(Tu)\right\|_1 \le \|a\|_1 + b\left\|T(u)\right\|_1 \le \|a\|_1 + L_1\|u\|_1^{1/\gamma} + L_2.$$

If we assume that $\|u\| = R$, since

$$\lim_{R\to\infty} \frac{\|a\|_1 + (L_1 + M)R^{1/\gamma} + L_2 + s}{R^{1/\gamma}} = L_1 + M \quad \text{and} \quad \lim_{R\to\infty} R^{1-1/\gamma} = +\infty,$$

we may find $R_0 > 0$ such that

$$\frac{\|a\|_1 + (L_1 + M)R_0^{1/\gamma} + L_2 + s}{R_0^{1/\gamma}} < R_0^{1-1/\gamma}$$

and, consequently, $\|a\|_1 + (L_1 + M)R_0^{\frac{1}{\gamma}} + L_2 + s < R_0$. The preceding calculations with $R = R_0$ show that $K(\mathbb{S}_{R_0}) \subseteq \mathbb{B}_{R_0}$, which ends the proof. ☐

8.2 Existence results for a nonlinear functional integral equation

In the present section, we are concerned with the solvability of the following quite general nonlinear functional integral equation:

$$\zeta(t) = f\left(t, \zeta(t), \int_\Omega \kappa(t,s)u(s,\zeta(s))\,ds\right), \quad t \in \Omega, \tag{8.3}$$

in $L^1(\Omega, X)$, the space of Lebesgue integrable functions on a measurable subset Ω of $\mathbb{R}^n$ with values in X. Here, $f : \Omega\times X\times X \to X$ and $u : \Omega\times X \to Y$ are given nonlinear functions, while X and Y are two finite-dimensional Banach spaces. The kernel κ is measurable on $\Omega \times \Omega$ and such that, for each $t \in \Omega$, the function $s \to \kappa(t,s)$ belongs to L^∞, and the Hammerstein integral operator K generated by the kernel $\kappa(\cdot,\cdot)$ is continuous from $L^1(\Omega, Y)$ into $L^1(\Omega, X)$.

We will assume that the functions involved in equation (8.3) satisfy the following conditions:

(H1) $u : \Omega \times X \to Y$ is a Carathéodory function, and there exist a function $a(\cdot) \in L^1_+(\Omega)$ and a constant $b > 0$ such that $\|u(t,x)\|_Y \leq a(t) + b\|x\|_X$.

(H2) The function $\kappa : \Omega \times \Omega \to \mathcal{L}(Y,X)$ is strongly measurable, where $\mathcal{L}(Y,X)$ refers to the space of linear bounded operators from Y to X.

(H3) For each $t \in \Omega$, the function

$$\rho(t) : \Omega \to \mathcal{L}(Y,X), \quad s \mapsto \rho(t)(s) := \kappa(t,s)$$

belongs to $L^\infty(\Omega, \mathcal{L}(Y,X))$, and the function

$$\rho : \Omega \to \mathcal{L}(Y,X), \quad t \mapsto \rho(t),$$

belongs to $L^1(\Omega, L^\infty(\Omega, \mathcal{L}(Y,X)))$, which we denote (for short) simply by $L^1(\Omega, L^\infty)$.

(H4) $f : \Omega \times X \times X \to X$ is a Carathéodory function, and there exist a function $g(\cdot) \in L^1_+(\Omega)$ and two positive real numbers α, β such that

$$\|f(t,x(t),y(t))\|_X \leq g(t) + \alpha\|x(t)\|_X + \beta\|y(t)\|_X,$$

for any $x, y \in L^1(\Omega, X)$.

(H5) $\alpha + b\beta\|K\| + \|g\| \leq 1$ if $g \neq 0$; otherwise $\alpha + b\beta\|K\| < 1$, where $\|K\|$ denotes the norm of the linear operator K generated by the function $\kappa(\cdot,\cdot)$.

(H6) There exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(r) < r$ for $r > 0$ such that

$$\int_\Omega \|f(t,x_1(t),y_1(t)) - f(t,x_2(t),y_2(t))\|_X \, dt \leq \phi(\|x_1 - x_2\|),$$

for any $x_1, x_2 \in L^1(\Omega, X)$ where $y(t) = \int_\Omega \kappa(t,s)u(s,z(s))\,ds$ with $z(\cdot) \in \mathbb{B}_{r_0}$, and r_0 satisfies

$$r_0 \geq \frac{\|g\| + \beta\|K\|\|a\|}{1 - \alpha - b\beta\|K\|}.$$

We note that equation (8.3) may be written in the abstract form

$$x = F(x, Ax),$$

where F is the superposition operator associated to f ($F = \mathcal{N}_f$):

$$\begin{aligned} &F : L^1(\Omega, X) \times L^1(\Omega, X) \to L^1(\Omega, X), \\ &(x,y) \mapsto F(x,y) : \Omega \to X, \quad F(x,y)(t) = f\big(t, x(t), y(t)\big), \end{aligned}$$

and $A := K \circ \mathcal{N}_u$ appears as the composition of the superposition operator associated to u with the linear operator defined by

$$\begin{aligned} &K : L^1(\Omega, Y) \to L^1(\Omega, X), \\ &\psi \mapsto K\psi : \Omega \to X; \quad K\psi(t) = \int_\Omega \kappa(t,s)\psi(s)\,ds. \end{aligned}$$

Our next task is to prove that the nonautonomous superposition operator $\mathcal{N}_f \circ A$ has a fixed point in $L^1(\Omega, X)$. Before starting to prove the solvability of equation (8.3), we give some remarks.

Remark 8.2.1.

(1) We note that assumptions (H2) and (H3) lead to the estimate

$$\left\| \int_\Omega \kappa(t,s)\phi(s)\,ds \right\|_X \leq \|\rho(t)\|_{L^1(\Omega, \mathcal{L}(\mathcal{Y}, \mathcal{X}))} \cdot \|\phi\|_{L^1(\Omega, Y)},$$

and so

$$\|K\phi\|_{L^1(\Omega,X)} = \int_\Omega \left\| \int_\Omega \kappa(t,s)\phi(s)\,ds \right\|_X dt \leq \|\rho\|_{L^1(\Omega, L^\infty)} \cdot \|\phi\|_{L^1(\Omega,Y)},$$

for any $\phi \in L^1(\Omega, Y)$. This shows that the linear operator K is continuous. Next, using Theorem 1.7.1, we conclude that K is weakly continuous from $L^1(\Omega, Y)$ into $L^1(\Omega, X)$ and that $\|K\| \leq \|\rho\|_{L^1(\Omega, L^\infty)}$.

(2) Considering the space $X \times X$ with the norm $\alpha\|x\|_X + \beta\|y\|_X$ for the product topology and using Theorem 1.15.2, we can see that assumption (H4) implies that the superposition operator $\mathcal{N}_f$ is continuous and maps bounded sets of $L^1(\Omega, X) \times L^1(\Omega, X)$ into bounded sets of $L^1(\Omega, X)$.

Lemma 8.2.1. *Let X, Y be two finite-dimensional Banach spaces and let Ω be a bounded domain in $\mathbb{R}^N$. If $u : \Omega \times X \to Y$ satisfies* (H1)*, then $\mathcal{N}_u$ is a ww-compact operator.*

Proof. According to (H1), for any measurable subset D of Ω, we have

$$\int_D \|u(t, \psi(t))\|_Y = \int_D \|\mathcal{N}_u \psi(t)\|_Y \leq \int_D a(t)\,dt + b \int_D \|\psi(t)\|_X\,dt.$$

This, together with (7.17), leads to

$$\omega(\mathcal{N}_u(S)) \leq b\omega(S) \tag{8.4}$$

for any bounded subset S of $L^1(\Omega; X)$. Next, let $(\psi_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence of $L^1(\Omega; X)$. Using (8.4), together with Proposition 7.7.2(4), we infer that $\omega(\{\mathcal{N}_u(\psi_n) : n \in \mathbb{N}\}) = 0$. This shows that the set $\{\mathcal{N}_u(\psi_n) : n \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega; Y)$, which ends the proof. ☐

Theorem 8.2.1. *Let X and Y be two finite-dimensional Banach spaces, and let Ω be a bounded domain of $\mathbb{R}^n$. If the hypotheses* (H1)–(H6) *are satisfied, then problem* (8.3) *has at least one solution $x(\cdot) \in L^1(\Omega, X)$.*

Proof. Define the operators A and F as follows:

$$(Az)(t) := \int\limits_{\Omega} \kappa(t,s)u(s,z(s))\,ds, \quad F(x,y)(t) := f(t,x(t),y(t)).$$

It is clear that any solution of problem (8.3) is a solution to the fixed point problem $x = F(x, Ax)$, and conversely. So, to solve problem (8.3), we only have to check that the assumptions of Theorem 7.6.3 are all fulfilled.

The proof is divided into several steps:

(i) By Remark 8.2.1, the operators

$$A : L^1(\Omega, X) \to L^1(\Omega, X) \quad \text{and} \quad F : L^1(\Omega, X) \times L^1(\Omega, X) \to L^1(\Omega, X)$$

are well defined and continuous.

Let S be a subset of $L^1(\Omega, X)$ and let $M > 0$ be a real number such that $S \subset \mathbb{B}_M$. For all $\psi \in S$, we have

$$\begin{aligned} \|A\psi(t)\| &\le \|\rho(t)\|_{L^\infty(\Omega,(X,Y))}\|\mathcal{N}_u\psi\|_{L^1(\Omega,Y)} \\ &\le \|\rho(t)\|_{L^\infty(\Omega,(X,Y))}(\|a\| + b\|\psi\|_{L^1(\Omega,X)}) \\ &\le \|\rho(t)\|_{L^\infty(\Omega,(X,Y))}(\|a\| + bM). \end{aligned} \tag{8.5}$$

We claim that the operator A is ws-compact. To see this, let $(\psi_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in $L^1(\Omega, X)$. According to Lemma 8.2.1, $\mathcal{N}_u$ is ww-compact, hence the sequence $(\mathcal{N}_u\psi_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in $L^1(\Omega, X)$, say $(\mathcal{N}_u\psi_{n_k})_{k\in\mathbb{N}}$. Let η be the weak limit of $(\mathcal{N}_u\psi_{n_k})_{k\in\mathbb{N}}$. Bearing in mind the boundedness of the function $\kappa(t,\cdot) = \rho(t)$, we get

$$(A\psi_{n_k})(t) = \int\limits_{\Omega} \kappa(t,s)u(s,\psi_{n_k}(s))\,ds \to \int\limits_{\Omega} \kappa(t,s)\eta(s)\,ds. \tag{8.6}$$

Hence, (8.5) and (8.6) allow us to apply the dominated convergence theorem to conclude that the sequence $(A\psi_{n_k})_{k\in\mathbb{N}}$ converges in $L^1(\Omega, X)$. This proves our claim. Hence assumption (a) of Theorem 7.6.3 is satisfied.

(ii) For any $y \in \overline{A(\mathbb{B}_{r_0})}$, there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{B}_{r_0}$ such that $\lim_{n\to+\infty} A(z_n) = y$. By the continuity of F and (H6), we obtain that

$$\begin{aligned}\|F(x_1,y)-F(x_2,y)\| &= \lim_{n\to+\infty}\|F(x_1,A(z_n))-F(x_2,A(z_n))\|\\ &= \lim_{n\to+\infty}\int_\Omega \|f(t,x_1(t),(Az_n)(t))-f(t,x_2(t),(Az_n)(t))\|_X\,dt\\ &\le \phi(\|x_1-x_2\|),\end{aligned}$$

for all $x_1, x_2 \in L^1(\Omega, X)$.

Hence assumption (b) of Theorem 7.6.3 is satisfied.

(iii) If there exists $x \in L^1(\Omega, X)$ such that $x(t) = f(t, x(t), (Az)(t))$ for $z \in \mathbb{B}_{r_0}$, then by (H4) we have

$$\begin{aligned}f\|(t,x(t),(Az)(t))\|_X &\le g(t)+\alpha\|x(t)\|_X+\beta\|(Az)(t)\|_X\\ &\le g(t)+\alpha\|x(t)\|_X+\beta\|K\|\|\mathcal{N}_u z(t)\|_Y\\ &\le g(t)+\alpha\|x(t)\|_X+\beta\|K\|(a(t)+b\|z(t)\|_X).\end{aligned}$$

It follows that

$$\|x\| = \int_\Omega \|f(t,x(t),(Az)(t))\|_X \le \|g\|+\alpha\|x\|+\beta\|K\|(\|a\|+b\|z\|),$$

which implies

$$\begin{aligned}\|x\| &\le (1-\alpha)^{-1}(\|g\|+\beta\|K\|\|a\|+\beta\|K\|\|z\|)\\ &\le (1-\alpha)^{-1}(\|g\|+\beta\|K\|\|a\|+\beta\|K\|r_0)\\ &\le r_0,\end{aligned}$$

since, by (H6), we have $\|g\| + \beta\|K\|\|a\| \le r_0(1 - \alpha - b\beta\|K\|)$. Thus, we obtain that $x \in \mathbb{B}_{r_0}$.

(iv) Let $\mathcal{P}_0 := \mathbb{B}_{r_0}$, and let

$$\mathcal{P}_n := \overline{\text{co}}\left\{x \in L^1(\Omega,X) : x(t) = f\left(t, x(t), \int_\Omega \kappa(t,s)u(s,z(s))\,ds\right), \quad z \in \mathcal{P}_{n-1}\right\}.$$

It is clear that $\mathcal{P}_n$ $(n = 0, 1, 2, \dots)$ are all nonempty, closed, convex sets, and therefore they are weakly closed. Moreover, we have $\mathcal{P}_1 \subset \mathbb{B}_{r_0} = \mathcal{P}_0$ from step (iii). Hence, by induction, we may infer that $\mathcal{P}_n \subset \mathcal{P}_{n-1}$ for all $n \in \mathbb{N}$.

On the other hand, for each $\varepsilon > 0$ and a nonempty measurable subset $D \subset \Omega$ such that $\text{meas}(D) \le \varepsilon$, we know that for any $z \in \mathcal{P}_{n-1}$ and $x \in \mathcal{P}_n$, if

$$x(t) = f\left(t, x(t), \int_\Omega \kappa(t,s)u(s,z(s))\,ds\right), \quad t \in \Omega,$$

then

$$\begin{aligned}\int_D \|x(t)\|\,dt &= \int_D \Big\|f\Big(t, x(t), \int_\Omega \kappa(t,s)u(s,z(s))\,ds\Big)\,ds\Big\|_X\,dt\\ &\le \int_D \|g(t)\|\,dt + \alpha\int_D \|x(t)\|\,dt + \beta\|K\|\Big(\int_D a(t)\,dt + b\int_D \|z(t)\|\,dt\Big),\\ \int_D \|x(t)\|d &\le \frac{\int_D \|g(t)\|\,dt + \beta\|K\|(\int_D a(t)\,dt + b\int_D \|z(t)\|\,dt)}{1-\alpha}.\end{aligned}$$

Since sets consisting of one element are weakly compact, the use of formula (7.17) leads to

$$\lim_{\varepsilon\to 0}\sup\Big(\int_D \|g(t)\|_X\,dt : \operatorname{meas}(D)\le\varepsilon\Big) = 0$$

and

$$\lim_{\varepsilon\to 0}\sup\Big(\int_D \|a(t)\|_X\,dt : \operatorname{meas}(D)\le\varepsilon\Big) = 0.$$

Consequently, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sup\Big(\int_D \|x(t)\|_X\,dt : \operatorname{meas}(D)\le\varepsilon\Big)\\ &\quad\le (1-\alpha)^{-1}b\beta\|K\|\lim_{\varepsilon\to 0}\sup\Big(\sup_{z\in\mathcal{P}_{n-1}}\int_D \|z(t)\|_X\,dt : \operatorname{meas}(D)\le\varepsilon\Big).\end{aligned}$$

So, plugging into formula (7.17), we conclude that

$$\omega(\mathcal{P}_n)\le\lambda\omega(\mathcal{P}_{n-1})\quad\text{with } \lambda = (1-\alpha)^{-1}b\beta\|K\|.$$

Thus, from (H6) we have that $\lambda < 1$.

Next, using the latter inequality we obtain

$$\omega(\mathcal{P}_n)\le\lambda\omega(\mathcal{P}_{n-1})\le\cdots\le\lambda^n\omega(\mathcal{P}_0)\quad\text{for } n\in\mathbb{N},$$

and therefore $\lim_{n\to+\infty}\omega(\mathcal{P}_n) = 0$ Setting $\mathcal{P} = \bigcap_{n=0}^{+\infty}\mathcal{P}_n$, Proposition 7.7.5 guarantees that $\mathcal{P}$ is a nonempty, weakly compact subset of $L^1(\Omega, X)$. Moreover, we infer that, for any $z\in\mathcal{P}$, if $x = F(x, Az)$ holds, then $x\in\mathcal{P}$. Now assumption (c) of Theorem 7.6.3 is satisfied, which ends the proof. □

8.3 A nonlinear transport equation with delayed neutrons

8.3.1 Introduction

This section deals with the existence of solutions to the following nonlinear boundary value problem which involves a multidimensional transport equation with delayed neutrons in a bounded spatial domain. More precisely, we shall discuss the existence of solutions to the nonlinear boundary value problem

$$\begin{cases} v \cdot \nabla_x f_0(x,v) + \sigma(x,v,f_0(x,v)) + \lambda f_0(x,v) \\ \quad = \int_{\mathbb{R}^N} \kappa_0(x,v,v')\Theta_0(x,v',f_0(x,v'))\,d\mu(v') + \sum_{i=1}^{d} \lambda_i\beta_i(x,v)f_i(x,v), \\ \lambda_i f_i(x,v) = \int_{\mathbb{R}^N} \kappa_i(x,v,v')\Theta_i(x,v',f_0(x,v'))\,d\mu(v'), \quad 1 \le i \le d, \end{cases} \tag{8.7}$$

where $(x,v) \in \mathsf{D} \times \mathsf{V}$. Here D is a smooth open bounded subset of $\mathbb{R}^N$ and $\mu(\cdot)$ is a positive Radon measure on $\mathbb{R}^N$ with support V. The function f_0 represents the neutron density, $f_i(x,v)$ represents the density of the ith group of delayed neutron emitters and $\{\lambda_i, 1 \le i \le d\}$ are radioactive decay constants [168]. The functions $\sigma(\cdot,\cdot,\cdot)$ and $\kappa_i(\cdot,\cdot,\cdot)$ are called, respectively, the collision frequency and the scattering kernel, which are, in general, nonlinear functions of f_0, while $\beta_i(x,v)$, $i = 1,2,\dots,d$, are essentially bounded functions which denote physical parameters related to the ith group of delayed neutrons. The maps Θ_i, $i = 0,1,\dots,d$ are nonlinear functions of the density of neutrons, f_0. The boundary conditions are modeled by

$$f_{0|\Gamma_-} = 0, \tag{8.8}$$

where $f_{0|\Gamma_-}$ is the restriction of f_0 to Γ_-, with Γ_- being the incoming part of the phase space boundary.

For the physical meaning of the problem, we refer to the introduction of Section 5.3 and the reference therein.

8.3.2 Notations and preliminaries

Let D be a smooth, open, bounded subset of $\mathbb{R}^N$ and let μ be a positive Radon measure on $\mathbb{R}^N$ supported by V (we recall that D denotes the set of particle positions and V is the set of admissible velocities). For the reader convenience, we recall the functional setting of the problem which was already introduced in Section 5.3.1.

The boundary of the phase space is written as $\partial\mathsf{D} \times \mathbb{R}^N := \Gamma_- \cup \Gamma_+$ where

$$\Gamma_- = \{(x,v) \in \partial\mathsf{D} \times \mathbb{R}^N : v \cdot \nu_x < 0\} \quad \text{and} \quad \Gamma_+ = \{(x,v) \in \partial\mathsf{D} \times \mathbb{R}^N : v \cdot \nu_x > 0\},$$

where ν_x stands for the outer unit normal vector at $x \in \partial\mathsf{D}$.

In the remainder of this section, for $1 \leq p < +\infty$, we use the notation

$$L^p(\mathsf{D} \times V) := L^p(\mathsf{D} \times V; dx\, d\mu(v)), \quad L^p(\mathsf{V}) := L^p(\mathsf{V}; d\mu(v)).$$

We now introduce the following functional space:

$$\mathsf{W}_p(\mathsf{D}) = \{\psi \in L^p(\mathsf{D} \times V) \text{ such that } v \cdot \nabla_x \psi \in L^p(\mathsf{D} \times V)\}.$$

It is well known (cf. [56, 57, 74] or [121]) that any function in $\mathsf{W}_p(\mathsf{D})$ possesses a trace $\psi_{|\Gamma_\pm}$ on $\Gamma_\pm$ belonging to $L^\pm_{p,\mathrm{loc}}(\Gamma_\pm; |v.v_x| d\gamma_x\, d\mu(v))$, where $d\gamma_x$ denotes the Lebesgue measure on $\partial\mathsf{D}$. In applications, suitable L^p-spaces for traces on $\Gamma_\pm$ are

$$L^\pm_p := L^p(\Gamma_\pm; |v \cdot v_x| d\gamma_x\, d\mu(v)).$$

So, we define the set

$$\widetilde{\mathsf{W}}_p(\mathsf{D}) = \{\psi \in \mathsf{W}_p : \psi_{|\Gamma_-} \in L^-_p\}.$$

It is well known that if $\psi \in \mathsf{W}_p(\mathsf{D})$ and $\psi_{|\Gamma_-} \in L^-_p$, then $\psi_{|\Gamma_+} \in L^+_p$, and vice versa [56, 57, 121]. More precisely, we have the identity

$$\widetilde{\mathsf{W}}_p(\mathsf{D}) = \{\psi \in \mathsf{W}_p : \psi_{|\Gamma_-} \in L^-_p\} = \{\psi \in \mathsf{W}_p : \psi_{|\Gamma_+} \in L^+_p\}.$$

Define the free streaming operator T

$$\begin{cases} T : D(T) \subseteq L^p(\mathsf{D} \times V) \to L^p(\mathsf{D} \times V), \\ T\psi(x, v) = -v \cdot \nabla_x \psi(x, v) \end{cases}$$

with domain

$$D(T) = \{f \in \widetilde{\mathsf{W}}_p(\mathsf{D}) \text{ such that } \psi_{|\Gamma_-} = 0\}.$$

In the rest of this section, we will use the following notation. For any real number τ, we set

$$\mathbb{C}_\tau = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda > \tau\}.$$

Let X be a Banach space and let X^* be its topological dual. We recall that the normalized duality map of X (see Section 1.10) is the mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 \text{ with } \|x^*\| = \|x\|\}.$$

A linear operator A with domain and range both in X is called dissipative if, for every $x \in D(A)$, there exists $x^* \in J(x)$ such that $\operatorname{Re}\langle Ax, x^*\rangle \leq 0$. In other words, the operator A is dissipative if and only if $-A$ is accretive.

Let $(\Omega, \sum, \mu)$ be a measure space and consider the Lebesgue space $L^1(\Omega, \sum, d\mu)$. It is well known that, for $\psi \in L^1(\Omega, \sum, d\mu)$, we have

$$\|\psi\|_1 \operatorname{sgn}(\psi) \in J(\psi).$$

So, to prove the dissipativity of the operator A (defined on $L^1(\Omega, \sum, d\mu)$), it suffices to show that $\operatorname{Re}\langle A\psi, \operatorname{sgn}(\psi)\rangle \le 0$ for all $\psi \in D(A)$ where

$$\operatorname{sgn}(\psi)(x) = \begin{cases} 1 & \text{when } \psi(x) > 0, \\ 0 & \text{when } \psi(x) = 0, \\ -1 & \text{when } \psi(x) < 0. \end{cases}$$

The following observation is required below. For any $\psi \in D(A)$, we have

$$\begin{aligned}\langle \|\psi\|_1 \operatorname{sgn}(\psi), \psi\rangle &= \int_\Omega \|\psi\|_1 \operatorname{sgn}(\psi)(x)\psi(x)\,dx = \|\psi\|_1 \int_\Omega \operatorname{sgn}(\psi)(x)\psi(x)\,dx \\ &= \|\psi\|_1 \int_\Omega |\psi(x)|\,dx = \|\psi\|_1^2.\end{aligned}$$

Hence, for any $\psi \in D(A)$, we have

$$\|\psi\|_1^2 = \langle \|\psi\|_1 \operatorname{sgn}(\psi), \psi\rangle.$$

Lemma 8.3.1. *Let* $1 \le p < +\infty$. *Then, for any* $\lambda \in \mathbb{C}_0$, *we have*

$$\|(\lambda - T)^{-1}\|_{L^p(\mathrm{D}\times\mathrm{V})} \le \frac{1}{\operatorname{Re}\lambda}.$$

Proof. We shall first show that T is dissipative on X_p with $1 < p < +\infty$. To do so, consider $\psi \in D(T)$, and then

$$\langle T\psi, |\psi|^{p-2}\psi\rangle = \int_{\mathrm{D}\times\mathrm{V}} |\psi|^{p-2}\psi(-v \cdot \nabla_x \psi)(x, v)\,dx\,d\mu(v).$$

Taking into account the fact that $v \cdot \nabla_x(|\psi|^p) = p|\psi|^{p-2}\psi v \cdot \nabla_x \psi$, we may write

$$\begin{aligned}\langle T\psi, |\psi|^{p-2}\psi\rangle &= -\frac{1}{p} \int_{\mathrm{D}\times\mathrm{V}} v \cdot \nabla_x(|\psi|^p)\,dx\,d\mu(v) \\ &= -\frac{1}{p} \int_{\partial\mathrm{D}\times\mathrm{V}} |\psi|^p v \cdot v_x\,d\gamma_x\,d\mu(v) \\ &= \frac{1}{p}[\|\psi_{\Gamma_-}\|_{L^{p,-}} - \|\psi_{\Gamma_+}\|_{L^{p,+}}] = -\frac{1}{p}\|\psi_{\Gamma_+}\|_{L^{p,+}} \le 0.\end{aligned}$$

Consider now the case $p = 1$. We note that

$$\begin{aligned}\langle T\psi, \operatorname{sgn}(\psi)\rangle &= -\int\limits_{\mathsf{D}\times\mathsf{V}} v\nabla_x \psi(x,v)\operatorname{sgn}(\psi)(x,v)\,dx\,d\mu(v)\\ &= -\int\limits_{\mathsf{D}\times\mathsf{V}} v\nabla_x\big(|\psi(x,v)|\big)\,dx\,d\mu(v)\\ &= -\int\limits_{\partial\mathsf{D}\times\mathsf{V}} |\psi| v v_x\,d\gamma(x)\,d\mu(v)\\ &= \|\psi_{\Gamma_-}\|_{L^{1,-}} - \|\psi_{\Gamma_+}\|_{L^{1,+}}\\ &= -\|\psi_{\Gamma_+}\|_{L^{1,+}} \le 0.\end{aligned}$$

Hence, T is dissipative on $L^p(\mathsf{D}\times\mathsf{V})$ for $1 \le p < +\infty$.

Let $\psi \in D(T_H)$ and set $\varphi = \lambda\psi - T_H\psi$. Using the equality $\operatorname{Re}\lambda\|\psi\|^2 = \operatorname{Re}[\lambda\langle\psi,\psi^*\rangle]$, we see that

$$\begin{aligned}\operatorname{Re}\lambda\|\psi\|^2_{L^p(\mathsf{D}\times\mathsf{V})} &\le \operatorname{Re}\big[\langle\lambda\psi,\psi^*\rangle\big] \le \operatorname{Re}\big[\langle\lambda\psi,\psi^*\rangle\big] - \langle\widetilde{T}_H\psi,\psi^*\rangle\\ &= \operatorname{Re}\langle\lambda\psi - T_H\psi,\psi^*\rangle\\ &\le \|\varphi\|_{L^p(\mathsf{D}\times\mathsf{V})}\|\psi\|_{L^p(\mathsf{D}\times\mathsf{V})}.\end{aligned}$$

This shows that, if $\lambda \in \mathbb{C}_0$, then $\|\psi\|_{L^p(\mathsf{D}\times\mathsf{V})} \le \frac{\|\varphi\|_{L^p(\mathsf{D}\times\mathsf{V})}}{\operatorname{Re}\lambda}$ and therefore

$$\big\|(\lambda - T)^{-1}\big\|_{\mathcal{L}(L^p(\mathsf{D}\times\mathsf{V}))} \le \frac{1}{\operatorname{Re}\lambda}. \qquad \square$$

Before going further, let us define the dual operator of T. By standard duality arguments, we see that T^* is given by

$$\begin{cases} T^* : D(T^*) \subseteq L^q(\mathsf{D}\times V) \to L^q(\mathsf{D}\times V),\\ T^*\psi(x,v) = v\cdot\nabla_x\psi(x,v)\end{cases}$$

with domain

$$D(T^*) = \{f \in \widetilde{\mathsf{W}}_q(\mathsf{D}) \text{ such that } \psi_{|\Gamma_+} = 0\},$$

where q is the conjugate exponent of p.

8.3.3 Regular collision operators

The physical collision operators, used in nuclear reactor theory for all types of moderators (gas, liquid, or solid), are in general *integral* operators with respect to velocities of the form

$$L^p(\mathsf{D}\times V) \ni \phi \mapsto (K\phi)(x,v) = \int\limits_{\mathsf{V}} \kappa(x,v,v')\phi(x,v')\,d\mu(v') \in L^p(\mathsf{D}\times V), \tag{8.9}$$

where $k(\cdot,\cdot,\cdot)$ is a nonnegative measurable function.

Note that K is local with respect to the space variable $x \in \mathsf{D}$. So, it may be regarded as an operator valued mapping from D into $\mathcal{L}(L^p(\mathsf{V}))$, that is,

$$\mathsf{D} \in x \mapsto K(x) \in \mathcal{L}(L^p(\mathsf{V})),$$

where

$$K(x) : L^p(\mathsf{V}) \ni \varphi \mapsto K(x)\varphi = \int\limits_{\mathsf{V}} \kappa(x,v,v')\varphi(v')\,d\mu(v') \in L^p(\mathsf{V}).$$

We assume that K is strongly measurable, i. e., for every $\varphi \in L^p(\mathsf{V})$,

$$\mathsf{D} \ni x \mapsto K(x)\varphi \in L^p(\mathsf{V})$$

is measurable, and that $\mathsf{D} \ni x \mapsto \|K(x)\|_{\mathcal{L}(L^p(\mathsf{V}))}$ is essentially bounded. We note that

$$\mathsf{D} \ni x \mapsto \big\|K(x)\big\|_{\mathcal{L}(L^p(\mathsf{V}))} = \sup_{\|\varphi\|_{L^p(\mathsf{V})}\leq 1} \big\|K(x)\varphi\big\|_{L^p(\mathsf{V})}$$

is measurable because of the separability of $L^p(\mathsf{V})$ $(1 \leq p < +\infty)$. Now we define a collision operator by

$$K : L^p(\mathsf{D}\times V) \ni \varphi \mapsto K(x)\varphi(x),$$

where we make the identification $L^p(\mathsf{D}\times V) = L^p(\mathsf{D}; L^p(\mathsf{V}))$. Clearly, $K \in \mathcal{L}(L^p(\mathsf{D}\times\mathsf{V}))$ and

$$\|K\|_{\mathcal{L}(L^p(\mathsf{D}\times\mathsf{V}))} \leq \operatorname{ess\,sup}_{x\in\mathsf{D}} \big\|K(x)\big\|_{\mathcal{L}(L^p(\mathsf{V}))}. \tag{8.10}$$

To see this, we choose arbitrary separable functions

$$\varphi(x,v) = \varphi_1(x)\varphi_2(v) \quad \text{with } \|\varphi_1\|_{\mathrm{L}^p(\mathsf{D},dx)} = \|\varphi_2\|_{\mathrm{L}^p(\mathsf{V})} = 1.$$

Hence, for all $\varphi_1 \in \mathrm{L}^p(\mathsf{D},dx)$ with $\|\varphi_1\|_{\mathrm{L}^p(\mathsf{D},dx)} = 1$, we have

$$\begin{aligned}\|K\|^p_{\mathcal{L}(L^p(\mathsf{D}\times\mathsf{V}))} \geq \|K\varphi\|^p_{L^p(\mathsf{D}\times\mathsf{V})} &= \int\limits_{\mathsf{D}} \big\|K(x)\varphi(x)\big\|^p_{L^p(\mathsf{V})} \\ &= \int\limits_{\mathsf{D}} \big|\varphi_1(x)\big|^p \big\|K(x)\varphi_2\big\|^2_{L^p(\mathsf{V})}\,dx,\end{aligned}$$

and therefore

$$\|K\|^p_{\mathcal{L}(L^p(\mathsf{D}\times\mathsf{V}))} \geq \operatorname{ess\,sup}_{x\in\mathsf{D}}\|K(x)\varphi_2\|^p_{L^p(\mathsf{V})}, \quad \forall\varphi_2 \in L^p(\mathsf{V}), \quad \|\varphi_2\|_{L^p(\mathsf{V})} = 1,$$

which implies the reversed inequality

$$\|K\|^p_{\mathcal{L}(L^p(\mathsf{D}\times\mathsf{V}))} \geq \operatorname{ess\,sup}_{x\in\mathsf{D}}\|K(x)\|^p_{\mathcal{L}(L^p(\mathsf{V}))}.$$

Definition 8.3.1. Let X and Y be two normed spaces. A set C of $\mathcal{L}(X,Y)$ is said to be collectively compact if and only if the set $C(\mathbb{B}_1) = \{U(x) : U \in C, x \in \mathbb{B}_1\}$ is relatively compact in Y.

Definition 8.3.2. Let $p \in (1,+\infty)$. A collision operator K is said to be regular on $L^p(\mathsf{D}\times V)$ if:

(a) $\{K(x) : x \in \mathsf{D}\}$ is a set of collectively compact operators on $L^p(\mathsf{V})$, i. e.,

$$\{K(x)\varphi : x \in \mathsf{D}, \|\varphi\|_p \leq 1\} \quad \text{is relatively compact in } L^p(\mathsf{V}).$$

(b) For every $\varphi' \in L^q(\mathsf{V})$,

$$\{K^*(x)\varphi' : x \in \mathsf{D}\} \quad \text{is relatively compact in } L^q(\mathsf{V}),$$

where $K^*(x)$ denotes the dual operator of $K(x)$ and $q = \frac{p}{p-1}$ (the conjugate exponent of p).

The class of regular collision operators is closed in $\mathcal{L}(L^p(\mathsf{D}\times V))$ and enjoys the useful approximation property.

Proposition 8.3.1. *The class of regular collision operators is the closure in the operator norm of the class of collision operator with kernels of the form*

$$\kappa(x,v,v') = \sum_{i\in I} \alpha_i(x)f_i(v)g_i(v'), \tag{8.11}$$

with

$$f_i \in L^p(\mathsf{V}), \quad g_i \in L^q(\mathsf{V}), \quad q = \frac{p}{p-1}, \quad \alpha_i \in L^\infty(\mathsf{D};dx) \quad (I \text{ finite}).$$

Proof. It is easy to see that a collision operator with kernel of the form (8.11) is regular, so the closure in $\mathcal{L}(L^p(\mathsf{D}\times V))$ of this class of collision operators is contained in the class of regular collision operators. Conversely, let K be a regular collision operator. Let $(P_n)_{n\in\mathbb{N}} \subset \mathcal{L}(L^p(\mathsf{V}))$ be a sequence of finite-dimensional projections converging strongly to the identity, $P_n\varphi \to \varphi$, $\forall\varphi \in L^p(\mathsf{V})$. We can construct such a sequence from a Schauder basis of $L^p(\mathsf{V})$ $(1 \leq p < \infty)$, for instance. Note that $P_n\varphi \to \varphi$ uniformly on compact subsets of $L^p(\mathsf{V})$, and $P_nK(x)\varphi \to K(x)\varphi$ in $L^p(\mathsf{V})$ uniformly in $\varphi \in \mathbb{B}_1$ ($\mathbb{B}_1$ is the unit ball of $L^p(\mathsf{V})$) and $x \in \mathsf{D}$, that is,

$$\sup_{x \in \mathsf{D}} \| K(x) - P_n K(x) \|_{\mathcal{L}(L^p(\mathsf{V}))} \to 0.$$

In view of (8.11), the collision operator K is approximated in $\mathcal{L}(L^p(\mathsf{D} \times \mathsf{V}))$ as close as we want by a finite-dimensional (with respect to velocities) collision operator of the form

$$L^p(\mathsf{D} \times \mathsf{V}) \ni \varphi \mapsto \sum_{i \in I} (K(x)\varphi(x) e_i') e_i,$$

where $\{e_i, i \in I\}$ is a basis of the finite-dimensional range of P_n and $\{e_i', i \in I\}$ is the dual basis in $L^q(\mathsf{V})$,

$$(e_i', e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the finite-dimensional collision operator is of the form

$$L^p(\mathsf{D} \times \mathsf{V}) \ni \varphi \mapsto \sum_{i \in I} (\varphi(x), K'(x) e_i') e_i = \sum_{i \in I} \int_{\mathsf{V}} e_i(v) \overline{e_i}(x, v') \varphi(x, v') \, d\mu(v'),$$

where

$$\overline{e_i}(x, v') = K'(x) e_i' \in L^\infty(\mathsf{D}; L^p(\mathsf{V})).$$

In view of (8.3.2),

$$\{K'(x) e_i' : x \in \mathsf{D}\} \quad \text{is relatively compact in } L^q(\mathsf{V})$$

and, consequently, $\overline{e_i}(x, v')$ can be approximated in $L^\infty(\mathsf{D}; L^p(\mathsf{V}))$ by degenerate functions of the form

$$\sum_{j \in J} \alpha_j(x) \beta_j(v'), \quad \alpha_j \in L^\infty(\mathsf{D}), \ \beta_j \in L^q(\mathsf{V}) \quad (J \text{ finite}).$$

This ends the proof. □

We point out that, for $p = 1$, there is a more appropriate definition of regular collision operators.

Definition 8.3.3. Let K be an operator defined by (8.9) on $L^1(\mathsf{D} \times \mathsf{V})$. We say that K is a regular collision operator if

$$\{\kappa(x, \cdot, v') : (x, v') \in \mathsf{D} \times \mathsf{V}\}$$

is a relatively weakly compact subset of $L^1(\mathsf{V})$.

Remark 8.3.1. Obviously Dunford–Pettis criterion of weak compactness [79, Theorem 15, p. 76] shows that if K is a regular collision operator, then $|K|$ is also a regular collision operator where $|K|$ is defined on $L^1(\mathsf{D} \times \mathsf{V})$ by

$$\phi \mapsto (|K|\phi)(x,v) = \int_{\mathsf{V}} |\kappa(x,v,v')|\phi(x,v')\,d\mu(v').$$

The class of regular collision operators on $L^1(\mathsf{D} \times \mathsf{V})$ enjoys the following nice approximation property.

Proposition 8.3.2. *Let $K \in \mathcal{L}(L^1(\mathsf{D} \times \mathsf{V}))$ be a regular and nonnegative operator. Then, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of operators of $\mathcal{L}(L^1(\mathsf{D} \times \mathsf{V}))$ such that*
(a) $0 \leq K_m \leq K$ *for any* $m \in \mathbb{N}$*;*
(b) *for any $m \in \mathbb{N}$, K_m is dominated by rank-one operator in $\mathcal{L}(L^1(\mathsf{V}))$;*
(c) $\lim_{m\to+\infty} \|K - K_m\| = 0.$

Remark 8.3.2. We note that item (b) of Proposition 8.3.2 asserts that, for any $m \in \mathbb{N}$, there exists a nonnegative function $f_m \in L^1(\mathsf{V})$ such that, for any $\varphi \in L^1(\mathsf{D} \times \mathsf{V})$, $\varphi \geq 0$,

$$K_m\varphi(x,v) \leq f_m(v) \int_{\mathbb{R}^{\mathbb{N}}} \varphi(x,v')\,d\mu(v). \tag{8.12}$$

Proof. According to Definition 8.3.3,

$$\begin{aligned} K &: L^1(\mathsf{D} \times \mathsf{V}) \to L^1(\mathsf{D} \times \mathsf{V}), \\ \varphi &\mapsto K\varphi(x,v) = \int_{\mathsf{V}} \kappa(x,v,v')\varphi(x,v')\,d\mu(v'), \end{aligned}$$

with

$$(dx \otimes d\mu) - \operatorname{ess\,sup}_{(x,v')\in\mathsf{D}\times\mathsf{V}} \int_{\mathsf{V}} \kappa(x,v,v')\,d\mu(v)\infty.$$

and $\{\kappa(x,\cdot,v') : (x,v') \in \mathsf{D} \times \mathsf{V}\}$ being a relatively weakly compact subset of $L^1(\mathsf{V})$. According to Takac's version of Dunford–Pettis criterion [227, Section 6, Remark 2],

$$\lim_{m\to\infty} \sup_{(x,v')\in\mathsf{D}\times\mathsf{V}} \int_{S_m(x,v')} \kappa(x,v,v')\,d\mu(v) = 0, \tag{8.13}$$

where

$$S_m(x,v') = \{v \in \mathsf{V}; |v| \geq m\} \cup \{v \in \mathbb{R}^N : \kappa(x,v,v') \geq m\}, \quad (x,v') \in \mathsf{D} \times \mathsf{V}.$$

For any $m \in \mathbb{N}$, let us define

$$K_m : L^1(\mathsf{D} \times \mathsf{V}) \ni \varphi \mapsto \int_{\mathsf{V}} \kappa_m(x, v, v')\varphi(x, v')\, d\mu(v') \in L^1(\mathsf{D} \times \mathsf{V}),$$

with

$$\kappa_m(x, v, v') = \inf\{\kappa(x, v, v') : m\chi_{\mathbb{B}_m}(\cdot)\}, \quad (x, v, v') \in \mathsf{D} \times L^1(\mathsf{V}) \times L^1(\mathsf{V}),$$

where $\chi_{\mathbb{B}_m}(\cdot)$ denotes the characteristic function of the set $\mathbb{B}_m = \{v \in \mathsf{V} : |v| \leq m\}$. It is obvious that

$$0 \leq K_m \leq K.$$

Moreover, one can easily check that

$$\|K - K_m\| \leq (dx \otimes d\mu(v)) - \operatorname{ess\,sup}_{(x,v')\in \mathsf{D}\times\mathsf{V}} \int_{\mathsf{V}} \left|\kappa(x, v, v') - \kappa_m(x, v, v')\right| d\mu(v).$$

On the other hand, for any $(x, v') \in \mathsf{D} \times \mathsf{V}$, the construction of $\kappa_m(x, \cdot, v')$ yields

$$\begin{aligned}\int_{\mathsf{V}} \left|\kappa(x, v, v') - \kappa_m(x, v, v')\right| d\mu(v) &= \int_{\{\kappa(x,v,v')\geq m\chi_{\mathbb{B}_m(v)}\}} \left|\kappa(x, v, v') - \kappa_m(x, v, v')\right| d\mu(v) \\ &\leq \int_{\{\kappa(x,v,v')\geq m\chi_{\mathbb{B}_m(v)}\}} \kappa(x, v, v')\, d\mu(v).\end{aligned}$$

So, by (8.13), we have

$$\lim_{m\to+\infty} \|K - K_m\| = 0.$$

Moreover, it is not difficult to prove that

$$K_m\varphi(x, v) \leq m\chi_{\mathbb{B}_m}(v) \int_{\mathsf{V}} \varphi(x, v')\, d\mu(v')$$

for any $\varphi \in L^1(\mathsf{D} \times \mathsf{V})^+$. This proves the second assertion, and completes the proof. □

8.3.4 Compactness

We recall that $\mu(\cdot)$ is a positive Radon measure on $\mathbb{R}^N$ and suppose that the hyperplanes have zero $d\mu$ measure, i. e.,
(H7) For each $e \in \mathbb{S}^{N-1}$, we have $\mu\{v \in \mathbb{R}^N : v \cdot e = 0\} = 0$.

To deal with boundary value problems, we proceed as in [120] by appealing to an extension operator. First of all, we recall the relevant functional spaces and some of their properties. Let Ω be a *smooth open bounded* subset of $\mathbb{R}^N$ and let

$$\begin{aligned} L^p(\Omega \times \mathbb{R}^N) &:= L^p(\Omega \times \mathbb{R}^N, dx\, d\mu(v)), \\ W^p(\Omega \times \mathbb{R}^N) &:= \{\varphi \in L^p(\Omega \times \mathbb{R}^N) : v \cdot \nabla_x \in L^p(\Omega \times \mathbb{R}^N)\} \end{aligned}$$

As seen above, elements of $W^p(\Omega \times \mathbb{R}^N)$ have traces on $\partial\Omega \times \mathbb{R}^N$ lying in suitable weighted space. Let

$$\Gamma_- = \{(x,v) \in \partial\Omega \times \mathbb{R}^N : v \cdot v_x < 0\}, \quad \Gamma_+ = \{(x,v) \in \partial\Omega \times \mathbb{R}^N : v \cdot v_x > 0\}$$

and

$$\widetilde{W^p}(\Omega \times \mathbb{R}^N) = \Big\{\varphi \in W^p(\Omega \times \mathbb{R}^N) : \int\limits_{\partial\Omega \times \mathbb{R}^N} |\varphi(x,v)|^p |v \cdot v_x|\, d\gamma(x)\, d\mu(v) < +\infty\Big\}.$$

As seen above, we have

$$\begin{aligned} \widetilde{W^p}(\Omega \times \mathbb{R}^N) &= \Big\{\varphi \in W^p(\Omega \times \mathbb{R}^N) : \int\limits_{\Gamma_+} |\varphi(x,v)|^p |v \cdot v_x|\, d\gamma(x)\, d\mu(v) < +\infty\Big\} \\ &= \Big\{\varphi \in W^p(\Omega \times \mathbb{R}^N) : \int\limits_{\Gamma_-} |\varphi(x,v)|^p |v \cdot v_x|\, d\gamma(x)\, d\mu(v) < +\infty\Big\}. \end{aligned}$$

Finally, we define the closed subspaces of $\widetilde{W^p}(\Omega \times \mathbb{R}^N)$ as

$$W^p_{0,+} = \{\varphi \in \widetilde{W^p}(\Omega \times \mathbb{R}^N) : \varphi_{\Gamma_+} = 0\}, \quad W^p_{0,-} = \{\varphi \in \widetilde{W^p}(\Omega \times \mathbb{R}^N) : \varphi_{\Gamma_-} = 0\}.$$

Now we can define the averaging operator

$$W^p_{0,\pm} \ni \varphi \mapsto \int\limits_{\mathbb{R}^N} \varphi(x,v)\, d\mu(v) \in L^p(\Omega, dx).$$

With these notations, we can recall the following result due to M. Mokhtar-Kharroubi (see [180, Theorem 3.2, p. 37]).

Proposition 8.3.3. *Let Ω be a smooth open bounded subset of $\mathbb{R}^N$ and let M be the averaging operator*

$$L^p(\mathbb{R}^N \times \mathbb{R}^N) \ni \varphi \mapsto \widetilde{\varphi}(x) = \int\limits_{\mathbb{R}^N} \varphi(x,v)\, d\mu(v) \in L^p(\mathbb{R}^N, dx)$$

where $1 < p < +\infty$. If the hypothesis (H7) *is satisfied, then the following assertions hold true:*

(a) $M : W^p_{0,-} \to L^p(\Omega)$ *is compact,*
(b) $M : W^p_{0,+} \to L^p(\Omega)$ *is compact*

Now we are ready to state

Theorem 8.3.1. *Let* $p \in (1, +\infty)$ *and let* (H7) *be satisfied. If* K *is a regular collision operator on* $L^p(\mathsf{D} \times \mathsf{V})$ *with nonnegative kernel, then, for any* $\lambda \in \mathbb{C}_0$*, the operators* $K(\lambda - T)^{-1}$ *and* $(\lambda - T)^{-1}K$ *are compact in* $L^p(\mathsf{D} \times \mathsf{V})$.

Proof. Since K is a regular collision operator on $L^p(\mathsf{D}\times\mathsf{V})$, according to Proposition 8.3.1 and by linearity, it suffices to establish the theorem for a collision operator with a kernel of the form

$$\kappa(x, v, v') = \alpha(x) f(v) g(v'),$$

where

$$\alpha(\cdot) \in L^\infty(\mathsf{D}), \quad f(\cdot) \in L^p(\mathsf{V}), \quad \text{and} \quad g(\cdot) \in L^q(\mathsf{V}) \quad \left(q = \frac{p}{p-1}\right).$$

Approximating f and g by continuous functions with compact support, we may suppose, without loss of generality, that f and g are continuous with compact supports. In such a case, $K(\lambda - T)^{-1}$ and $(\lambda - T)^{-1}K$ map $L^\eta(\mathsf{D} \times \Omega)$ into itself for all $\eta \in [1, \infty]$. Using interpolation arguments (cf. Theorem 3.10 in [147, p. 57]), we can restrict ourselves to the case $p = 2$. We first consider the operator $K(\lambda - T)^{-1}$.

Let Λ_g be the averaging operator

$$\Lambda_g : \varphi \in L^2(\mathsf{D} \times \mathsf{V}) \to \int \varphi(x, v') g(v')\, d\mu(v') \in L^2(\mathsf{D}).$$

It suffices to show that $\Lambda_g(\lambda - T)^{-1}$ is a compact operator from $L^2(\mathsf{D} \times \mathsf{V})$ into $L^2(\mathsf{D})$. This amounts to

$$\Lambda_g : D(T) = \{\varphi \in \widetilde{W}_2(\mathsf{D}) : \varphi_{|\Gamma_-} = 0\} \to L^2(\mathsf{D}) \text{ is compact.}$$

Note that, since $g = g^+ - g^-$ where g^+ and g^- are the positive and negative parts of g, we may assume that g is nonnegative. Hence the result follows from Proposition 8.3.3(a).

Similarly, by Proposition 8.3.3(b), the operator

$$\Lambda_f : D(T^*) = \{\varphi \in \widetilde{W}_2(\mathsf{D}) : \varphi_{|\Gamma_+} = 0\} \to L^2(\mathsf{D}) \text{ is compact,}$$

and consequently,

$$K^*(\bar{\lambda} - T^*)^{-1} \text{ is compact.}$$

Moreover, due to the duality property $[K^*((\bar{\lambda} - T)^{-1})^*]^* = (\lambda - T)^{-1}K$, it follows from Schauder theorem (see, for example, [89, Theorem 2, p. 485] or [173, Theorem 3.4.15, p. 323]) that $(\lambda - T)^{-1}K$ is compact. ☐

Note that the operator $K(\lambda - T)^{-1}$ is not weakly compact on $L^1(\mathsf{D} \times \mathsf{V})$ even for regular collision operators (see [120, p. 123]). The weak compactness of $(\lambda - T)^{-1}K$ on $L^1(\mathsf{D} \times \mathsf{V})$ is an open problem (cf. [180, Chapter 4]).

Let us recall the following comparison result for positive operators owing to D. Aliprantis and O. Burkinshaw [11] (see also P. Dood and D. H. Fremlin [84]) required in the proof of Lemma 8.3.3.

Lemma 8.3.2. *Let* (Ω, Σ, σ) *be a positive measure space and* $E := L_p(\Omega, \Sigma, d\sigma)$. *Let* S *and* T *be two bounded linear operators on* E *such that* $0 \leq S \leq T$. *If* $1 < p < \infty$ *and* T *is compact, then* S *is compact.*

Lemma 8.3.3. *Assume that* $\mu(V) < \infty$ *and* (H7) *holds true. Let* K *be a regular collision operator in* $L^1(\mathsf{D} \times \mathsf{V})$. *Then, for any* λ *in* $\mathbb{C}_0$, $(\lambda - T)^{-1}K$ *is a Dunford–Pettis operator, i. e.,* $(\lambda - T)^{-1}K$ *maps weakly compact sets of* $L^1(\mathsf{D} \times \mathsf{V})$ *into norm compact sets of* $L^1(\mathsf{D} \times \mathsf{V})$.

Proof. Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}(\lambda) > 0$. According to Remark 8.3.2, it suffices to establish the result for an operator K dominated by a collision operator B of the form (8.12). Let $\mathcal{O}$ be a weakly compact subset of $L^1(\mathsf{D} \times \mathsf{V})$. For any $\phi \in \mathcal{O}$ and $c > 0$, we set

$$\phi_c = (\lambda - T)^{-1}K(\phi\chi_{\{|\phi| \geq c\}}),$$

where χ denotes the characteristic function, and denote by $\tilde{\phi}$ the zero extension of ϕ outside $\mathsf{D} \times V$, that is,

$$\tilde{\phi}(x, v) = \begin{cases} \phi(x, v) & \text{if}(x, v) \in \mathsf{D} \times V, \\ 0 & \text{if}(x, v) \notin \mathsf{D} \times V. \end{cases}$$

It is clear that, for any $h_1, h_2 \in \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |\tilde{\phi}_c(x + h_1, v + h_2) - \tilde{\phi}_c(x, v)|\, dx\, d\mu(v) \leq \frac{2\|K\|}{\operatorname{Re}\lambda} \int_{D \times V} |\phi\chi_{\{|\phi| \geq c\}}(x, c)|\, dx\, d\mu(v).$$

Since $\mathcal{O}$ is weakly compact and the measure of the subset $\{(x, v) \in \mathsf{D} \times V : |\phi|(x, v) \geq c\}$ tends to 0 as $c \to \infty$, it follows from [79, Theorem 15, p. 76] that, for any $\varepsilon > 0$, there exists a positive number $c > 0$ such that

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |\tilde{\phi}_c(x + h_1, v + h_2) - \tilde{\phi}_c(x, v)|\, dx\, d\mu(v) < \varepsilon,$$

uniformly in $\phi \in \mathcal{O}$ and $(h_1, h_2) \in \mathbb{R}^{2N}$.

Let $c > 0$ be chosen as above. According to [2, Theorem 2.21], it suffices to prove that

$$\mathcal{O}_c = \{(\lambda - T)^{-1} K(\phi \chi_{\{|\phi| < c\}}) : \phi \in \mathcal{O}\}$$

is relatively compact in $L^1(\mathsf{D} \times \mathsf{V})$. Indeed, since $\mathcal{O}$ is bounded in $L^1(\mathsf{D} \times \mathsf{V})$, the set $\mathcal{O}_c$ is bounded in $L^2(\mathsf{D} \times \mathsf{V})$. By hypotheses, it is enough to prove that $(\lambda - T)^{-1}K$ is compact in $L^2(\mathsf{D} \times \mathsf{V})$. It is clear that $|(\lambda - T)^{-1}K| \le (\operatorname{Re}\lambda - T)^{-1}|K|$, so, according to Lemma 8.3.2, it suffices to proves that $(\operatorname{Re}\lambda - T)^{-1}|K|$ is compact on $L^2(\mathsf{D} \times \mathsf{V})$. Since $|K|$ is a nonnegative regular collision operator (see Remark 8.3.1), by Proposition 8.3.2, it is enough to prove the result for a rank-one operator B defined by (8.12). It is clear that B is a regular collision operator on $L^2(\mathsf{D} \times \mathsf{V})$. Hence the use of Theorem 8.3.1 proves the compactness of $(\operatorname{Re}\lambda - T)^{-1}B$, which ends the proof. ☐

Theorem 8.3.2. *Let K be a regular collision operator in $L^1(\mathsf{D}\times\mathsf{V})$ with nonnegative kernel and let λ be a complex number in $\mathbb{C}_0$. If V is unbounded and the hypothesis* (H7) *is satisfied, then $(\lambda - T)^{-1}K$ is a Dunford–Pettis operator on $L^1(\mathsf{D} \times \mathsf{V})$.*

Proof. Let us show that $(\lambda - T)^{-1}K$ is a Dunford–Pettis operator. Let $\mathcal{O}$ be a weakly compact subset of $L^1(\mathsf{D} \times \mathsf{V})$. For $m > 0$ and $\psi \in \mathcal{O}$, we set $\psi_m^1 = \psi\chi_{(|v|<m)}$ and $\psi_m^2 = \psi\chi_{(|v|\ge m)}$. Clearly,

$$\lim_{m\to+\infty} \|(\lambda - T)^{-1}K\psi - (\lambda - T)^{-1}K\psi_m^1\| = \lim_{m\to+\infty} \|(\lambda - T)^{-1}K\psi_m^2\|.$$

Since

$$\|(\lambda - T)^{-1}K\psi_m^2\| \le \|(\lambda - T)^{-1}K\| \int_{\mathsf{D}} dx \int_{|v'|\ge m} |\psi(x, v')|\, dx\, d\mu(v'),$$

we get

$$\lim_{m\to+\infty} \|(\lambda - T)^{-1}K\psi - (\lambda - T)^{-1}KE_m\psi\| = 0$$

uniformly in $\psi \in \mathcal{O}$, where E_m denotes the operator from $L^1(\mathsf{D}\times\mathsf{V})$ into itself defined by

$$E_m(\psi) = \psi\chi_{\{v\in\mathbb{R}^N : |v|<m\}} \quad \text{for all } \psi \in L^1(\mathsf{D} \times \mathsf{V}).$$

Since the class of weakly compact operators is closed, it suffices to show that $(\lambda - T)^{-1}KE_m$ is a Dunford–Pettis operator on $L^1(\mathsf{D}\times\mathsf{V}_m,)$. Now, since E_m maps continuously $L^1(\mathsf{D}\times\mathsf{V})$ into $L^1(\mathsf{D}\times\mathsf{V}_m,)$ where $\mathsf{V}_m = \{v \in \mathsf{V}$ such that $|v| < m\}$, it suffices to show that $(\lambda - T)^{-1}K$ is a Dunford–Pettis operator on $L^1(\mathsf{D}\times\mathsf{V}_m)$, where V_m is a subset of V such that $\mu(\mathsf{V}_m) < +\infty$. This was already established in Lemma 8.3.3. ☐

8.3.5 A measure of weak noncompactness

Let $\mathbf{B}(L^1(\mathsf{D} \times \mathsf{V}))$ denote the set of all nonempty bounded subsets of $L^1(\mathsf{D} \times \mathsf{V})$ and let $\mathsf{W}(L^1(\mathsf{D} \times \mathsf{V}))$ be the subset of $\mathbf{B}(L^1(\mathsf{D} \times \mathsf{V}))$ consisting of all weakly compact subsets of $L^1(\mathsf{D} \times \mathsf{V})$. Inspired by the work [27], we define a function $\gamma : \mathbf{B}(L^1(\mathsf{D} \times \mathsf{V})) \to [0, \infty)$ by

$$\gamma(M) = \gamma_1(M) + \gamma_2(M),$$

where

$$\gamma_1(M) = \limsup_{\varepsilon \to 0} \left\{ \sup_{\psi \in M} \left[\int_E |\psi(x,v)|\, dx\, d\mu(v), |E| \le \varepsilon \right] \right\} \tag{8.14}$$

with $|E|$ denotes the measure of $E \subset \mathsf{D} \times \mathsf{V}$ with respect to $dx\, d\mu(v)$ and

$$\gamma_2(M) = \lim_{m \to +\infty} \left\{ \sup_{\psi \in M} \left[\int_{\mathsf{D}} \int_{|v| \ge m} |\psi(x,v)|\, dx\, d\mu(v) \right] \right\}. \tag{8.15}$$

Remark 8.3.3. We notice that, in the definition of $\gamma(\cdot)$, the velocity space V is assumed to be unbounded. In the case where V is bounded, we define $\gamma(\cdot)$ as $\gamma_1(\cdot)$.

We will prove now that the mapping $\gamma(\cdot)$ is a measure of weak noncompactness on $L^1(\mathsf{D} \times \mathsf{V})$.

Proposition 8.3.4. *Let M, M_1, and M_2 be three elements of $\mathbf{B}(L^1(\mathsf{D} \times \mathsf{V}))$. Then the following properties are satisfied:*

1. *if $M_1 \subseteq M_2$, then $\gamma(M_1) \le \gamma(M_2)$;*
2. *$\gamma(M_1 \cup M_2) = \max\{\gamma(M_1), \gamma(M_2)\}$;*
3. *$\gamma(\alpha M) = |\alpha| \gamma(M)$ for any $\alpha \in \mathbb{R}$;*
4. *$\gamma(M_1 + M_2) \le \gamma(M_1) + \gamma(M_2)$;*
5. *$\gamma(M) = 0$ if and only if $\overline{M}^w \in \mathsf{W}$ where $\overline{M}^w$ is the weak closure of M;*
6. *$\gamma(\mathrm{co}(M)) = \gamma(M)$;*
7. *$\gamma(\overline{M}^w) = \gamma(M)$;*
8. *if $(M_n)_{n \ge 1}$ is a decreasing sequence of nonempty, weakly closed subsets of $\mathbf{B}(L^1(\mathsf{D} \times \mathsf{V}))$ where $\lim_{n \to +\infty} \gamma(M_n) = 0$, then $\bigcap_{n \ge 1} M_n$ is nonempty and weakly relatively compact.*

Proof. Statements (1), (2), (3), and (4) are clear and immediate.

(5) The fact that $\gamma(M) = 0$ implies $\overline{M}^w \in \mathsf{W}(L^1(\mathsf{D} \times \mathsf{V}))$ is a consequence of Theorem 1.7.6. Now suppose that $\overline{M}^w \in \mathsf{W}(L^1(\mathsf{D} \times \mathsf{V}))$. It follows from Theorem 1.7.6(a) that $\gamma_1(M) = 0$. It remains to show that $\gamma_2(M) = 0$. Let $E_n = \mathsf{D} \times \{|v| \ge T_n\}$ where $(T_n)_n$ is a nondecreasing sequence such that $\lim_{n \to +\infty} T_n = +\infty$. It is clear that $(E_n)_n$ is a decreasing sequence of subsets and $\bigcap_{n \ge 0} E_n = \emptyset$. It follows from Theorem 1.7.5 that

$$\lim_{n\to+\infty}\sup_{\psi\in M}\int_{\mathsf{D}}\int_{\{|v|\geq T_n\}}\big|\psi(x,v)\big|\,dx\,d\mu(v)=0,$$

and consequently $\gamma_2(M)=0$.

(6) Since $M\subseteq \operatorname{co}(M)$, using (1), we get $\gamma(M)\leq\gamma(\operatorname{co}(M))$. To prove the converse inequality, let A be a measurable subset of $\mathsf{D}\times\mathsf{V}$ such that $\mu(A)<\varepsilon$. For $\psi\in\operatorname{co}(M)$ there exists $(a_i)_{1\leq i\leq n}$ with $a_i\geq 0$ and $(\varphi_i)_{1\leq i\leq n}$ with $\varphi_i\in M$ such that $\sum_{i=1}^{n}a_i=1$ and $\psi=\sum_{i=1}^{n}a_i\varphi_i$. Clearly,

$$\begin{aligned}\int_A\big|\psi(x,v)\big|\,dx\,d\mu(v)&\leq\sum_{i=1}^{n}a_i\int_A\big|\phi_i(x,v)\big|\,dx\,d\mu(v)\\&\leq\sum_{i=1}^{n}a_i\sup_{1\leq i\leq n}\int_A\big|\varphi_i(x,v)\big|\,dx\,d\mu(v)\\&=\sup_{1\leq i\leq n}\int_A\big|\varphi_i(x,v)\big|\,dx\,d\mu(v)\leq\sup_{\varphi\in M}\int_A\big|\varphi(x,v)\big|\,dx\,d\mu(v).\end{aligned}$$

Therefore

$$\sup_{\psi\in co(M)}\int_A\big|\psi(x,v)\big|\,dx\,d\mu(v)\leq\sup_{\varphi\in M}\int_A\big|\varphi(x,v)\big|\,dx\,d\mu(v),$$

which proves that $\gamma_1(\operatorname{co}(M))\leq\gamma_1(M)$.

Note also that, for any $T>0$, similar computations as above give

$$\sup_{\psi\in co(M)}\int_{\mathsf{D}}\int_{|v|>T}\big|\psi(x,v)\big|\,dx\,d\mu(v)\leq\sup_{\varphi\in M}\int_{\mathsf{D}}\int_{|v|>T}\big|\psi(x,v)\big|\,dx\,d\mu(v).$$

Now passing to the limit as T goes to $+\infty$, we get $\gamma_2(\operatorname{co}(M))\leq\gamma_2(M)$.

Accordingly,

$$\gamma\big(\operatorname{co}(M)\big)\leq\gamma(M).$$

(7) We first observe that the inclusion $M\subset\overline{M}^{w}$, together with (1), implies $\gamma(M)\leq\gamma(\overline{M}^{w})$.

Conversely, using (1), the inclusion $M\subset\overline{M}^{w}\subset\overline{\operatorname{co}}(M)$ we get $\gamma(\overline{M}^{w})\leq\gamma(\overline{\operatorname{co}}(M))$. Thus we only have to prove that $\gamma(M)=\gamma(\overline{M})$ (here $\overline{M}$ denotes the strong closure of M). By (1), we already have the inequality $\gamma(M)\leq\gamma(\overline{M})$. Next, for $\psi\in\overline{M}$ there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in M such that $\lim_{n\to+\infty}\psi_n=\psi$. Hence, for any $\varepsilon>0$ and any subset A of $\mathsf{D}\times\mathsf{V}$ such that $\mu(A)<\varepsilon$, we have

$$\begin{aligned}\int_A\big|\psi(x,v)\big|\,dx\,d\mu(v)&\leq\int_A\big|\psi(x,v)-\psi_n(x,v)\big|\,dx\,d\mu(v)+\int_A\big|\psi_n(x,v)\big|\,dx\,d\mu(v)\\&\leq\|\psi-\psi_n\|_{L^1(\mathsf{D}\times\mathsf{V})}+\gamma_1(M).\end{aligned}$$

Since, for n large enough, $\|\psi - \psi_n\|_{L^1(\mathsf{D}\times\mathsf{V})}$ is very small, we infer that $\gamma_1(\overline{M}) \le \gamma_1(M)$. In the same way, we also prove that $\gamma_2(\overline{M}) \le \gamma_2(M)$ and therefore $\gamma(\overline{M}) = \gamma(M)$. Next, the use of (6) gives $\gamma(\overline{\mathrm{co}}(M)) = \gamma(\mathrm{co}(M)) = \gamma(M)$ and, consequently,

$$\gamma(M) \le \gamma(\overline{M}^{w}) \le \gamma(M),$$

which ends the proof of (7).

(8) Let $(M_n)_{n\ge1}$ be a decreasing sequence of nonempty, weakly closed subsets of $\mathbf{B}(L^1(\mathsf{D}\times\mathsf{V}))$. Since $\lim_{n\to+\infty}\omega(M_n) = 0$, let $(\psi_n)_n$ be a sequence such that, for each $n \in \mathbb{N}$, $\psi_n \in M_n$ and put $\widetilde{M}_n := \{\psi_k : k \ge n\}$. It is clear that $(\widetilde{M}_n)_n$ is a decreasing sequence of nonempty subsets. Since, for all $n \in \mathbb{N}$, $\widetilde{M}_n \subset M_n$ and $\widetilde{M}_1\backslash\widetilde{M}_n$ is a finite set, we have

$$\gamma(\widetilde{M}_1) = \gamma(\widetilde{M}_n) \le \gamma(M_n).$$

Hence, $\gamma(\widetilde{M}_1) \le \lim_{n\to+\infty}\gamma(M_n) = 0$ and therefore $\widetilde{M}_1$ is relatively weakly compact. Consequently, $(\psi_n)_n$ converges weakly to a function ψ. Hence $\psi \in \bigcap_{n\ge1} M_n$ because, for each $n \in \mathbb{N}$, M_n is weakly closed. This completes the proof. ☐

The same arguments as in Corollary 7.8.1 show that, for every bounded linear operator L on $L^1(\mathsf{D}\times\mathsf{V})$, we have

$$\gamma(L(M)) \le \|L\|_{\mathcal{L}(L^1(\mathsf{D}\times\mathsf{V}))}\gamma(M) \quad \text{for all } M \in \mathbf{B}(L^1(\mathsf{D}\times\mathsf{V})). \tag{8.16}$$

8.3.6 Existence results

For $i \in \{1, 2, \dots, d\}$, let Λ_i denote the bounded multiplication operators from $L^p(\mathsf{D}\times\mathsf{V})$ into itself defined by $\Lambda_i f_i = \lambda_i\beta_i(x,v)f_i$. Since the functions $\beta_i(\cdot,\cdot)$, $i = 1,\dots,d$, belong to $L^\infty(\mathsf{D}\times\mathsf{V})$, the operators Λ_i, $i = 1,\dots,d$ are bounded. We also denote by K_i, $0 \le i \le d$, the operator defined from $L^p(\mathsf{D}\times\mathsf{V})$ into $L^p(\mathsf{D}\times\mathsf{V})$ by

$$f \mapsto K_i f(x,v) := \int_{\mathbb{R}^N} \kappa_i(x,v,v')f(x,v')\,d\mu(v'), \quad 0 \le i \le d, \tag{8.17}$$

where $\kappa_i : \mathsf{D}\times\mathsf{V}\times\mathsf{V}$. Moreover, $\mathcal{N}_g$ stands for the Nemytskii operator generated by the function g (for $g = -\sigma, \Theta_0, \dots, \Theta_d$).

Now problem (8.7)–(8.8) may be written abstractly as

$$\begin{cases} (\lambda - T)f_0 = \mathcal{N}_{-\sigma} + K_0\mathcal{N}_{\Theta_0}f_0 + \sum_{i=1}^d \lambda_i\Lambda_i f_i, \\ \lambda_i f_i = K_i\mathcal{N}_{\Theta_i}f_0, \quad 1 \le i \le d. \end{cases} \tag{8.18}$$

Since, for $i = 1,\dots,d$, the operator Λ_i is bounded, we can replace $\lambda_i f_i(x,\xi)$ of the first equation of (8.18) by $K_i\mathcal{N}_{\Theta_i}f_0$. Moreover, each $\lambda \in \mathbb{C}_0$ belongs to $\rho(T)$ and therefore, for such λ, problem (8.7)–(8.8) reduces to the following fixed point problem:

$$f_0 = \mathsf{F}(\lambda)f_0 + \mathsf{B}(\lambda)f_0, \quad f_{0|\Gamma_-} = 0, \tag{8.19}$$

where

$$\begin{cases} \mathsf{F}(\lambda) = (\lambda - T)^{-1}K_0\mathcal{N}_{\Theta_0} \quad \text{and} \\ \mathsf{B}(\lambda) = (\lambda - T)^{-1}\mathcal{N}_{-\sigma} + \sum_{i=1}^{d}(\lambda - T)^{-1}\Lambda_i K_i \mathcal{N}_{\Theta_i}. \end{cases} \tag{8.20}$$

Because, for each $i \in \{1, \dots, d\}$, $f_i = \frac{K_i\mathcal{N}_{\Theta_i}f_0}{\lambda_i}$, to solve problem (8.7)–(8.8), it suffices to search for the solutions of the fixed point problem (8.19).

Let $r > 0$, $p \in [0, +\infty)$, and denote by $\mathbb{B}_r^p$ the ball defined by

$$\mathbb{B}_r^p := \{f \in L^p(\mathsf{D} \times \mathsf{V}) \text{ such that } \|f\|_{L^p(\mathsf{D}\times\mathsf{V})} \le r\}.$$

We now introduce the following assumptions:

(H8) For each $i \in \{0, 1, \dots, d\}$, the map $\Theta_i(\cdot,\cdot,\cdot)$ is a Carathéodory function and $\mathcal{N}_{\Theta_i}$ acts from $L^p(\mathsf{D} \times \mathsf{V})$ into itself.

(H9) For each $r > 0$, the function $\sigma(\cdot,\cdot,\cdot)$ satisfies

$$\left|\sigma(x, v, \psi_1(x, v)) - \sigma(x, v, \psi_2(x, v))\right| \le \sigma_0(x, v)|\psi_1 - \psi_2|,$$

$\forall \psi_1, \psi_2 \in \mathbb{B}_r^p$, $\sigma(\cdot,\cdot,\cdot)$ is a Carathéodory function, and $\mathcal{N}_{-\sigma}$ acts from $L^p(\mathsf{D} \times \mathsf{V})$ into itself, where $\sigma_0(\cdot,\cdot) \in L^\infty(\mathsf{D} \times \mathsf{V})$.

(H10) For each $r > 0$ and each $i \in \{1, 2, \dots, d\}$, the function $\Theta_i(\cdot,\cdot,\cdot)$ satisfies

$$\left|\Theta_i(x, v, \psi_1(x, v)) - \Theta_i(x, v, \psi_2(x, v))\right| \le \rho_i(x, v)|\psi_1 - \psi_2|$$

$\forall \psi_1, \psi_2 \in \mathbb{B}_r^p$, where $\rho_i(\cdot,\cdot) \in L^\infty(\mathsf{D} \times \mathsf{V})$.

Theorem 8.3.3. *Let* K_0 *be a regular collision operator on* $L^p(\mathsf{D} \times \mathsf{V})$ *and assume that* (H7)–(H10) *hold true.*

(a) *If* $1 < p < +\infty$, *then, for each* $r > 0$, *there exists a real* $\theta(r) > 0$ *such that, for all* $\lambda \in \mathbb{C}_{\theta(r)}$, *problem* (8.7)–(8.8) *has at least one solution with* $f_0 \in \mathbb{B}_r^p$.

(b) *If* $p = 1$, *then for each* $r > 0$, *there exists a real* $\theta(r) > 0$ *such that, for all* $\lambda \in \mathbb{C}_{\theta(r)}$, *problem* (8.7)–(8.8) *has at least one solution with* $f_0 \in \mathbb{B}_r^1$.

Proof. Let $\lambda \in \mathbb{C}_0$, $p \in [1, +\infty)$, and $r > 0$. It is clear that the operators $\mathsf{B}(\lambda)$ and $\mathsf{F}(\lambda)$ are well defined and continuous.

Now let $\varphi, \psi \in \mathbb{B}_r^p$. Since the Nemytskii operators generated by $-\sigma$, $\Theta_0, \dots, \Theta_d$ map $\mathbb{B}_r^p$ into bounded sets, easy calculations using Lemma 8.3.1 yield

$$\|\mathsf{F}(\lambda)\varphi + \mathsf{B}(\lambda)\psi\|_{L^p(\mathsf{D}\times\mathsf{V})} \le \frac{1}{\operatorname{Re}\lambda}\left(\|K_0\|\mathcal{M}_{r,0} + \mathcal{M}_{r,\sigma} + \sum_{i=1}^{d}|\lambda_i|\|\beta_i\|_\infty\|K_i\|\mathcal{M}_{r,i}\right),$$

where $\mathcal{M}_{r,i}$, $0 \le i \le \mathrm{d}$ (resp. $\mathcal{M}_{r,\sigma}$) denotes the upper bound of $\mathcal{N}_{\Theta_i}$ (resp. $\mathcal{N}_{-\sigma}$) on $\mathbb{B}_r^p$.

Let $\mathcal{M}_r = \max(\mathcal{M}_{r,0}, \dots, \mathcal{M}_{r,d}, \mathcal{M}_{r,\sigma})$ and let τ_1 be a nonnegative real number such that

$$\frac{1}{\tau_1}\left(1 + \|K_0\| + \sum_{i=1}^{d} |\lambda_i| \|\beta_i\|_\infty \|K_i\|\right) \le \frac{r}{\mathcal{M}_r}.$$

Hence, for all $\lambda \in \mathbb{C}_{\tau_1}$, we have

$$\left\|\mathsf{F}(\lambda)\varphi + \mathsf{B}(\lambda)\psi\right\|_{L^p(\mathsf{D}\times\mathsf{V})} \le r,$$

and therefore

$$\mathsf{F}(\lambda)(\mathbb{B}_r^p) + \mathsf{B}(\lambda)(\mathbb{B}_r^p) \subset \mathbb{B}_r^p.$$

Next, we show that, for appropriate value of $\lambda \in \mathbb{C}_0$, the operator $\mathbf{B}(\lambda)$ is a contraction on $L^p(\mathsf{D} \times \mathsf{V})$. Indeed, let $\psi_1, \psi_2 \in \mathsf{X}_p$ and $\lambda \in \mathbb{C}_0$. Using Lemma 8.3.1 and the hypotheses (H9)–(H10), we obtain

$$\left\|\mathsf{B}(\lambda)\psi_1 - \mathsf{B}(\lambda)\psi_2\right\|_{L^p(\mathsf{D}\times\mathsf{V})} \le \frac{1}{\operatorname{Re}\lambda}\left(\|\sigma_0\|_\infty + \sum_{i=1}^{d} |\lambda_i| \|\beta_i\|_\infty \|K_i\| \|\rho_i\|_\infty\right)\|\psi_1 - \psi_2\|_{L^p(\mathsf{D}\times\mathsf{V})}.$$

Let τ_2 be a nonnegative real number such that

$$k := \frac{1}{\tau_2}\left(\|\sigma_0\|_\infty + \sum_{i=1}^{d} |\lambda_i| \|\beta_i\|_\infty \|K_i\| \|\rho_i\|_\infty\right) < 1.$$

Hence, for all $\lambda \in \mathbb{C}_{\tau_2}$, $\mathsf{B}(\lambda)$ is k-contractive.

From now, we treat separately the cases $p \in (1, +\infty)$ and $p = 1$.

(a) Let $p \in (1, +\infty)$. We show that the operator $\mathsf{F}(\lambda)$ is compact. Since K_0 is a regular collision operator, according to Theorem 8.3.1, the operator $(\lambda - T)^{-1}K_0$ is compact. Therefore, by Theorem 1.15.2 and assumption (H8), we infer that $\mathcal{N}_{\Theta_0}$ is continuous and, consequently, $\mathsf{F}(\lambda)$ is compact on $L^p(\mathsf{D} \times \mathsf{V})$.

Set $\theta(r) = \max(\tau_1, \tau_2)$. Then, for any $\lambda \in \mathbb{C}_{\theta(r)}$, the operators $\mathsf{B}(\lambda)$ and $\mathsf{F}(\lambda)$ satisfy the conditions of the classical Krasnosel'skii fixed point theorem (cf. Proposition 7.12.1). So, the fixed point problem (8.19) has a solution $f_0 \in \mathbb{B}_r^p$ and therefore the boundary value problem (8.7)–(8.8) has at least one solution $(f_0, \dots, f_d)$ with $f_0 \in \mathbb{B}_r^p$.

(b) Let $p = 1$. We shall show that the operators $\mathsf{F}(\lambda)$ and $\mathsf{B}(\lambda)$ satisfy the hypotheses of Corollary 7.12.8. To do so, we first observe that, according to Theorem 8.3.2, $(\lambda - T)^{-1}K_0$ is a Dunford–Pettis operator on $L^1(\mathsf{D}\times\mathsf{V})$. Hence condition (a) of Corollary 7.12.8 is satisfied.

Let $S \subset \mathbb{B}_r^1$. Since the maps Θ_i, $i = 0, \dots, d$, satisfy condition (H8), by Theorem 1.15.2, there exist $\eta_i > 0$ and $h_i(\cdot) \in L^1(\mathsf{D} \times \mathsf{V})^+$ (the positive cone of $L^1(\mathsf{D} \times \mathsf{V})^+$) such that

$$\left|\Theta_i(x, v, f(x, v))\right| \le h_i(x, v) + \eta_i \left|f(x, v)\right|,$$

for all $f \in S$ and for almost all $(x,v) \in \mathsf{D} \times \mathsf{V}$. Accordingly,

$$\int_E \left|(\mathcal{N}_{\Theta_i} f)(x,v)\right| dx\, d\mu(v) \le \int_E h_i(x,v)\, dx\, d\mu(v) + \eta_i \int_E \left|f(x,v)\right| dx\, d\mu(v) \tag{8.21}$$

and

$$\int_{\mathsf{D}} \int_{|v|\ge m} \left|(\mathcal{N}_{\Theta_i} f)(x,v)\right| dx\, d\mu(v) \le \int_{\mathsf{D}} \int_{|v|\ge m} h_i(x,v)\, dx\, d\mu(v) + \int_{\mathsf{D}} \int_{|v|\ge m} \left|f(x,v)\right| dx\, d\mu(v), \tag{8.22}$$

for all measurable subsets E of $\mathsf{D} \times \mathsf{V}$, $m > 0$, and $0 \le i \le d$. This, together with the estimate (8.16), implies that

$$\begin{aligned}
\gamma_1\big(\mathsf{F}(\lambda)(S) + \mathsf{B}(\lambda)(S)\big) &\le \gamma_1\big(\mathsf{F}(\lambda)(S)\big) + \gamma_1\big(\mathsf{B}(\lambda)(S)\big) \\
&\le \left\|(\lambda - T)^{-1}\right\|_{L^1(\mathsf{D}\times\mathsf{V})} \\
&\quad \times \left(\|K_0\|\gamma_1(\mathcal{N}_{\Theta_0} S) + \gamma_1(\mathcal{N}_{-\sigma} S) + \sum_{i=1}^{d} \|\Lambda_i K_i\|\gamma_1(\mathcal{N}_{\Theta_i} S) \right) \\
&\le \frac{1}{\operatorname{Re}\lambda}\left(\eta_0\|K_0\| + \eta_\sigma + \sum_{i=1}^{d} \eta_i\|\beta_i\|_\infty\|K_i\| \right)\gamma_1(S).
\end{aligned}$$

Now, let τ_3 be a nonnegative real number such that

$$\nu := \frac{1}{\tau_3}\left(\eta_0\|K_0\| + \eta_\sigma + \sum_{i=1}^{d} \eta_i|\lambda_i|\|\beta_i\|_\infty\|K_i\| \right) < 1.$$

Hence, for all $\lambda \in \mathbb{C}_{\tau_3}$ and for all measurable subsets S of $\mathbb{B}_r^1$, we have

$$\gamma_1\big(\mathsf{F}(\lambda)(S) + \mathsf{B}(\lambda)(S)\big) \le \nu\gamma_1(S).$$

Similarly, by using (8.22), we obtain

$$\gamma_2\big(\mathsf{F}(\lambda)(S) + \mathsf{B}(\lambda)(S)\big) \le \nu\gamma_2(S).$$

Hence, the latter two estimates yield

$$\gamma\big(\mathsf{F}(\lambda)(S) + \mathsf{B}(\lambda)(S)\big) \le \nu\gamma(S),$$

for all subsets $S \subset \mathbb{B}_r^1$.

Next, since Θ_0 satisfies (H8), it satisfies the hypotheses of Lemma 8.2.1 and therefore $\mathcal{N}_{\Theta_0}$ is ww-compact.

Set $\eta(r) = \max(\theta(r), \tau_3)$. The steps above show that, for all $\lambda \in \mathbb{C}_{\eta(r)}$, the assumptions of Corollary 7.12.8 are satisfied, and so the fixed point problem (8.19) has a solution $f_0 \in \mathbb{B}_r^1$. Therefore, the boundary value problem (8.7)–(8.8) has at least one solution $(f_0, \dots, f_d)$ with $f_0 \in \mathbb{B}_r^1$. □

8.4 A nonlinear singular neutron transport equation

8.4.1 Introduction

This section deals with existence of solutions to the following boundary value problem

$$\begin{cases} v \cdot \nabla_x \psi(x,v) + \sigma(v)\psi(x,v) + \lambda\psi(x,v) = \int_{\mathbb{R}^N} \kappa(v,v')f(x,v',\psi(x,v'))\,dv', \\ \psi_{|\Gamma_-} = 0, \end{cases} \tag{8.23}$$

where $(x,v) \in \mathsf{D} \times \mathbb{R}^N$ and D denotes a smooth open bounded subset of $\mathbb{R}^N$. This equation describes transport of particles (neutrons, photons, molecules of gas, etc.) in the domain D. The function $\psi(x,v)$ represents the density (or probability) of gas particles having position x and velocity v; $\sigma(\cdot)$ and $\kappa(\cdot,\cdot)$ are measurable functions called, respectively, the collision frequency and scattering kernel. The map $f(\cdot,\cdot,\cdot)$ is a nonlinear function of the density of gas particles ψ, and λ is a complex number. By $\psi_{|\Gamma_-}$ we denote the restriction of ψ to Γ_-, where Γ_- is the incoming part of the phase space.

We shall study the existence of the solutions to a multidimensional neutron transport equation involving unbounded collision frequencies $\sigma(\cdot)$ and unbounded collision operators.

It is worth noticing that the pioneering works in this direction, for multidimensional bounded geometry and slab geometry in L^1-spaces, motivated by free gas models, were done by B. Montagnini, M. L. Demuru [82] and A. Suhadolc [223] (in the linear case). The spectral theory of linear singular transport operators and the well-posedness of Cauchy problems governed by linear singular transport operators were discussed by many authors. We refer, for example, to the works [58–60, 165] or [180, Chapter 9]. In [155], a boundary value problem involving a nonlinear singular transport operator in slab geometry was discussed in L^1-space.

8.4.2 Notations and preliminaries

Let

$$\mathsf{X}_p := L^p(\mathsf{D} \times \mathbb{R}^N; dx\,dv), \quad 1 \le p < +\infty$$

where D is a smooth bounded open convex subset of $\mathbb{R}^N$. We recall (cf. Section 8.3.2) that

$$\Gamma_- = \{(x,v) \in \partial\mathsf{D} \times \mathbb{R}^N : v \cdot \nu_x < 0\},$$

where ν_x stands for the outer unit normal vector at $x \in \partial \mathsf{D}$. For $(x,v) \in \overline{\mathsf{D}} \times \mathbb{R}^N$, set

$$t^-(x,v) = \sup\{t > 0 : x - sv \in \mathsf{D}, 0 < s < t\}.$$

The number $t^-(x,v)$ is the time required for a particle having position $x \in \mathsf{D}$ and velocity $v \in \mathbb{R}^N$ to go out of D.

We introduce the following spaces:

$$\mathsf{X}_p^\sigma := L^p(\mathsf{D} \times \mathbb{R}^N; \sigma(v)\,dx\,dv), \quad \mathsf{L}^p := L^p(\mathbb{R}^N; dv), \quad \text{and} \quad \mathsf{L}^{p,\sigma} := L^p(\mathbb{R}^N; \sigma(v)\,dv).$$

Now, we are in position to define the streaming operator T with nonentry boundary conditions, i. e., $\psi_{|\Gamma_-} = 0$, as follows:

$$\begin{cases} T : D(T) \subseteq \mathsf{X}_p^\sigma \to \mathsf{X}_p, \\ \psi \mapsto T\psi(x,v) = -v \cdot \nabla_x \psi(x,v) - \sigma(v)\psi(x,v), \\ D(T) = \{\psi \in \mathsf{X}_p^\sigma \text{ such that } v \cdot \nabla_x \psi \in \mathsf{X}_p \text{ and } \psi_{|\Gamma_-} = 0\}. \end{cases}$$

The collision frequency $\sigma(\cdot)$ satisfies the following assumption:

(H11) There exist a closed subset $\mathcal{O} \subset \mathbb{R}^N$ with zero Lebesgue measure and a constant $\sigma_0 > 0$ such that $\sigma(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \mathcal{O})$, and $\sigma(v) > \sigma_0$ for a. e. $v \in \mathbb{R}^N$.

Let $\varphi \in \mathsf{X}_p^\sigma$ and consider the problem

$$(\lambda - T)\psi = \varphi, \tag{8.24}$$

where λ is a complex number and the unknown function ψ must be sought in $D(T)$.

As in the previous section, for any real number a, we denote by $\mathbb{C}_a$ the complex half-plane

$$\mathbb{C}_a := \{\lambda \in \mathbb{C} \quad \text{such that } \operatorname{Re}\lambda > a\}.$$

It is clear that the resolvent set of T contains $\mathbb{C}_{-\sigma_0}$ and, for $\lambda \in \mathbb{C}_{-\sigma_0}$, the resolvent of T is given by

$$(\lambda - T)^{-1}\varphi(x,v) = \int_0^{t^-(x,v)} e^{-t(\lambda+\sigma(v))}\varphi(x - tv, v)\,dt.$$

Lemma 8.4.1. *Let* $1 \le p < +\infty$ *and* $\lambda \in \mathbb{C}_{\sigma_0}$. *If the hypothesis* (H11) *holds true, then* $(\lambda - T)^{-1}$ *belongs to* $\mathcal{L}(\mathsf{X}_p, \mathsf{X}_p^\sigma)$ *and satisfies*

$$\|(\lambda - T)^{-1}\|_{\mathcal{L}(\mathsf{X}_p,\mathsf{X}_p^\sigma)} \le \sup_{v \in \mathbb{R}^N} \frac{\sigma(v)^{\frac{1}{p}}}{\operatorname{Re}\lambda + \sigma(v)}. \tag{8.25}$$

Proof. For $p = 1$, the result is immediate. We give the proof for $p > 1$. Letting $\varphi \in \mathsf{X}_p$, we have

$$\begin{aligned}\|(\lambda - T)^{-1}\varphi\|^p_{\mathsf{X}^\sigma_p} &= \int\limits_{\mathsf{D}\times\mathbb{R}^N} \left| \int\limits_0^{t^-(x,v)} e^{-t(\lambda+\sigma(v))}\varphi(x - tv, v)\,dt \right|^p \sigma(v)\,dx\,dv \\ &\leq \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \left(\int\limits_0^{+\infty} e^{-\frac{t(\operatorname{Re}\lambda+\sigma(v))}{q}} \times e^{-\frac{t(\operatorname{Re}\lambda+\sigma(v))}{p}} |\tilde{\varphi}(x - tv, v)|\,dt \right)^p \sigma(v)\,dx\,dv,\end{aligned}$$

where $\tilde{\varphi}$ denotes the zero extension of φ outside D and $q = \frac{p}{p-1}$ is the conjugate of p. According to Hölder inequality and Fubini theorem, we obtain

$$\|(\lambda - T)^{-1}\varphi\|^p_{\mathsf{X}^\sigma_p} \leq \int\limits_{\mathbb{R}^N} \frac{\sigma(v)}{(\operatorname{Re}\lambda + \sigma(v))^{\frac{p}{q}}}\,dv \int\limits_0^{+\infty} e^{-t(\operatorname{Re}\lambda+\sigma(v))}\,dt \int\limits_{\mathbb{R}^N} |\tilde{\varphi}(x - tv, v)|^p\,dx.$$

Now, the change of variables $y = x - tv$ leads to

$$\begin{aligned}\|(\lambda - T)^{-1}\varphi\|^p_{\mathsf{X}^\sigma_p} &\leq \int\limits_{\mathbb{R}^N} \frac{\sigma(v)}{(\operatorname{Re}\lambda + \sigma(v))^{\frac{p}{q}}} \int\limits_0^{+\infty} e^{-t(\operatorname{Re}\lambda+\sigma(v))}\,dt \int\limits_{\mathbb{R}^N} |\tilde{\varphi}(y, v)|^p\,dy\,dv \\ &= \int\limits_{\mathsf{D}\times\mathbb{R}^N} \frac{\sigma(v)}{(\operatorname{Re}\lambda + \sigma(v))^{\frac{p}{q}+1}} |\varphi(y, v)|^p\,dy\,dv \\ &\leq \sup_{v\in\mathbb{R}^N} \frac{\sigma(v)}{(\operatorname{Re}\lambda + \sigma(v))^p} \|\varphi\|^p_{\mathsf{X}_p}.\end{aligned}$$

This yields

$$\|(\lambda - T)^{-1}\|_{\mathcal{L}(\mathsf{X}_p,\mathsf{X}^\sigma_p)} \leq \sup_{v\in\mathbb{R}^n} \frac{\sigma(v)^{\frac{1}{p}}}{\operatorname{Re}\lambda + \sigma(v)}. \qquad \square$$

Remark 8.4.1.

(a) For $p = 1$, one can derive from the estimate (8.25) that

$$\|(\lambda - T)^{-1}\|_{\mathcal{L}(\mathsf{X}_1,\mathsf{X}^\sigma_1)} \leq 1, \tag{8.26}$$

because, for any $\lambda \in \mathbb{C}_{\sigma_0}$, we have $\sup_{v\in\mathbb{R}^n} \frac{\sigma(v)}{\operatorname{Re}\lambda+\sigma(v)} \leq 1$ regardless of λ.

(b) For $p > 1$, we have

$$\|(\lambda - T)^{-1}\|_{\mathcal{L}(\mathsf{X}_p,\mathsf{X}^\sigma_p)} \leq \sigma_0^{-\frac{1}{q}}, \tag{8.27}$$

where $q = \frac{p}{p-1}$ is the conjugate of p.

(c) Note that, for any real λ such that $\lambda > -\sigma_0$ and $1 \le p < +\infty$, the operator $(\lambda - T)^{-1}$ is positive (in the lattice sense) on X_p, that is, it maps the positive cone of X_p into the positive cone of X_p^σ.

8.4.3 Preparatory results

Let $\mathbf{B}(\mathsf{X}_1^\sigma)$ denote the collection of all nonempty bounded subsets of X_1^σ and let $\mathbf{W}(\mathsf{X}_1^\sigma)$ be the subset of $\mathbf{B}(\mathsf{X}_1^\sigma)$ consisting of all weakly compact subsets of X_1^σ. As in Section 8.3.5, we define the map $\rho : \mathbf{B}(\mathsf{X}_1^\sigma) \to [0, +\infty)$ by

$$\rho(M) = \rho_1(M) + \rho_2(M),$$

where

$$\rho_1(M) = \limsup_{\varepsilon \to 0} \left\{ \sup_{\psi \in M} \left[\iint_E |\psi(x,v)| \sigma(v)\, dx\, dv,\ |E| \le \varepsilon \right] \right\} \tag{8.28}$$

with $|E|$ denotes the measure of $E \subset \mathsf{D} \times \mathbb{R}^N$ with respect to $\sigma(v)\, dx\, dv$ and

$$\rho_2(M) = \lim_{m \to +\infty} \left\{ \sup_{\psi \in M} \left[\int_D \int_{|v| \ge m} |\psi(x,v)| \sigma(v)\, dx\, dv \right] \right\}. \tag{8.29}$$

Arguing as in the proof of Proposition 8.3.4, we show that $\rho(\cdot)$ is a measure of weak noncompactness satisfying properties (1)–(8) of Proposition 8.3.4. In fact, $\rho(\cdot)$ is a regular measure of weak noncompactness on X_1^σ.

As for $\gamma(\cdot)$ (cf. (8.16)), for every bounded linear operator L on X_1^σ, we have

$$\rho\big(L(M)\big) \le \|L\|_{\mathcal{L}(\mathsf{X}_1^\sigma)} \|\rho(M), \quad \text{for all } M \in \mathbf{B}(\mathsf{X}_1^\sigma). \tag{8.30}$$

Now, let K be the linear operator defined by

$$K : \mathsf{X}_p^\sigma \to \mathsf{X}_p, \quad \psi \mapsto \int_{\mathbb{R}^n} \kappa(v, v') \psi(x, v')\, dv',$$

where the kernel $\kappa(\cdot,\cdot)$ is a measurable, nonnegative, real-valued function on $\mathbb{R}^N \times \mathbb{R}^N$.

Now we introduce the following hypothesis:

(H12) K is a compact operator from $\mathsf{L}^{p,\sigma}$ into L^p.

Lemma 8.4.2. *The operator T is dissipative on* X_2.

This result is well known in neutron transport theory. It is valid on X_p with $1 \le p < +\infty$ even if $\sigma(\cdot)$ satisfies the hypothesis (H11).

Proof. Let $\psi \in D(T)$. Since X_2 is a Hilbert space, the duality mapping $J : \mathsf{X}_2 \to 2^{\mathsf{X}_2}$ is single-valued and $J(\psi) = \{\psi\}$ (cf. Section 1.10). So, using the boundary conditions, we can write

$$\begin{aligned}\langle T\psi, J(\psi)\rangle &= \int\limits_{D\times\mathbb{R}^N} \psi(-v\cdot\nabla_x\psi)\,dx\,dv = \frac{-1}{2}\int\limits_{D\times\mathbb{R}^N} v\cdot\nabla_x|\psi|^2\,dx\,dv = \frac{-1}{2}\int\limits_{\partial D\times\mathbb{R}^N}|\psi|^2 v\cdot\nu_x\,d\gamma_x\,dv\\ &= \frac{1}{2}\big[\|\psi_{|\Gamma_-}\|^2_{L^{2,-}} - \|\psi_{|\Gamma_+}\|^2_{L^{2,+}}\big] = -\frac{1}{2}\|\psi_{|\Gamma_+}\|^2_{L^{2,+}}\\ &\le 0,\end{aligned}$$

where $\|\psi_{|\Gamma_\pm}\|^2_{L^{2,\pm}} = \int_{\Gamma_\pm}|\psi(x,v)|^2|v\cdot\nu_x|\,d\gamma_x\,dv$. Hence, for every $\psi \in D(T)$, we have $\langle T\psi, \psi\rangle \le 0$, which ends the proof. ☐

Lemma 8.4.3. *Let K be an operator from X_2^σ into X_2 with kernel $\kappa(v,v') = k_1(v)k_2(v')$ with $k_1(\cdot) \in \mathsf{L}^2$ and $\frac{k_2(\cdot)}{\sigma(\cdot)^{\frac{1}{2}}} \in \mathsf{L}^2$. For all $\varphi \in \mathsf{X}_2$ and $\psi \in \mathsf{X}_2^\sigma$, we have*

$$\langle K\psi, \varphi\rangle_{\mathsf{X}_2} = \langle \psi, K^*\varphi\rangle_{\mathsf{X}_2^\sigma},$$

where $\langle\cdot,\cdot\rangle_{\mathsf{X}_2^\sigma}$ (respectively, $\langle\cdot,\cdot\rangle_{\mathsf{X}_2}$) denotes the inner product on X_2^σ (respectively, on X_2) and

$$(K^*\varphi)(x,v) := \int\limits_{\mathbb{R}^N}\frac{k_2(v)k_1(v')}{\sigma(v)}\varphi(x,v')\,dv'. \tag{8.31}$$

Proof. Let $\varphi \in \mathsf{X}_2$ and $\psi \in \mathsf{X}_2^\sigma$. By Fubini theorem, we have

$$\begin{aligned}\langle K\psi,\varphi\rangle_{\mathsf{X}_2} &= \int\limits_{D\times\mathbb{R}^N} K\psi(x,v)\varphi(x,v)\,dx\,dv\\ &= \int\limits_{D\times\mathbb{R}^N}\Bigg[\int\limits_{\mathbb{R}^N}\frac{1}{\sigma(v')}k_1(v)k_2(v')\varphi(x,v)\,dv\Bigg]\psi(x,v')\sigma(v')\,dx\,dv'.\end{aligned}$$

It suffices to prove that $f \in \mathsf{X}_2^\sigma$, where $f(x,v') := \int_{\mathbb{R}^N}\frac{k_1(v)k_2(v')}{\sigma(v')}\varphi(x,v)\,dv$. By Hölder inequality, we have

$$\begin{aligned}\|f\|^2_{\mathsf{X}_2^\sigma} &\le \int\limits_{D\times\mathbb{R}^N}\Bigg[\int\limits_{\mathbb{R}^N}\frac{1}{\sigma(v')}|k_1(v)||k_2(v')||\varphi(x,v)|\,dv\Bigg]^2\sigma(v')\,dx\,dv'\\ &\le \left\|\frac{k_2}{\sqrt{\sigma(\cdot)}}\right\|^2_{\mathsf{L}^2}\|k_1\|^2_{\mathsf{L}^2}\int\limits_{D\times\mathbb{R}^N}|\varphi(x,v)|^2\,dx\,dv < \infty.\end{aligned}$$

☐

Lemma 8.4.4. *Let $\lambda \in \mathbb{C}_{\sigma_0}$ and assume that the hypotheses* (H11) *and* (H12) *are satisfied. Denote by K^* the dual operator of K (it is defined by* (8.31)*). Then $K^*(\lambda - T)^{-1}K$ is a compact operator on X_2^σ.*

Proof. Let $\lambda \in \mathbb{C}_{\sigma_0}$. Since K is assumed to be a compact operator from X_2^σ into X_2, approximating K by a sequence of finite-rank operators and by linearity, we may restrict ourselves to collision operators K with kernels of the form

$$\kappa(v, v') = k_1(v)k_2(v'),$$

where $k_1 \in \mathsf{L}^2$ and $\frac{k_2}{\sigma^{\frac{1}{2}}} \in \mathsf{L}^2$. Thus, using (H11), one has $h_2 = \frac{k_2}{\sigma} \in \mathsf{L}^2$. By density, we can assume that $k_1, h_2 \in \mathcal{C}_c(\mathbb{R}^N \setminus \mathcal{O})$. According to Lemma 8.4.3,

$$\begin{cases} K^* : \mathsf{X}_2 \to \mathsf{X}_2^\sigma, \\ \psi \mapsto K^*\psi(x, v) = \int_{\mathbb{R}^N} h_2(v)k_1(v')\psi(x, v')\,dv'. \end{cases}$$

For $\varphi \in \mathsf{X}_2^\sigma$, we have

$$\begin{aligned}
&(K^*(\lambda - T)^{-1}K\varphi)(x, v) \\
&\quad = \int\limits_{\mathbb{R}^N} h_2(v)k_1(v')((\lambda - T)^{-1}K\varphi)(x, v')\,dv' \\
&\quad = \int\limits_{\mathbb{R}^N} h_2(v)k_1(v') \int\limits_0^{t^-(x,v')} e^{-t(\lambda+\sigma(v'))} K\varphi(x - tv', v')\,dt\,dv' \\
&\quad = \int\limits_{\mathbb{R}^N} h_2(v)k_1(v')\,dv' \int\limits_0^{t^-(x,v')} e^{-t(\lambda+\sigma(v'))}\,dt \int\limits_{\mathbb{R}^N} k_1(v')k_2(v'')\varphi(x - tv', v'')\,dv''.
\end{aligned}$$

The change of variables $y = x - tv'$ leads to

$$(K^*(\lambda - T)^{-1}K\varphi)(x, v) = \int\limits_{\mathsf{D}\times\mathbb{R}^N} h_2(v) \int\limits_0^{+\infty} k_1^2\Big(\frac{x - y}{t}\Big)k_2(v'')e^{-t(\lambda+\sigma(\frac{x-y}{t}))}\varphi(y, v'')\frac{dt}{t^n}\,dy\,dv''.$$

Thus the operator $K^*(\lambda - T)^{-1}K$ can be factorized as $K^*(\lambda - T)^{-1}K = \mathcal{A}\Xi_\lambda\mathcal{F}$, where

$$\begin{cases} \mathcal{A} : \mathsf{X}_2^\sigma \to L^2(\mathsf{D}), \\ \mathcal{A}\varphi(x, v) := \int_{\mathbb{R}^N} k_2(v)\varphi(x, v)\,dv, \end{cases} \qquad \begin{cases} \mathcal{F} : L^2(\mathsf{D}) \to \mathsf{X}_2^\sigma, \\ \mathcal{F}\varphi(x) := h_2(v)\varphi(x), \end{cases}$$

and

$$\begin{cases} \Xi_\lambda : L^2(\mathsf{D}) \to L^2(\mathsf{D}), \\ \Xi_\lambda\varphi(x) := \int_D n_\lambda(x, y)\varphi(y)\,dy, \end{cases}$$

where

$$n_\lambda(x,y) := \int_0^{+\infty} k_1^2\left(\frac{x-y}{t}\right) e^{-t(\lambda+\sigma(\frac{x-y}{t}))} \frac{dt}{t^n}.$$

Because the operators $\mathcal{A}$ and $\mathcal{F}$ are bounded, it suffices to establish that Ξ_λ is a compact operator. Indeed, let $\varphi \in L^2(\mathsf{D})$. According to Hölder inequality, we have

$$\begin{aligned}
\|\Xi_\lambda \varphi\|_{L^2(\mathsf{D})}^2 &= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \int_0^{+\infty} k_1\left(\frac{x-y}{t}\right)^2 e^{-t(\lambda+\sigma(\frac{x-y}{t}))} \tilde{\varphi}(y) \frac{dt}{t^n}\, dy \right|^2 dx \\
&\leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \int_0^{+\infty} \left| k_1\left(\frac{x-y}{t}\right)\right|^2 e^{-t(\operatorname{Re}\lambda+\sigma(\frac{x-y}{t}))} \frac{dt}{t^n}\, dy \right) \\
&\quad \times \left(\int_{\mathbb{R}^N} \int_0^{+\infty} \left| k_1\left(\frac{x-y}{t}\right)\right|^2 e^{-t(\operatorname{Re}\lambda+\sigma(\frac{x-y}{t}))} |\tilde{\varphi}(y)|^2 \frac{dt}{t^n}\, dy \right) dx,
\end{aligned}$$

where $\tilde{\varphi}$ denotes the zero extension of φ outside D.

Using the change of variables, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \int_0^{+\infty} \left| k_1\left(\frac{x-y}{t}\right)\right|^2 e^{-t(\operatorname{Re}\lambda+\sigma(\frac{x-y}{t}))} \frac{dt}{t^n}\, dy &= \int_{\mathbb{R}^N} \int_0^{+\infty} \left| k_1\left(\frac{z}{t}\right)\right|^2 e^{-t(\operatorname{Re}\lambda+\sigma(\frac{z}{t}))} \frac{dt}{t^n}\, dz \\
&= \int_{\mathbb{R}^N} \int_0^{+\infty} |k_1(v)|^2 e^{-t(\operatorname{Re}\lambda+\sigma(v))}\, dt\, dv \\
&\leq \frac{1}{\operatorname{Re}\lambda + \sigma_0} \|k_1\|_{\mathsf{L}^2}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Xi_\lambda \varphi\|_{L^2(\mathsf{D})}^2 &\leq \frac{1}{\operatorname{Re}\lambda + \sigma_0} \|k_1\|_{\mathsf{L}^2}^2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_0^{+\infty} |k_1(v)|^2 e^{-t(\operatorname{Re}\lambda+\sigma(v))} |\tilde{\varphi}(x-tv)|^2\, dt\, dv\, dx \\
&= \frac{1}{\operatorname{Re}\lambda + \sigma_0} \|k_1\|_{\mathsf{L}^2}^2 \int_{\mathbb{R}^N} \int_0^{+\infty} |k_1(v)|^2 e^{-t(\operatorname{Re}\lambda+\sigma(v))} \int_{\mathbb{R}^N} |\tilde{\varphi}(x-tv)|^2\, dx\, dv\, dt \\
&= \frac{1}{\operatorname{Re}\lambda + \sigma_0} \|k_1\|_{\mathsf{L}^2}^2 \int_{\mathbb{R}^N} \int_0^{+\infty} |k_1(v)|^2 e^{-t(\operatorname{Re}\lambda+\sigma(v))}\, dt\, dv \int_{\mathsf{D}} |\varphi(y)|^2\, dy \\
&= \frac{\|\varphi|_{L^2(\mathsf{D})}^2}{(\operatorname{Re}\lambda + \sigma_0)^2} \|k_1\|_{\mathsf{L}^2}^4.
\end{aligned}$$

Accordingly,

$$\|\Xi_\lambda\|_{\mathcal{L}(L^2(\mathrm{D}))} \leq \frac{\|k_1\|^2_{\mathrm{L}^2}}{\operatorname{Re}\lambda + \sigma_0}.$$

Hence, Ξ_λ depends continuously on $k_1 \in \mathrm{L}^2$. Now, since Ξ_λ is an integral operator, and the set of bounded functions which vanish in a neighborhood of $v = 0$ is dense in L^2, it follows that Ξ_λ is a limit, in the uniform topology, of integral operators with bounded kernels. Hence Ξ_λ is a compact operator, which completes the proof. ☐

Theorem 8.4.1. *Assume that the hypotheses* (H11)–(H12) *are satisfied. Then, for any $\lambda \in G_{\sigma_0}$, the following two assertions hold true:*
(a) *$(\lambda - T)^{-1}K$ is compact on X^σ_p, for all $p \in (1, +\infty)$,*
(b) *$(\lambda - T)^{-1}K$ is a Dunford–Pettis operator (cf. Definition* 1.9.8*) from X^σ_1 into itself.*

It was shown in [155] that, in slab geometry, $(\lambda - T)^{-1}K$ is weakly compact on X^σ_1. However, for multidimensional bounded geometry, the weak compactness of $(\lambda - T)^{-1}K$ in X^σ_1 is an open problem.

Proof. (a) We know from Lemma 8.4.4 that the operator $K^*(\lambda - T)^{-1}K$ is compact in X^σ_2. We shall now prove the compactness of $(\lambda - T)^{-1}K$ in X^σ_2. To this end, let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of X^σ_2 which converges weakly to 0, then it suffices to establish that the sequence $((\lambda - T)^{-1}K\psi_n)_{n\in\mathbb{N}}$ converges strongly in X^σ_2 to 0. Arguing as in the proof of Lemma 8.4.4, we may restrict ourselves to collision operators K with kernels of the form

$$\kappa(v, v') = k_1(v)k_2(v')$$

where $k_1, h_2 = \frac{k_2}{\sigma} \in \mathcal{C}_c(\mathbb{R}^N \setminus \mathcal{O})$.

For each $n \in \mathbb{N}$, put $\varphi_n = (\lambda - T)^{-1}K\psi_n$.

We claim that $(\varphi_n)_{n\in\mathbb{N}}$ converges strongly in X^σ_2 to 0. Indeed, using the fact that $\mathrm{X}^\sigma_2 \subset \mathrm{X}_2$ and the dissipativity of T, for each $n \in \mathbb{N}$, we get

$$\left|\langle(\lambda - T)\varphi_n, \varphi_n\rangle_{\mathrm{X}_2}\right| \geq \operatorname{Re}\langle(\lambda - T)\varphi_n, \varphi_n\rangle_{\mathrm{X}_2} \geq \operatorname{Re}\lambda\|\varphi_n\|^2_{\mathrm{X}_2}.$$

Applying Lemma 8.4.3, one can write

$$\begin{aligned}\operatorname{Re}\lambda\|\varphi_n\|^2_{\mathrm{X}_2} &\leq \left|\langle(\lambda - T)\varphi_n, \varphi_n\rangle_{\mathrm{X}_2}\right| \\ &= \left|\langle K\psi_n, (\lambda - T)^{-1}K\psi_n\rangle_{\mathrm{X}_2}\right| \\ &= \left|\langle\psi_n, K^*(\lambda - T)^{-1}K\psi_n\rangle_{\mathrm{X}^\sigma_2}\right|.\end{aligned}$$

According to Lemma 8.4.4, the operator $K^*(\lambda - T)^{-1}K$ is compact in X^σ_2, so we have

$$\lim_{n\to+\infty}\|K^*(\lambda - T)^{-1}K\psi_n\|_{\mathrm{X}^\sigma_2} = 0$$

and therefore, $\varphi_n \to 0$ as $n \to +\infty$ in X_2. On the other hand, we have

$$\begin{aligned}
\|\varphi_n\|^2_{X_2^\sigma} &= \int\limits_{D\times\mathbb{R}^N} |\varphi_n(x,v)|^2 \sigma(v)\,dx\,dv \\
&= \int\limits_{D\times\mathbb{R}^N} \left| \int\limits_0^{t^-(x,v)} (K\varphi_n)(x-tv,v)\,dt \right|^2 \sigma(v)\,dx\,dv \\
&= \int\limits_{D\times\mathbb{R}^N} \left| \int\limits_0^{t^-(x,v)} \int\limits_{\mathbb{R}^n} k_1(v)k_2(v')\varphi_n(x-tv,v')\,dv'\,dt \right|^2 \hat{\sigma}(v)\,dx\,dv \\
&= \|\varphi_n\|^2_{X_2^{\hat{\sigma}}} \\
&\le \|\sigma\|^2_{L^\infty(\operatorname{supp}(k_1))} \|\varphi_n\|^2_{X_2},
\end{aligned}$$

where

$$\hat{\sigma}(v) = \begin{cases} \sigma(v) & \text{if } v \in \operatorname{supp}(k_1), \\ 0 & \text{if } v \notin \operatorname{supp}(k_1). \end{cases}$$

Consequently, $(\varphi_n)_{n\in\mathbb{N}}$ converges strongly to 0 in X_2^σ. This proves that $(\lambda - T)^{-1}K$ is compact in X_2^σ. By an interpolation argument (see [147, Theorem 3.10]), we conclude that $(\lambda - T)^{-1}K$ is compact in X_p^σ with $1 < p < +\infty$.

(b) Now we will prove that $(\lambda - T)^{-1}K$ is a Dunford–Pettis operator on X_1^σ. Let $\mathcal{O}$ be a weakly compact subset of X_1^σ. Our goal is to establish that $(\lambda - T)^{-1}K(\mathcal{O})$ is a compact subset of X_1^σ for the norm topology. To do this, let $\varphi \in \mathcal{O}$, $c > 0$ and write

$$\varphi_c = (\lambda - T)^{-1}K(\varphi\chi_{\{|\varphi|\ge c\}}) \quad \text{and} \quad \hat{\varphi}_c = (\lambda - T)^{-1}K(\varphi\chi_{\{|\varphi|< c\}})$$

where $\chi(\cdot)$ denotes the characteristic function. For any $(y,w) \in \mathbb{R}^{2N}$, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N\times\mathbb{R}^N} |\tilde{\varphi}_c(x+y,v+w) - \tilde{\varphi}_c(x,v)|\sigma(v)\,dx\,dv \\
&\qquad \le \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} |\tilde{\varphi}_c(x+y,v+w)|\sigma(v)\,dx\,dv + \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} |\tilde{\varphi}_c(x,v)|\sigma(v)\,dx\,dv,
\end{aligned}$$

where $\tilde{\varphi}_c$ denotes the zero extension of φ_c outside $D \times \mathbb{R}^N$. It follows from the estimate (8.26) that

$$\begin{aligned}
\int\limits_{\mathbb{R}^N\times\mathbb{R}^N} |\tilde{\varphi}_c(x,v)|\tilde{\sigma}(v)\,dx\,dv &\le \|K(\varphi\chi_{\{|\varphi|\ge c\}})\|_{X^\sigma} \\
&\le \|K\|_{\mathcal{L}(X_1^\sigma,X_1)} \int\limits_{D\times\mathbb{R}^N} |\varphi\chi_{\{|\varphi|\ge c\}}(x,v)|\sigma(v)\,dx\,dv.
\end{aligned}$$

On the other hand, considering (8.26), one can write

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \big|\tilde{\varphi}_c(x+y, v+w)\big|\sigma(v)\,dx\,dv \\
&\quad = \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \big|\tilde{\varphi}_c(x, v)\big|\sigma(v-w)\,dx\,dv \\
&\quad = \int\limits_{\mathsf{D}\times\mathbb{R}^N} \big|\varphi_c(x, v)\big|\sigma(v-w)\,dx\,dv \\
&\quad \le \sup_{v\in\mathbb{R}^N} \frac{\sigma(v)}{\operatorname{Re}\lambda+\sigma(v)} \int\limits_{\mathsf{D}\times\mathbb{R}^N} \big|K(\varphi\chi_{\{|\varphi|\ge c\}})(x, v)\big|\sigma(v-w)\,dx\,dv \\
&\quad \le \int\limits_{\mathbb{R}^N} \big|k_1(v)\big|\sigma(v-w)\,dv \int\limits_{\mathsf{D}\times\mathbb{R}^N} \big|k_2(v')\big|\big|\varphi\chi_{\{|\varphi|\ge c\}}(x, v')\big|\,dx\,dv' \\
&\quad \le \int\limits_{(\operatorname{supp}(k_1)-\{w\})} \big|k_1(v+w)\big|\sigma(v)\,dv\|h_2\|_{L^\infty(\mathbb{R}^N)} \int\limits_{\mathsf{D}\times\mathbb{R}^N} \big|\varphi\chi_{\{|\varphi|\ge c\}}(x, v')\big|\sigma(v')\,dx\,dv'.
\end{aligned}
$$

There exists $\eta > 0$ such that $(\operatorname{supp}(k_1) - \mathbb{B}_\eta) \subset \mathbb{R}^N \setminus S$. Then, we get

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \big|\tilde{\varphi}_c(x+y, v+w)\big|\tilde{\sigma}(v)\,dx\,dv \\
&\quad \le \|\sigma\|_{L^\infty(\operatorname{supp}(k_1)-\mathbb{B}_\eta)}\|k_1\|_{\mathsf{L}^1}\|h_2\|_{L^\infty(\mathbb{R}^n)} \int\limits_{\mathsf{D}\times\mathbb{R}^N} \big|\varphi\chi_{\{|\varphi|\ge c\}}(x, v')\big|\sigma(v')\,dx\,dv'.
\end{aligned}
$$

Consequently,

$$
\int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \big|\tilde{\varphi}_c(x+y, v+w) - \tilde{\varphi}_c(x, v)\big|\tilde{\sigma}(v)\,dx\,dv \le C \int\limits_{\mathsf{D}\times\mathbb{R}^N} \big|\varphi\chi_{\{|\varphi|\ge c\}}(x, v')\big|\sigma(v)\,dx\,dv,
$$

where

$$
C = \big(\|K\|_{\mathcal{L}(\mathsf{X}_1^\sigma,\mathsf{X}_1)} + \|\sigma\|_{L^\infty(\operatorname{supp}(k_1)-\mathbb{B}_\eta)}\|k_1\|_{\mathsf{L}^1}\|h_2\|_{L^\infty(\mathbb{R}^N)}\big). \tag{8.32}
$$

Since $\mathcal{O}$ is weakly compact and the measure of the set $\{(x, v) \in \mathsf{D}\times\mathbb{R}^N : |\varphi(x, v)| \ge c\}$ tends to 0 as $c \to +\infty$, it follows from [79, Theorem 15, p. 76] that, for any $\varepsilon > 0$, there exist real numbers $c > 0$ and $\eta > 0$ such that, for all $w \in \mathbb{B}_\eta$, we have

$$
\int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \big|\tilde{\varphi}_c(x+y, v+w) - \tilde{\varphi}_c(x, v)\big|\sigma(v)\,dx\,dv \le \varepsilon,
$$

uniformly in $\varphi \in \mathcal{O}$.

For $c > 0$ chosen as above, according to [2, Theorem 2.21], it suffices to prove that

$$
\mathcal{O}_c = \big\{(\lambda - T)^{-1}K(\varphi\chi_{\{|\varphi|<c\}}) : \varphi \in \mathcal{O}\big\}
$$

is relatively compact in X_1^σ. To do this, we first observe that $\mathcal{O}_c$ is bounded in X_2^σ because $\mathcal{O}$ is bounded in X_1^σ and $\varphi\chi_{\{|\varphi|<c\}} \in \mathsf{X}_2^\sigma$. Hence, it suffices to prove that $(\lambda - T)^{-1}K$ is a compact operator from X_2^σ into itself. This was established in part (a) of the proof. □

8.4.4 Existence results

To derive existence results for the boundary value problem (8.23), we introduce the following assumptions:

(H13) $f(\cdot,\cdot,\cdot)$ is a Carathéodory function satisfying the growth condition

$$\left|f(x,v,u)\right| \le \alpha(x,v)\sigma(v)^{-\frac{1}{p}} + \beta|u|, \tag{8.33}$$

for a. e. $(x,v) \in \mathsf{D} \times \mathbb{R}^N$, every $u \in \mathbb{C}$ with a fixed nonnegative function $\alpha \in \mathsf{X}_p$, and a fixed nonnegative constant β.

Remark 8.4.2. As it was seen in Proposition 1.15.1, under assumption (H13), the superposition operator $\mathcal{N}_f$ is continuous. Moreover, the fact that it maps bounded sets of X_p^σ into bounded sets of X_p^σ follows immediately from the estimate (8.33) (cf. Section 1.15).

(H14) $\beta\sigma_0^{-\frac{1}{q}}\|K\|_{\mathcal{L}(\mathsf{X}_p^\sigma,\mathsf{X}_p)} < 1$, where β is the constant appearing in the growth condition (8.33) and q denotes the conjugate exponent of p.

For $r > 0$, we denote by $\mathbb{B}_r^{\sigma,p}$ the set

$$\mathbb{B}_r^{\sigma,p} = \{\psi \in \mathsf{X}_p^\sigma : \|\psi\|_{\mathsf{X}_p^\sigma} \le r\}.$$

Now, we are in a position to state the following existence result.

Theorem 8.4.2. *Let* $1 \le p < +\infty$, $K \in \mathcal{L}(\mathsf{X}_p^\sigma,\mathsf{X}_p)$, *and assume that the hypotheses* (H11)–(H14) *are satisfied. Then, for each* $\lambda \in \mathbb{C}_{\sigma_0}$ *and* $r \in \mathbb{R}$ *satisfying*

$$r \ge \frac{\sigma_0^{-\frac{1}{q}}\|K\|_{\mathcal{L}(\mathsf{X}_p^\sigma,\mathsf{X}_p)}\|\alpha\|_{\mathsf{X}_p}}{1 - \beta\sigma_0^{-\frac{1}{q}}\|K\|_{\mathcal{L}(\mathsf{X}_p^\sigma,\mathsf{X}_p)}}, \tag{8.34}$$

problem (8.23) *has at least one solution in* $\mathbb{B}_r^{\sigma,p}$.

Remark 8.4.3. It is clear that if $r \in \mathbb{R}$ satisfies equation (8.34) and hypothesis (H14) holds, then $r > 0$.

Proof. Let λ be a complex number such that $\operatorname{Re}\lambda > -\sigma_0$. Then, according to Lemma 8.4.1, the operator $\lambda - T$ is invertible, and therefore problem (8.23) my be written in the form

$$\psi = \amalg(\lambda)\psi, \quad \psi_{|\Gamma_-} = 0,$$

where $\amalg(\lambda) = (\lambda - T)^{-1}K\mathcal{N}_f$.

It is clear that $\amalg(\lambda)$ is continuous. We shall now check that $\amalg(\lambda)$ leaves $\mathbb{B}_r^{\sigma,p}$ invariant. According to the estimate (8.33), there exist a constant $\beta > 0$ and a function $\alpha \in \mathsf{X}_p$ such that

$$\left|f(x,v,\psi(x,v))\right| \le \frac{\alpha(x,v)}{\sigma(v)^{\frac{1}{p}}} + \beta\left|\psi(x,v)\right| \quad \text{a.e. in } (x,v) \text{ for all } \psi \in \mathsf{X}_p^\sigma. \tag{8.35}$$

For any $\psi \in \mathbb{B}_r^{\sigma,p}$, we have

$$\begin{aligned}
\|\amalg(\lambda)\psi\|_{\mathsf{X}_p^\sigma} &= \|(\lambda - T)^{-1}K\mathcal{N}_f\psi\|_{\mathsf{X}_p^\sigma} \\
&\le \|(\lambda - T)^{-1}\|_{\mathcal{L}(\mathsf{X}_p,\mathsf{X}_p^\sigma)}\|K\|_{\mathcal{L}(\mathsf{X}_p^\sigma,\mathsf{X}_p)}\|\mathcal{N}_f\psi\|_{\mathsf{X}_p^\sigma} \\
&\le \sigma_0^{-\frac{1}{q}} \sup_{v\in V} \frac{\sigma(v)}{\operatorname{Re}\lambda + \sigma(v)}\|K\|_{\mathcal{L}(\mathsf{X}_p^\sigma,\mathsf{X}_p)}\left(\|\alpha\|_{\mathsf{X}_p} + \beta r\right) \\
&\le \sigma_0^{-\frac{1}{q}}\|K\|_{\mathcal{L}(\mathsf{X}_p^\sigma,\mathsf{X}_p)}\left(\|\alpha\|_{\mathsf{X}_p} + \beta r\right).
\end{aligned}$$

Hence, in order to show that $\amalg(\lambda)$ maps $\mathbb{B}_r^{\sigma,p}$ into itself, it suffices to choose r satisfying condition (8.34). Therefore, for such an r and for all $p \in [1,+\infty)$, we have $\amalg(\lambda)(\mathbb{B}_r^{\sigma,p}) \subset \mathbb{B}_r^{\sigma,p}$.

To complete the proof, we must treat separately the cases $p \in (1,+\infty)$ and $p = 1$.

- Let $p \in (1,+\infty)$. It follows from Theorem 8.4.1(a) that $(\lambda - T)^{-1}K$ is a compact operator of $\mathcal{L}(\mathsf{X}_p^\sigma, \mathsf{X}_p^\sigma)$ and therefore $\amalg(\lambda)$ is compact too (because $\mathcal{N}_f$ is continuous). This shows that $\amalg(\lambda)$ satisfies the hypotheses of Corollary 7.2.3. So, problem (8.23) has at least one solution in $\mathbb{B}_r^{\sigma,p}$.
- Let $p = 1$. In this case, we shall show that $\amalg(\lambda)$ satisfies the conditions of Theorem 7.8.2(a).

We claim that the operator $\amalg(\lambda)$ is ws-compact on X_1^σ. To see this, let $(\psi_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in X_1^σ. Hence, the set $\Psi = \{\psi_n : n \in \mathbb{N}\}$ is a relatively weakly compact set of X_1^σ. Let A be a measurable subset of $\mathsf{D} \times \mathbb{R}^N$. It follows that

$$\left|\int_A (\mathcal{N}_f\psi)(x,v)\,dx\sigma(v)\,dv\right| \le \int_A \left|\alpha(x,v)\right| dx\,dv + \beta\int_A \left|\psi(x,v)\right| dx\sigma(v)\,dv$$

for all $\psi \in \Psi$, and therefore,

$$\rho_1\left(\mathcal{N}_f(\Psi)\right) \le \beta\rho_1(\Psi),$$

uniformly in $\psi \in \Psi$. Now, let $m > 0$. We have

$$\left|\int_{D}\int_{|v|\geq m}(\mathcal{N}_f\psi)(x,v)\,dx\sigma(v)\,dv\right|\leq\int_{D}\int_{|v|\geq m}|\alpha(x,v)|\,dx\,dv+\beta\int_{D}\int_{|v|\geq m}|\psi(x,v)|\,dx\sigma(v)\,dv$$

for all $\psi\in\Psi$, and therefore,

$$\rho_2(\mathcal{N}_f(\Psi))\leq\beta\rho_2(\Psi).$$

Accordingly,

$$\rho(\mathcal{N}_f(\Psi))\leq\beta\rho(\Psi). \tag{8.36}$$

Since Ψ is relatively weakly compact, we conclude that $\rho(\Psi)=0$ and therefore, $\rho(\mathcal{N}_f(\Psi))=0$, so that $(\mathcal{N}_f\psi_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(\mathcal{N}_f\psi_{n_k})_{k\in\mathbb{N}}$. Let Φ be its weak limit. Since the set

$$\mathsf{Z}:=\{\mathcal{N}_f\psi_{n_k}:k\in\mathbb{N}\}\cup\{\Phi\}$$

is weakly compact on X_1^σ and X_1^σ has Dunford–Pettis property, it follows from Theorem 8.4.1 that the set $(\lambda-T)^{-1}K(\mathsf{Z})$ is a compact subset of X_1^σ. Thus, $(\amalg(\lambda)\psi_{n_k})_{k\in\mathbb{N}}$ has a strongly convergent subsequence in X_1^σ. Hence $\amalg(\lambda)$ is ws-compact on X_1^σ.

We claim that $\amalg(\lambda)$ is a ρ-$\beta\|K\|$-contractive. Let us notice that if S is a bounded subset of X_1^σ and f satisfies (H18), then

$$\rho(\mathcal{N}_f(S))\leq\beta\rho(S). \tag{8.37}$$

It follows from Eqs. (8.26), (8.30), and (8.37) that

$$\begin{aligned}\rho(\amalg(\lambda)(S))&=\rho((\lambda-T)^{-1}K\mathcal{N}_f(S))\\&\leq\|(\lambda-T)^{-1}K\|_{\mathcal{L}(\mathsf{X}_1^\sigma)}\rho(\mathcal{N}_f(S))\\&\leq\beta\|K\|_{\mathcal{L}(\mathsf{X}_1^\sigma,\mathsf{X}_1)}\rho(S).\end{aligned}$$

Next, making use of the hypothesis (H14) for $p=1$, we conclude that the operator $\amalg(\lambda)$ is ρ-$\beta\|K\|$-contractive.

Hence, for any $\lambda\in\mathbb{C}_{\sigma_0}$ and for each r satisfying condition (8.34), $\amalg(\lambda)$ satisfies the conditions of Theorem 7.8.2(a). Hence, problem (8.23) has at least one solution in $\mathbb{B}_r^{\sigma,1}$. □

The remainder of this section is devoted to the existence of positive solutions to the boundary value problem (8.23). Note that X_p^σ is a Banach lattice space; its positive cone will be denoted by $\mathsf{X}_p^{\sigma,+}$.

Let $r>0$. We define the set $\mathbb{B}_{r,+}^{\sigma,p}$ by

$$\mathbb{B}_{r,+}^{\sigma,p}:=\mathbb{B}_r^{\sigma,p}\cap\mathsf{X}_p^{\sigma,+}.$$

According to Remark 8.4.1(c), for any real $\lambda > -\sigma_0$, $(\lambda - T)^{-1}$ is a positive operator (in the lattice sense). If further the hypotheses of Theorem 8.4.2 are satisfied, $\mathcal{N}_f(X_p^{\sigma,+}) \subseteq X_p^{\sigma,+}$ and K is positive, then $(\lambda - T)^{-1}K\mathcal{N}_f$ leaves the positive cone $X_p^{\sigma,+}$ invariant. Now, we replace $\mathbb{B}_r^{\sigma,p}$ by $\mathbb{B}_{r,+}^{\sigma,p}$ and, by the same reasoning as in the proof of Theorem 8.4.2, we obtain the following result.

Corollary 8.4.1. *Let* $1 \leq p < \infty$ *and assume that the hypotheses of Theorem* 8.4.2 *are satisfied. Further, if K is positive and* $\mathcal{N}_f(X_p^{\sigma,+}) \subseteq X_p^{\sigma,+}$, *then problem* (8.23) *has at least one solution in* $\mathbb{B}_{r,+}^{\sigma,p}$.

Next, we shall pay attention on the existence of nontrivial positive solutions.

Corollary 8.4.2. *Let* $1 < p < \infty$, $K \in \mathcal{L}(X_p^{\sigma}, X_p)$, $\lambda \in \mathbb{C}_{\sigma_0}$, $r > 0$, *and assume that the hypotheses* (H11)–(H14) *are satisfied. Furthermore, suppose that K is positive and* $\mathcal{N}_f(X_p^{\sigma,+}) \subseteq X_p^{\sigma,+}$. *If there is* $\zeta > 0$ *and* $0 \neq \psi_0 \in \mathbb{B}_{r,+}^{\sigma,p}$ *such that*

(a) $\psi_0 \notin N(K)$ *where* $N(K)$ *denotes the null space of K,*

(b) $(\mathcal{N}_f\psi)(x,v) \geq \zeta\psi_0(x,v)$ *for all* $\psi \in \mathbb{B}_{r,+}^{\sigma,p}$,

then there exists $\eta > 0$ *such that the problem*

$$\begin{cases} v\cdot\nabla_x\psi(x,v) + \sigma(v)\psi(x,v) + \lambda\psi(x,v) = \eta\int_{\mathbb{R}^n}\kappa(v,v')f(x,v',\psi(x,v'))\,dv', \\ \psi_{|\Gamma_-} = 0, \end{cases} \tag{8.38}$$

has at least one solution in $\mathbb{B}_{r,+}^{\sigma,p}$ *satisfying* $\|\psi^*\|_{X_p^{\sigma}} = r$.

Proof. Arguing as in the proofs of Theorems 8.4.2 and Corollary 8.4.1, we infer that, for any real number λ such that $\lambda > -\sigma_0$ and for all r satisfying (8.34), the operator $\amalg(\lambda)$ is continuous and maps $\mathbb{B}_{r,+}^{\sigma,p}$ into itself. On the other hand, using the positivity of $(\lambda - T)^{-1}$ and K, together with assumptions (a) and (b), we see that

$$\amalg(\lambda)(\psi) \geq \zeta(\lambda - T)^{-1}K(\psi_0) \geq 0 \quad \text{and} \quad (\lambda - T)^{-1}K(\psi_0) \neq 0 \quad \text{for all } \psi \in \mathbb{B}_{r,+}^{\sigma,p}.$$

Accordingly, for all $\psi \in \mathbb{B}_{r,+}^{\sigma,p}$, we have

$$\|\amalg(\lambda)(\psi)\|_{X_p^{\sigma}} \geq \zeta\|(\lambda - T)^{-1}(\psi_0)\|_{X_p^{\sigma}} > 0,$$

and consequently,

$$\inf\{\|\amalg(\lambda)(\psi)\|_{X_p^{\sigma}} : \psi \in \mathbb{B}_{r,+}^{\sigma,p}\} > 0.$$

Hence, for $\lambda \geq -\sigma_0$, one can define the operator H_λ on $\mathbb{B}_{r,+}^{\sigma,p}$ by

$$\mathsf{H}_\lambda(\psi) = r\frac{\amalg(\lambda)(\psi)}{\|\amalg(\lambda)(\psi)\|_{X_p^{\sigma}}} \qquad \forall\psi \in \mathbb{B}_{r,+}^{\sigma,p}.$$

Using the same arguments as in the proof of Theorem 8.4.2 for $p \in (1, +\infty)$, we see that H_λ is continuous, compact on X_p^σ, and maps $\mathbb{B}_{r,+}^{\sigma,p}$ into itself. So, applying Corollary 7.2.3, we infer that H_λ has a fixed point ψ^* in $\mathbb{B}_{r,+}^{\sigma,p}$ satisfying $\|\psi^*\|_{\mathsf{X}_p^\sigma} = r$. Putting $\eta = \frac{r}{\|\amalg(\lambda)(\psi^*)\|_{\mathsf{X}_p^\sigma}}$, we get $\amalg(\lambda)(\psi^*) = \eta^{-1}\psi^*$. Consequently, $\psi^* \in D(T) \cap \mathbb{B}_{r,+}^{\sigma,p}$ and

$$v.\nabla_x \psi^*(x,v) + \sigma(v)\psi^*(x,v) + \lambda\psi^*(x,v) = \eta \int\limits_{\mathbb{R}^n} \kappa(v,v')f(x,v',\psi^*(x,v'))\,dv'.$$

This completes the proof. □

Remark 8.4.4. We note that the result of Corollary 8.4.2 holds true without any restriction on the size of the ball $\mathbb{B}_r$. However, the problem is open for $p = 1$. In fact, the operator H_λ in the proof above is ws-compact in X_1^σ but we did not succeed in showing that it is ρ-k-contractive for some $k \in [0,1)$.

8.5 Bibliographical remarks

The first example (cf. Section 8.1) is from [99] and it was rewritten in terms of ws-compact operators. The second example is dedicated to the existence results for a nonlinear functional integral equation. This example is from [236]. In Section 8.3 we discussed the existence of solutions of nonlinear boundary value problem which involves a multidimensional transport equation with delayed neutrons in a bounded spatial domain. To simplify the presentation, we considered the case of vacuum boundary conditions (the incoming flux is zero). For general boundary conditions, we refer to the work [1]. The concept of regular collision operators is due to M. Mokhtar-Kharroubi [179] (see also [180, 181]). Proposition 8.3.1, established in [181], provides an approximation property for this class of operators in L^p-spaces $1 < p < \infty$. The same proposition for L^1-spaces, Proposition 8.3.2, is due to B. Lods [165]. Compactness properties of the operators $K(\lambda - T)^{-1}$ and $(\lambda - T)^{-1}K$ were established in [179, 180]. Similar compactness results for general boundary condition, in slab geometry as well as bounded geometry, may be found in [149, 150]. Theorem 8.3.2 is due to K. Latrach and A. Zeghal [160] (see also [42]) while Theorem 8.4.2 was established in [1]. In Section 8.4, the existence of solutions to a stationary multidimensional transport equation involving unbounded collision frequencies and unbounded collision operators was discussed. Here again, and for simplicity, we considered the case of vacuum boundary conditions (the incoming flux is zero). The results of this section may be found in [156], and we followed the presentation given in this paper.

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